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Edge singularities and structure of the 3-D Williams expansion

Thomas Apel [a], Dominique Leguillon [b,*], Cornelia Pester [c], Zohar Yosibash [d]

[a] Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, 85577 Neubiberg, Germany
[b] Institut Jean Le Rond d'Alembert, CNRS UMR 7190, Université Pierre et Marie Curie, case 162, 4, place Jussieu, 75252 Paris cedex 05, France
[c] CST GmbH – Computer Simulation Technology, Bad Nauheimer Str. 19, 64289 Darmstadt, Germany
[d] Perlstone Center for Aeronautical Engineering Studies, Dept. of Mechanical Engrg., Ben-Gurion University of the Negev, Beer-Sheva, 84105, Israel

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Abstract

The elastic solution in a vicinity of a re-entrant wedge can be described by a Williams like expansion in terms of powers of the distance to a point on the edge. This expansion has a particular structure due to the invariance of the problem by translation parallel to the edge. We show here that some terms, so-called primary solutions, derive directly from solutions to the 2-D corner problem posed in the orthogonal cross section of the domain. The others, baptized shadow functions, derive of the primary solutions by integration along the axis parallel to the edge. This 3-D Williams expansion is shown to be equivalent to the edge expansion proposed by Costabel et al. [M. Costabel, M. Dauge, Z. Yosibash, A quasidual function method for extracting edge stress intensity functions, SIAM J. Math. Anal. 35 (5) (2004) 1177–1202]. ***To cite this article: T. Apel et al., C. R. Mecanique 336 (2008).***

Résumé

Singularités d'arête et structure du développement de Williams tridimensionnel. Les solutions élastiques au voisinage d'un dièdre rentrant peuvent être décrites par un développement de type Williams composé de termes en puissance de la distance à un point de l'arête du dièdre. Ce développement a une structure particulière due à l'invariance du problème par translation parallèle à l'arête. Certains termes, appelés solutions particulières, viennent directement des solutions du problème bidimensionnel autour d'un coin entrant, posé sur la section droite du dièdre. Les autres, baptisés ombres, sont déduits des solutions particulières par intégration le long de l'axe parallèle à l'arête du dièdre. Nous montrons que le développement de Williams tridimensionnel est alors équivalent au développement le long de l'arête proposé par Costabel et al. [M. Costabel, M. Dauge, Z. Yosibash, A quasidual function method for extracting edge stress intensity functions, SIAM J. Math. Anal. 35 (5) (2004) 1177–1202)]. ***Pour citer cet article : T. Apel et al., C. R. Mecanique 336 (2008).***

Keywords: Elasticity; Edge singularities; Generalized stress intensity factors

Mots-clés : Elasticité ; Singularités d'arête ; Facteurs d'intensité des contraintes généralisés

* Corresponding author.
E-mail address: dominique.leguillon@upmc.fr (D. Leguillon).

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1. Introduction

The behaviour of linear elastic solutions near an interior point of a re-entrant straight edge with opening ω and traction-free or clamped faces is usually described by expansions in power terms of the distance either to the edge r or to a point on the edge R (Fig. 1). They are 3-D expansions that make use of 2-D solutions associated with the re-entrant corner (same opening ω) in the orthogonal cross section of the domain (throughout this paper 2-D holds for generalized plane strain linear elasticity).

1.1. The 2-D modes and shadow functions

In [1,2] an expansion of the elastic 3-D solution $\underline{U}$ in terms of 2-D solutions with additional terms so-called "shadow functions" was proposed in the form

$$\underline{U}(r,\varphi,z)=\sum_{i\geqslant 1}\sum_{j\geqslant 0}\partial_z^j A_i(z)\underline{\Phi}_j^i(r,\varphi) \tag{1}$$

The space variables r, φ and z are the cylindrical coordinates (Fig. 1). The expansion (1) is derived from the splitting of the partial differential Navier operator in derivatives with respect to z on the one hand and with respect to r and φ on the other, allowing a separation of variables. The function $\underline{\Phi}_0^i$ is the i-th mode of the 2-D corner problem, so-called primary mode of the actual 3-D problem

$$\underline{\Phi}_0^i(r,\varphi)=r^{\alpha_i}\underline{\phi}_0^i(\varphi) \tag{2}$$

Here $\alpha_i \geqslant 0$ and $\underline{\phi}_0^i(\varphi)$ are respectively the eigenvalues and eigenfunctions of the elastic operator defined on the 2-D domain with a re-entrant corner with opening ω [3,4].

In Eq. (1), the $\underline{\Phi}_j^i$'s ($j \geqslant 1$) are the shadow functions to $\underline{\Phi}_0^i$, following the terminology employed in [1,2]. They depend on the 2-D space variable r and φ only and are solutions to a partial differential system with a right hand side member depending on $\underline{\Phi}_{j-1}^i$. They write

$$\underline{\Phi}_j^i(r,\varphi)=r^{\alpha_i+j}\underline{\phi}_j^i(\varphi) \tag{3}$$

However, they are not solutions to the Navier operator neither in 2-D nor in 3-D.

The terms $\partial_z^j A_i(z)$ denote the weights of the corresponding primary or shadow functions (∂_z^j is the j-th derivative with respect to z). They depend on z and derive from a single edge stress intensity function (ESIF) $A_i(z)$. Expansion (1) holds at any interior point of the edge, i.e. at any point except the ends.

1.2. The 3-D Williams expansion

On the other hand, the 3-D solution in the vicinity of any given point O on the edge can be represented by a Williams like expansion [5]

$$\underline{U}(R,\varphi,\theta)=\sum_{i\geqslant 0}K_i R^{\beta_i}\underline{v}^i(\varphi,\theta) \tag{4}$$

where R, φ and θ are the spherical coordinates with origin at O (Fig. 1). The constant coefficients K_i are the generalized stress intensity factors (GSIF) associated with the different modes characterized by an eigenvalue $\beta_i \geqslant 0$ and an eigenfunction $\underline{v}^i(\varphi,\theta)$ [6]. The expansion (4) holds true for any selected origin O along the edge (Fig. 1), the β_i's and $\underline{v}^i$'s are independent of the choice of O if it is an interior point but differ at the two ends [7].

It is worth noting that the 3-D terms and the GSIF's involved in (4) are generally numerically known using algorithms dedicated to general 3-D edges and corners situations [6–9].

2. The structure of the 3-D Williams expansion

The three following properties allow us to define a structure in the Williams expansion (4) in the vicinity of interior points of the edge:

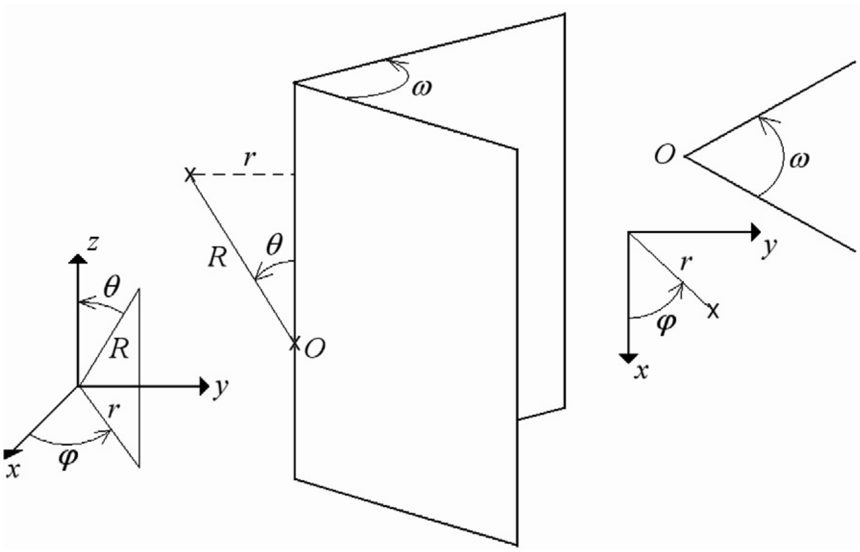

Fig. 1. The Cartesian, cylindrical and spherical coordinates along the edge and the polar coordinates in the cross section.

Fig. 1. Les coordonnées cartésiennes, cylindriques et sphériques le long de l'arête et les coordonnées polaires dans la section droite.

Property 1. If α_i is an eigenvalue of the 2-D problem, then $-\alpha_i$ is also an eigenvalue with its own eigenfunction $\underline{\xi}^i_0(\varphi)$. If β_i is an eigenvalue of the 3-D problem, then $-\beta_i - 1$ is also an eigenvalue with its own eigenfunction $\underline{w}^i(\varphi,\theta)$. These pairs are baptized dual modes. In both cases, it is a consequence of the operators involved in the eigenvalue problems [3,6,10]. Moreover, the strip $]-1,0[$ is free of 3-D eigenvalues [11].

Property 2. The 2-D eigenfunctions (2) are also 3-D eigenfunctions. From Fig. 1, it is clear that:

$$r^{\alpha_i}\underline{\phi}^i_0(\varphi) = R^{\alpha_i}\sin^{\alpha_i}\theta\,\underline{\phi}^i_0(\varphi) = R^{\alpha_i}\underline{v}^i(\varphi,\theta) \quad \text{with } \underline{v}^i(\varphi,\theta) = \sin^{\alpha_i}\theta\,\underline{\phi}^i_0(\varphi) \tag{5}$$

On the contrary, it must be pointed out that a dual 2-D mode (involving a negative exponent α_i) is not, since $\sin\theta$ vanishes at some points, which is contrary to the H^1 smoothness of $\underline{v}^i$ if $\alpha_i \leqslant -1$.

Property 3. Let $\underline{V}(x,y,z)$ be any solution to the 3-D problem in the infinite domain of Fig. 1 (unbounded in r and z, i.e. in R), then $\partial_z^k \underline{V}$ is also a solution for any $k \geqslant 0$. The problem is invariant under a translation parallel to z.

Let us start from a pure 3-D eigenfunction $R^{\alpha_j}\underline{v}^j(\varphi,\theta)$ (there is at least one otherwise the 3-D solutions would be described by a Williams expansion in terms of 2-D eigenfunctions only, which makes no sense). According to Property 3, its derivatives are eigenfunctions too and since the strip $]-1,0[$ is free of eigenvalues, it exists $p \geqslant 1$ such that the $p+1$-th derivative with respect to z vanishes and then the p-th one is a 2-D eigenfunction

$$\partial_z^p\big(R^{\alpha_j}\underline{v}^j(\varphi,\theta)\big) = R^{\alpha_i}\underline{v}^i(\varphi,\theta) = r^{\alpha_i}\underline{\phi}^i_0(\varphi) \quad \text{with } \alpha_j = \alpha_i + p \tag{6}$$

According to Property 1, the dual mode $R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta)$ and its derivatives with respect to z,

$$\partial_z^k\big(R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta)\big) = R^{-\alpha_i-1-k}\underline{w}^i_k(\varphi,\theta),$$

are 3-D eigenfunctions for any $k > 0$. Using again Property 1, they are the dual functions to the $R^{\alpha_i+k}\underline{v}^i_k(\varphi,\theta)$'s which coincide in particular with the derivatives of the above mentioned eigenfunction $R^{\alpha_j}\underline{v}^j(\varphi,\theta)$ we started from. Then by integration it comes for any $k > 0$

$$\begin{cases} R^{\alpha_i+1}\underline{v}_1^i(\varphi,\theta) = zR^{\alpha_i}\underline{v}^i(\varphi,\theta) + \underline{c}_1^i(r,\varphi) \\ R^{\alpha_i+2}\underline{v}_2^i(\varphi,\theta) = \dfrac{z^2}{2}R^{\alpha_i}\underline{v}^i(\varphi,\theta) + z\underline{c}_1^i(r,\varphi) + \underline{c}_2^i(r,\varphi) \\ \cdots \\ R^{\alpha_i+k}\underline{v}_k^i(\varphi,\theta) = \dfrac{z^k}{k!}R^{\alpha_i}\underline{v}^i(\varphi,\theta) + \dfrac{z^{k-1}}{(k-1)!}\underline{c}_1^i(r,\varphi) + \cdots + \underline{c}_k^i(r,\varphi) \\ \cdots \end{cases} \tag{7}$$

Notice that each of the expressions in (7) is solution to the 3-D problem. The arbitrary additive functions $\underline{c}_k^i$ are independent of z, thus they are functions of r and φ only. To be consistent with the Williams expansion (4), they can be written as

$$\underline{c}_k^i(r,\varphi) = r^{\alpha_i+k}\underline{\psi}_k^i(\varphi) \tag{8}$$

Since each term in (7) is an eigensolution to the 3-D problem, they are multiplied each by a GSIF and added to provide the Williams series (with a slight modification of the numbering of the GSIF's)

$$\begin{cases} \underline{U}(R,\varphi,\theta) = K_i r^{\alpha_i}\underline{\phi}_0^i(\varphi) + K_i^1 r^{\alpha_i}\big(z\underline{\phi}_0^i(\varphi) + r\underline{\psi}_1^i(\varphi)\big) \\ \quad + K_i^2 r^{\alpha_i}\left(\dfrac{z^2}{2}\underline{\phi}_0^i(\varphi) + zr\underline{\psi}_1^i(\varphi) + r^2\underline{\psi}_2^i(\varphi)\right) + \cdots \\ \quad + K_i^k r^{\alpha_i}\left(\dfrac{z^k}{k!}\underline{\phi}_0^i(\varphi) + \dfrac{z^{k-1}}{(k-1)!}r\underline{\psi}_1^i(\varphi) + \cdots + r^k\underline{\psi}_k^i(\varphi)\right) + \cdots \end{cases} \tag{9}$$

The same reasoning can be carried out with the other primary solutions (2) to complete the expansion. To summarize, the exponents involved in the Williams series are the exponents of the 2-D corner problem and these exponents plus integers.

3. The 3-D vs. 2-D analysis

To make the identification easier, (9) is rearranged to yield

$$\begin{cases} \underline{U}(R,\varphi,\theta) = \left(K_i + zK_i^1 + \dfrac{z^2}{2}K_i^2 + \cdots + \dfrac{z^k}{k!}K_i^k + \cdots\right) r^{\alpha_i}\underline{\phi}_0^i(\varphi) \\ \quad + \left(K_i^1 + zK_i^2 + \cdots + \dfrac{z^{k-1}}{(k-1)!}K_i^k + \cdots\right) r^{\alpha_i+1}\underline{\psi}_1^i(\varphi) \\ \quad + (K_i^2 + \cdots + \dfrac{z^{k-2}}{(k-2)!}K_i^k + \cdots) r^{\alpha_i+2}\underline{\psi}_2^i(\varphi) + \cdots + K_i^k r^{\alpha_i+k}\underline{\psi}_k^i(\varphi) + \cdots \end{cases} \tag{10}$$

From (1), (2) and (10), the ESIF $A_i(z)$, i.e. the term that multiplies $r^{\alpha_i}\underline{\phi}_0^i(\varphi)$, can be expressed in terms of the GSIF's

$$A_i(z) = K_i + zK_i^1 + \frac{z^2}{2}K_i^2 + \cdots + \frac{z^k}{k!}K_i^k + \cdots \tag{11}$$

A truncation of this series leads to a polynomial approximation of the ESIF comparable to that employed in [2]. Continuing the identification shows that the shadow functions are

$$\underline{\Phi}_1^i(r,\varphi) = r^{\alpha_i+1}\underline{\psi}_1^i(\varphi); \ldots \underline{\Phi}_k^i(r,\varphi) = r^{\alpha_i+k}\underline{\psi}_k^i(\varphi)\ldots \quad \text{i.e.} \quad \underline{\psi}_k^i(\varphi) = \underline{\phi}_k^i(\varphi) \quad \text{for } k \geqslant 1 \tag{12}$$

Also it is clear from (10) that the corresponding weights are successive derivatives of $A_i(z)$. The shadow functions are solution to the following system

Table 1
3-D exponents of the Williams expansion for $\omega = \pi/2$, bold-face numbers are the primary eigenvalues

Tableau 1
Exposants du développement de Williams 3-D pour $\omega = \pi/2$, les nombres en gras sont les valeurs propres particulières

n°	α_i	mult.	n°	α_i	mult.	n°	α_i	mult.
1	**0.**	3	9	**1.629 ± 0.23 i**	2	17	2.629 ± 0.231 i	2
2	**0.545**	1	10	1.667	1	18	2.667	1
3	**0.667**	1	11	1.909	1	19	**2.667**	1
4	**0.909**	1	12	2.	4	20	2.909	1
5	1.	3	13	**2.**	1	21	**2.972 ± 0.374 i**	2
6	**1.**	1	14	**2.301 ± 0.316 i**	2	22	3.	5
7	**1.333**	1	15	2.333	1	23	3.301 ± 0.316 i	2
8	1.545	1	16	2.546	1	24	3.333	1

$$\begin{cases} \underline{v}_1^i(\varphi,\theta) = \sin^{\alpha_i}\theta\big(\sin\theta\underline{\phi}_1^i(\varphi) + \cos\theta\underline{\phi}_0^i(\varphi)\big) \\ \underline{v}_2^i(\varphi,\theta) = \sin^{\alpha_i}\theta\left(\sin^2\theta\underline{\phi}_2^i(\varphi) + \cos\theta\sin\theta\underline{\phi}_1^i(\varphi) + \frac{1}{2}\cos^2\theta\underline{\phi}_0^i(\varphi)\right) \\ \vdots \\ \underline{v}_k^i(\varphi,\theta) = \sin^{\alpha_i}\theta\left(\sin^k\theta\underline{\phi}_k^i(\varphi) + \cdots + \frac{1}{(k-1)!}\cos^{k-1}\theta\sin\theta\underline{\phi}_1^i(\varphi) + \frac{1}{k!}\cos^k\theta\underline{\phi}_0^i(\varphi)\right) \end{cases} \tag{13}$$

If the primary eigenfunctions and their shadow functions, both 2-D functions, are analytically known (see [1,2]), Eq. (13) offers a method to determine the 3-D terms of the Williams series. Once the eigenfunctions are known, the GSIF's can be computed using a FE approximation $\underline{U}^{\mathrm{FE}}$ of $\underline{U}$ and a path independent integral making use of the already mentioned dual modes [3,12] (using the notations of Eq. (4) in Section 1)

$$K_i = \frac{H(\underline{U}^{\mathrm{FE}}, R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta))}{H(R^{\alpha_i}\underline{v}^i(\varphi,\theta), R^{-\alpha_i-1}\underline{w}^i(\varphi,\theta))} \quad \text{with } H(\underline{X},\underline{Y}) = \int_\Gamma \big(\underline{T}(\underline{X})\cdot\underline{Y} - \underline{T}(\underline{Y})\cdot\underline{X}\big)\,\mathrm{d}s \tag{14}$$

The integral H is independent of the surface Γ surrounding the point of interest and $\underline{T}(\underline{X})$ denotes the traction vector, derived from the displacement field $\underline{X}$ and acting on this surface. It is a straightforward extension of the 2-D case.

4. A numerical example

Table 1 gives the 40 smaller exponents of the 3-D problem with their multiplicity for an isotropic body with an opening $\omega = \pi/2$ and traction-free faces. They are obtained solving the eigenvalue problem using a p-version FEM and an algorithm based on a structured eigenvalue method [13]. Complex values are counted twice for the value and its conjugate. The bold-face numbers are the primary eigenvalues, solution to the 2-D problem. Due to the isotropic material the set of 2-D solutions is twofold: the in-plane $(U_x, U_y, 0)$ and the out-of-plane vectors $(0, 0, U_z)$. N° 2 and 4 in Table 1 enter the first class whereas n° 3 enters the second one for instance, but this plays no role in the present reasoning.

The multiplicity 3 associated with $\alpha_i = 0$ corresponds to 3 rigid translations, i.e. 3 constants in the directions x, y and z. The integration with respect to z (see (7)) gives 3 terms associated with $\alpha_i = 1$: a rigid rotation around $Ox(U_x = 0, U_y = z, U_z = -y = -r\sin\varphi)$, another one around $Oy(U_x = z, U_y = 0, U_z = -x = -r\cos\varphi)$, and the uniform tension in the direction $z(U_x = -\nu x = -\nu r\cos\varphi, U_y = -\nu y = -\nu r\sin\varphi, U_z = z)$, where ν holds for the Poisson's ratio of the material. The fourth term associated with $\alpha_i = 1$ is a primary eigenfunction: the in-plane rotation $(U_x = -y = -r\sin\varphi, U_y = x = r\cos\varphi, U_z = 0)$.

More generally, one can point out the particular cases $\alpha_i = 1, \alpha_i = 2$ and $\alpha_i = 2.667$. The set of eigenfunctions splits into two parts: the primary eigenfunctions and the others resulting of an integration operation.

The property reported in Section 2 (primary eigenvalues plus integers) can be ascertained for n° 1-5-12-22, 2-8-16, 3-10-18 etc., for real eigenvalues. It holds also true in the complex case, with 9-17 and 14-23.

5. Application to fracture mechanics

The knowledge of the Williams expansion terms is of a very high interest when studying local perturbations of the edge like the nucleation of small lens-shaped cracks under mode I + III loading for instance [14,15]. A matched asymptotic expansions procedure, carried out using the diameter d of the new lens-shaped crack as a small parameter, allows writing the leading terms of an expansion of the change in potential energy δW between an initial state prior to the small crack onset and following the onset (up to the sign)

$$-\delta W = K_1^2 B_{11} d^{2\alpha_1+1} + K_1 K_2 B_{12} d^{\alpha_1+\alpha_2+1} + \cdots \tag{15}$$

The coefficients B_{11} and B_{12} depend on the local geometry of the structure (the opening ω) and on the shape of the perturbation (the newly created crack). Their computation makes use of the 3-D modes involved in the Williams series and can be computed using the path integral (14) as well [16].

Eq. (15) can be used in the Griffith criterion

$$-\delta W \geqslant G_c \delta S \tag{16}$$

where G_c is the toughness of the material and δS the surface of the newly created crack. It is worth noting that (14) differs slightly from the 2-D case, the exponents are increased by 1, but the energy release rate $G = -\delta W/\delta S$ must now be taken with respect to the surface of the new crack which is proportional to d^2 ($\delta S = \pi d^2/4$ for a circular crack) leading to the classical exponents: $2\alpha_1 - 1, \alpha_1 + \alpha_2 - 1 \ldots$ [16].

6. Conclusion

The analysis extends to non-isotropic materials [17], provided some quite entangled calculations are performed to get an explicit form of expansion (1). The 3-D Williams expansion keeps the same form for non-homogeneous structures, composite laminates for instance, if the interfaces fulfil the cylindrical geometry of Fig. 1 (i.e. the interfaces are vertical planes emanating at the edge) [8,9]. The present results are valid at any interior point of the edge. Of course, it is a completely different situation at the two ends where the problem is a full 3-D problem [6–9], no local 2-D solutions exist.

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