Shared-state Monad

Denotational semantics for parallel execution

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Algebraic Effects

Roadmap to Contextual Equivalence via Denotational Semantics

- Equational Theory UU-last $U_{x,v} U_{x,u} \chi = U_{x,u} \chi$
- Monad $States \to X \times States$
- Denotations $[\![\mathtt{x} := \mathtt{z}?]\!] = \tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u} \tilde{\star}) = \lambda\sigma.\ \left\langle \star, \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ \sigma_{\mathtt{z}} & \sigma_{\mathtt{y}} & \sigma_{\mathtt{z}} \end{smallmatrix} \right) \right\rangle$
- Adequacy $[\![M]\!] = [\![N]\!] \implies \forall C[\_].\quad C[M] \cong C[N]$

Research Question

How could this work with shared-state concurrency?

Example – Global-state

$Locs = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$; $Vals = \{0, 1\}$ (simplification for talk)

$$[\![\mathrm{x} := \mathrm{y?} \,;\, \mathrm{x} := \mathrm{z?}]\!] = \tilde{L}_{\mathrm{y}}(v.\ \tilde{U}_{\mathrm{x},v}\tilde{L}_{\mathrm{z}}(u.\ \tilde{U}_{\mathrm{x},u}\tilde{\star}))$$

$$\text{(LU-comm)}\longrightarrow = \tilde{L}_{\mathrm{y}}(v.\ \tilde{L}_{\mathrm{z}}(u.\ \tilde{U}_{\mathrm{x},v}\tilde{U}_{\mathrm{x},u}\tilde{\star}))$$

$$\text{(UU-last)}\longrightarrow = \tilde{L}_{\mathrm{y}}(v.\ \tilde{L}_{\mathrm{z}}(u.\ \tilde{U}_{\mathrm{x},u}\tilde{\star}))$$

$$\text{(L-noop)}\longrightarrow = \tilde{L}_{\mathrm{z}}(u.\ \tilde{U}_{\mathrm{x},u}\tilde{\star}) = [\![\mathrm{x} := \mathrm{z?}]\!]$$

Fact (Adequacy for global-state [Folklore?])

$$[\![M]\!] = [\![N]\!] \implies \forall C[\_].\quad C[M] \cong C[N]$$

Consider context $C[M] := M\ ;\mathrm{x?}$

$$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad \mathrm{x} := \mathrm{y?} \,;\, \mathrm{x} := \mathrm{z?}\ ;\mathrm{x?} \quad \to^* \quad \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad 1$$

$$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad \mathrm{x} := \mathrm{z?}\ ;\mathrm{x?} \quad \to^* \quad \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad 1$$

Global-state – Equations

x, y, z – meta-variables for locations (note the font: x vs. $\mathtt{x}$)

The equations come from the global-state theory [Plotkin & Power, 2002].

Axioms

- LU-noop $L_x(v.\ U_{x,v}\chi) = \chi$
- UL-det $U_{x,u}L_x(v.\ \chi_v) = U_{x,u}\chi_u$
- UU-last $U_{x,v}U_{x,u}\chi = U_{x,u}\chi$
- LU-comm $L_x(v.\ U_{y,u}\chi_v) = U_{y,u}L_x(v.\ \chi_v) \quad (x \neq y)$
- LL-comm UU-comm (Different locations commute)

Notable consequences

- U-noop $L_x(v.\ U_{x,v}\chi_v) = L_x(v.\ \chi_v)$
- L-noop $L_x(v.\ \chi) = \chi$
- LL-diag $L_x(v.\ L_x(u.\ \chi_{v,u})) = L_x(v.\ \chi_{v,v})$ [Melliès, 2014]

Global-state Monad

Global-state Normal Form

$$L_{\mathtt{z}}(w.\ U_{\mathtt{x},w}\star) = L_{\mathtt{x}}(v.\ L_{\mathtt{y}}(u.\ L_{\mathtt{z}}(w.\ U_{\mathtt{x},w}U_{\mathtt{y},u}U_{\mathtt{z},w}\star)))$$

Fact

Every equational theory induces a (bijectively)-unique monad over Set.

- $States := Locs \to Vals$
- $\underline{T}X := States \to X \times States$
- $\tilde{L}_x(v.\ t_v) := \lambda\sigma.\ t_{\sigma_x}\sigma$
- $\tilde{U}_{x,v}t := \lambda\sigma.\ t\,(\sigma\,[x := v])$

Theorem (Representation for global-state [Plotkin & Power, 2002])

The free global-state model on X is $FX := \langle \underline{T}X, \tilde{L}, \tilde{U}\rangle$ with

$$\mathrm{ret} : X \to \underline{T}X$$

$$\mathrm{ret}\,\xi := \lambda\sigma.\ \langle \xi, \sigma\rangle \qquad (\tilde{\xi} := \mathrm{ret}\,\xi)$$

Shared-state – Intro

Desired (Adequacy for shared-state)

$[\![M]\!] = [\![N]\!] \implies \forall C[\_].\quad C[M] \cong C[N]$

Consider context $C[M] = M \parallel \mathtt{x?}$

$$\left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad \mathtt{x} := \mathtt{y?}\ ;\ \mathtt{x} := \mathtt{z?} \parallel \mathtt{x?} \quad \to^* \quad \left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad (\langle\rangle, 0)$$

$$\left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad \mathtt{x} := \mathtt{z?} \parallel \mathtt{x?} \quad \not\to^* \quad \left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 1 & 0 & 1\end{smallmatrix}\right),\quad (\langle\rangle, 0)$$

Question

Should we invalidate equations or change the denotations?

Approach

$$[\![\mathtt{x} := \mathtt{y?}\ ;\ \mathtt{x} := \mathtt{z?}]\!] \neq \tilde{L}_{\mathtt{y}}(v.\ \tilde{U}_{\mathtt{x},v}\tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u}\tilde{\star}))$$
$$= \tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u}\tilde{\star}) \neq [\![\mathtt{x} := \mathtt{z?}]\!]$$

Algebraic Effects

Roadmap to Contextual Equivalence via Denotational Semantics

- Equational Theory
- Monad
- Denotations
- Adequacy (WIP)

New operators reflecting meaning of programs:

Expose E_x : another thread did $L_x(v.\ \ldots)$

Interfere $I_{x,v}$: another thread did $U_{x,v}$

Choice Σ : non-deterministic choice

- $\tilde{\mathbb{E}} := \tilde{\sum}_{s\in(\{\tilde{I}_{x,v} | x\in Locs, v\in Vals\}\cup\{\tilde{E}_x | x\in Locs\})^*} s$

$$[\![\mathtt{x} := \mathtt{z}?]\!] = \tilde{\mathbb{E}}\tilde{L}_{\mathtt{z}}(u.\ \tilde{\mathbb{E}}\tilde{U}_{\mathtt{x},u}\tilde{\mathbb{E}}\tilde{\star})$$

$$[\![\mathtt{x} := \mathtt{y}?\ ;\ \mathtt{x} := \mathtt{z}?]\!] = \tilde{\mathbb{E}}\tilde{L}_{\mathtt{y}}(v.\ \tilde{\mathbb{E}}\tilde{U}_{\mathtt{x},v}\tilde{\mathbb{E}}\tilde{L}_{\mathtt{z}}(u.\ \tilde{\mathbb{E}}\tilde{U}_{\mathtt{x},u}\tilde{\mathbb{E}}\tilde{\star}))$$

- We will see: $[\![\mathtt{x} := \mathtt{y}?\ ;\ \mathtt{x} := \mathtt{z}?]\!] \neq [\![\mathtt{x} := \mathtt{z}?]\!]$

Shared-state – Equations

Axioms

- Global-state
- IL-det $I_{x,u} L_x(v.\, \chi_v) = I_{x,u} \chi_u$
- II-last $I_{x,v} I_{x,u} \chi = I_{x,u} \chi$
- EE-idem $E_x E_x \chi = E_x \chi$
- Different locations commute
- L, E commute even in same location
- Standard ND-theory
- ND and other operators commute

Notable consequences

- IU-redr $I_{x,v} U_{x,v} \chi = I_{x,v} \chi$
- UEU-redr $U_{x,v} E_x U_{x,v} \chi = U_{x,v} E_x \chi$

Shared-state – Equations (cont.)

Notable non-consequences

- UI-redI $U_{x,v} I_{x,v} \chi = I_{x,v} \chi$
  - Similar to IU-redr $I_{x,v} U_{x,v} \chi = I_{x,v} \chi$
  - Implies out-of-thin-air behaviour $I_{\mathtt{x},0} U_{\mathtt{x},0} \star \parallel U_{\mathtt{x},0} I_{\mathtt{x},0} \star \Rightarrow U_{\mathtt{x},0} U_{\mathtt{x},0} \langle \star, \star \rangle$

$$\left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 1 & 0 & 1 \end{smallmatrix}\right), \quad \langle\rangle \parallel \langle\rangle \quad \not\to^{*} \quad \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad (\langle\rangle, \langle\rangle)$$

- LI-noop $L_x(v.\ I_{x,v} \chi) = \chi$
  - Analogous to LU-noop $L_x(v.\ U_{x,v} \chi) = \chi$
  - The only U vs. I global-state analogue not included
  - Perhaps worrying that I can appear from nothing
  - Work-in-Progress
- UI-gobr $I_{x,v} U_{x,u} \chi = I_{x,u} \chi$

Shared-state Monad – Carrier

- $\underline{T}X := States \to P_{Countable}(X \times Histories)$
  - $P_{Countable}$ for ND
  - *Histories* instead of *States* for intervening effects
- $Histories := Locs \to Chronicles$
- *Chronicles* – certain sequences of U_v, I_v, E
  - No adjacent same-type operators — UU-last II-last EE-idem
  - Start with U — U-noop
  - U after every I and every E — UEU-redr IU-redr
  - No $I_v U_v I_u$ — IU-redr II-last
  - No $U_v E U_v E$ — UEU-redr EE-idem

Example

- $U_0 E U_1 \in Chronicles$
- $U_0 I_0 E U_1 \notin Chronicles$
- $\left(\overset{\mathtt{x}}{U_0 E U_1} \; \overset{\mathtt{y}}{U_0} \; \overset{\mathtt{z}}{U_1} \right) \in Histories$
- $\lambda\sigma.\ \left\{ \left\langle \star, \left(\overset{\mathtt{x}}{U_{\sigma_{\mathtt{y}}} E U_{\sigma_{\mathtt{z}}}} \; \overset{\mathtt{y}}{U_{\sigma_{\mathtt{y}}}} \; \overset{\mathtt{z}}{U_{\sigma_{\mathtt{z}}}} \right) \right\rangle \right\} \in \underline{T}\{\star\}$

Shared-state Monad – Algebraic Operations

- $\tilde{L}_x(v.\ t_v) := \lambda\sigma.\ t_{\sigma_x}\sigma$
- $\tilde{U}_{x,v} t := \lambda\sigma.\ t\,(\sigma\,[x := v])$
- $\tilde{E}_x t := \lambda\sigma.\ \mathsf{map}_{\mathtt{x}}\,\overline{\mathtt{U}_{\sigma_x}\mathtt{E}}\,(t\sigma)$
- $\tilde{I}_{x,v} t := \lambda\sigma.\ \mathsf{map}_{\mathtt{x}}\,\overline{\mathtt{U}_{\sigma_x}\mathtt{I}_v}\,(t\,(\sigma\,[x := v]))$
- $\tilde{\sum}_{i<\alpha} t_i := \lambda\sigma.\ \bigcup_{i<\alpha} t_i\sigma$

Example

$$\begin{aligned}
&\tilde{L}_{\mathtt{y}}(v.\ \tilde{U}_{\mathtt{x},v}\ \tilde{E}_{\mathtt{x}}\ \tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u}\tilde{\star}))\left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 1 & 0 & 1\end{smallmatrix}\right)\\
&= \tilde{E}_{\mathtt{x}}\tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u}\tilde{\star})\left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 0 & 0 & 1\end{smallmatrix}\right)\\
&= \mathsf{map}_{\mathtt{x}}\,\overline{\mathtt{U}_0\mathtt{E}}\left(\tilde{L}_{\mathtt{z}}(u.\ \tilde{U}_{\mathtt{x},u}\tilde{\star})\left(\begin{smallmatrix}\mathtt{x} & \mathtt{y} & \mathtt{z}\\ 0 & 0 & 1\end{smallmatrix}\right)\right)\\
&= \mathsf{map}_{\mathtt{x}}\,\overline{\mathtt{U}_0\mathtt{E}}\left\{\left\langle\star,\left(\begin{smallmatrix}x & y & z\\ U_1 & U_0 & U_1\end{smallmatrix}\right)\right\rangle\right\}\\
&= \left\{\left\langle\star,\left(\begin{smallmatrix}x & y & z\\ U_0EU_1 & U_0 & U_1\end{smallmatrix}\right)\right\rangle\right\}
\end{aligned}$$

Shared-state Monad – Representation Theorem

Theorem (Representation for shared-state)

The free shared-state model on X is $FX := \left\langle \underline{T}X, \tilde{L}, \tilde{U}, \tilde{E}, \tilde{I}, \tilde{\Sigma} \right\rangle$ with

$$\mathsf{ret} : X \to \underline{T}X$$

$$\mathsf{ret}\,\xi := \lambda\sigma.\ \left\{\left\langle \xi, \left(\begin{smallmatrix} \cdots & & \cdots \\ \vdots & U^{\mathtt{x}}_{\sigma_{\mathtt{x}}} & \vdots \\ \cdots & & \cdots \end{smallmatrix} \right) \right\rangle\right\} \qquad \left(\tilde{\xi} := \mathsf{ret}\,\xi\right)$$

Shared-state – Denotational Semantics

$\Gamma \vdash M : A \implies [\![M]\!] : [\![\Gamma]\!] \to \underline{T}[\![A]\!]$ (in our examples $\Gamma = \varnothing$)

Standard sequential semantics [Moggi, 1991], e.g.

$$[\![\langle\rangle]\!]\gamma := \tilde{\star}$$
$$[\![M \mathbin{;} N]\!]\gamma := [\![M]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda\_.\ [\![N]\!]\gamma$$

State effects including interference

$$[\![M?]\!]\gamma := [\![M]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda x.\ \tilde{\mathbb{E}}\tilde{L}_x(v.\ \tilde{\mathbb{E}}\tilde{v})$$
$$[\![M := N]\!]\gamma := [\![M]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda x.\ [\![N]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda v.\ \tilde{\mathbb{E}}\tilde{U}_{x,v}\tilde{\mathbb{E}}\tilde{\star}$$
$$[\![M \parallel N]\!]\gamma := \ldots ? \ldots$$

Example

$$[\![\mathtt{x} := \mathtt{z}?]\!]\gamma = [\![\mathtt{x}]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda x.\ [\![\mathtt{z}?]\!]\gamma \mathbin{\rangle\!\!\rangle\!\!=} \lambda v.\ \tilde{U}_{x,v}\tilde{\mathbb{E}}\tilde{\star} = \tilde{\mathbb{E}}\tilde{L}_{\mathtt{z}}(v.\ \tilde{\mathbb{E}}\tilde{U}_{\mathtt{x},v}\tilde{\mathbb{E}}\tilde{\star})$$

Shared-state – Back to the Example

Let's show: $[\![\mathtt{x} := \mathtt{y}?\ ;\ \mathtt{x} := \mathtt{z}?]\!] \neq [\![\mathtt{x} := \mathtt{z}?]\!]$

- $\left\langle \star, \left(\begin{smallmatrix} & x & & y & z \\ U_0 & E & U_1 & U_0 & U_1 \end{smallmatrix} \right) \right\rangle \in \tilde{\mathbb{E}} \tilde{L}_{\mathtt{y}}(v.\ \tilde{\mathbb{E}} \tilde{U}_{\mathtt{x},v} \tilde{\mathbb{E}} \tilde{L}_{\mathtt{z}}(u.\ \tilde{\mathbb{E}} \tilde{U}_{\mathtt{x},u} \tilde{\mathbb{E}} \tilde{\star})) \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 1 & 0 & 1 \end{smallmatrix} \right)$
  $= [\![\mathtt{x} := \mathtt{y}?\ ;\ \mathtt{x} := \mathtt{z}?]\!] \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 1 & 0 & 1 \end{smallmatrix} \right)$
- $\left\langle \star, \left(\begin{smallmatrix} & x & & y & z \\ U_0 & E & U_1 & U_0 & U_1 \end{smallmatrix} \right) \right\rangle \notin \tilde{\mathbb{E}} \tilde{L}_{\mathtt{z}}(u.\ \tilde{\mathbb{E}} \tilde{U}_{\mathtt{x},u} \tilde{\mathbb{E}} \tilde{\star}) \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 1 & 0 & 1 \end{smallmatrix} \right) = [\![\mathtt{x} := \mathtt{z}?]\!] \left(\begin{smallmatrix} \mathtt{x} & \mathtt{y} & \mathtt{z} \\ 1 & 0 & 1 \end{smallmatrix} \right)$
  - Can't take $\tilde{I}_{\mathtt{x},\_}$ in any of the $\tilde{\mathbb{E}}$'s – get stuck with it
  - Essentially 8 options remain (where to take $\tilde{E}_{\mathtt{x}}$) – easy to check

Sync

Example

$$\tilde{L}_{\mathrm{y}}(u.\ \tilde{U}_{\mathrm{x},u}\tilde{E}_{\mathrm{x}} \quad \tilde{L}_{\mathrm{z}}(w.\ \tilde{U}_{\mathrm{x},w} \quad \tilde{\star})) \quad \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right) \subseteq \tilde{\mathbb{E}}[\![\mathrm{x} := \mathrm{y}?\ ;\ \mathrm{x} := \mathrm{z}?]\!] \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$$

$$\tilde{E}_{\mathrm{y}} \quad \tilde{I}_{\mathrm{x},0}\tilde{L}_{\mathrm{x}}(v.\ \tilde{E}_{\mathrm{z}} \quad \tilde{I}_{\mathrm{x},1} \quad \tilde{v}) \quad \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right) \subseteq \tilde{\mathbb{E}}[\![\mathrm{x}?]\!] \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$$

$$\tilde{L}_{\mathrm{y}}(u.\ \tilde{U}_{\mathrm{x},u}\tilde{L}_{\mathrm{x}}(v.\ \tilde{L}_{\mathrm{z}}(w.\ \tilde{U}_{\mathrm{x},w}\ \widetilde{\langle\star, v\rangle})))) \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right) \subseteq [\![\mathrm{x} := \mathrm{y}?\ ;\ \mathrm{x} := \mathrm{z}? \parallel \mathrm{x}?]\!] \left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$$

$\mathrm{y} = 0$	L_{y}	E_{y}	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$	L_{y}
	$U_{\mathrm{x},0}$	$I_{\mathrm{x},0}$	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$	$U_{\mathrm{x},0}$
$\mathrm{x} = 0$	E_{x}	L_{x}	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 0 & 0 & 1\end{smallmatrix}\right)$	L_{x}
$\mathrm{z} = 1$	L_{z}	E_{z}	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 0 & 0 & 1\end{smallmatrix}\right)$	L_{z}
	$U_{\mathrm{x},1}$	$I_{\mathrm{x},1}$	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 0 & 0 & 1\end{smallmatrix}\right)$	$U_{\mathrm{x},1}$
	$\star$	0	$\left(\begin{smallmatrix}\mathrm{x} & \mathrm{y} & \mathrm{z}\\ 1 & 0 & 1\end{smallmatrix}\right)$	$\langle\star, 0\rangle$

$$[\![M \parallel N]\!]\gamma := \lambda\sigma.\ \bigcup\left\{\tilde{r}\sigma \;\middle|\; \begin{array}{ll} \exists t, s. & t \parallel s \xRightarrow{\sigma} r \\ \quad\wedge & \tilde{t}\sigma \subseteq \tilde{\mathbb{E}}[\![M]\!]\gamma\sigma \\ \quad\wedge & \tilde{s}\sigma \subseteq \tilde{\mathbb{E}}[\![N]\!]\gamma\sigma \end{array}\right\}$$

Shared-state Monad – Soundness Theorem

Theorem (Soundness for shared-state)

Assuming $\vdash M : A$,

$$\sigma, M \to \sigma', M' \implies [\![M]\!]\sigma \supseteq [\![M']\!]\sigma'$$

Conclusion

Summary

- Definition of algebraic theory for shared-state
- Representation theorem: free model defined explicitly
- Soundness theorem: assuming $\vdash M : A$,

$$\sigma, M \to \sigma', M' \implies [\![M]\!]\sigma \supseteq [\![M']\!]\sigma'$$

Future Prospects

- Prove Adequacy: $[\![M]\!] = [\![N]\!] \implies \forall C[\_].\quad C[M] \cong C[N]$
- Type-and-effect systems (modular effects)
- Weak-memory models

References

Eugenio Moggi
Notions of Computation and Monads

Gordon D. Plotkin and John Power
Notions of Computation Determine Monads

Paul-André Melliès
Local States in String Diagrams

Syntactic Sync Rules

$$\frac{t \parallel s \xRightarrow{\sigma[\mathrm{x}:=v]} r}{I_{\mathrm{x},v}\, t \parallel U_{\mathrm{x},v}\, s \xRightarrow{\sigma} U_{\mathrm{x},v}\, r}\ \textsc{U-right}$$

$$\frac{t \parallel s_{\sigma_{\mathrm{x}}} \xRightarrow{\sigma} r_{\sigma_{\mathrm{x}}}}{E_{\mathrm{x}}\, t \parallel L_{\mathrm{x}}(v.\ s_v) \xRightarrow{\sigma} L_{\mathrm{x}}(v.\ r_v)}\ \textsc{L-right}$$

$$\frac{t \parallel s \xRightarrow{\sigma[\mathrm{x}:=v]} r}{U_{\mathrm{x},v}\, t \parallel I_{\mathrm{x},v}\, s \xRightarrow{\sigma} U_{\mathrm{x},v}\, r}\ \textsc{U-left}$$

$$\frac{t_{\sigma_{\mathrm{x}}} \parallel s \xRightarrow{\sigma} r_{\sigma_{\mathrm{x}}}}{L_{\mathrm{x}}(v.\ t_v) \parallel E_{\mathrm{x}}\, s \xRightarrow{\sigma} L_{\mathrm{x}}(v.\ r_v)}\ \textsc{L-left}$$

$$\frac{t \parallel s \xRightarrow{\sigma[\mathrm{x}:=v]} r}{I_{\mathrm{x},v}\, t \parallel I_{\mathrm{x},v}\, s \xRightarrow{\sigma} I_{\mathrm{x},v}\, r}\ \mathrm{I}$$

$$\frac{t \parallel s \xRightarrow{\sigma} r}{E_{\mathrm{x}}\, t \parallel E_{\mathrm{x}}\, s \xRightarrow{\sigma} E_{\mathrm{x}}\, r}\ \mathrm{E}$$

$$\frac{}{\xi \parallel \zeta \xRightarrow{\sigma} \langle \xi, \zeta \rangle}\ \textsc{Var}$$

Shared-state Monad – Representation Theorem

Theorem (Representation for shared-state)

The free shared-state model on X is $FX := \langle \underline{T}X, \tilde{L}, \tilde{U}, \tilde{E}, \tilde{I}, \tilde{\Sigma} \rangle$ with

$$\mathrm{ret} : X \to \underline{T}X$$
$$\mathrm{ret}\,\xi := \lambda\sigma.\ \left\{\left\langle \xi, \left(\begin{smallmatrix} \cdots & \mathtt{x} & \cdots \\ \cdots & U_{\sigma_{\mathtt{x}}} & \cdots \end{smallmatrix} \right) \right\rangle\right\} \qquad \left(\tilde{\xi} := \mathrm{ret}\,\xi\right)$$

$$\_ \ \gg\!\!= f : \underline{T}X \to \underline{A}$$
$$t \gg\!\!= f := \overrightarrow{\tilde{L}^A}(\vec{v}.\ \tilde{\sum}^A_{\langle \xi, \eta \rangle \in t\vec{v}} \lceil \eta \rceil^A (f\xi))$$

- A is a shared-state model
- $f : X \to \underline{A}$