Supplements to “Dynamic Competition with Network Externalities: How History Matters”

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1 Heterogeneous consumers

In this appendix, we extend our model to show:

i) how to extend the concept of focal platform to the case of heterogenous consumers.

ii) that a Markov equilibrium may exist where a low-quality platform A stays or becomes focal in all states while platform B also obtains positive profits. Hence, platform B has an incentive to remain active in the market even though it does not win the focal position.

1.1 Static analysis

Let us assume that the market stays covered but the perception of the quality differential between the two platforms varies across consumers. More precisely, consider our base model and suppose that the quality differential $q = q^B - q^A$ is heterogenous, distributed in the population (of size 1) according to a distribution F on a support $\left(\underline{q}, \bar{q}\right)$ which may be infinite. Then if all consumers with quality differential below $\hat{q}$ join platform A, the relative gain in value of joining platform B is $q + \beta\left(1 - F\left(\hat{q}\right)\right) - \beta F\left(\hat{q}\right)$. Thus, at any period, possible allocations of consumers are the solutions to $D_A = F(\hat{q})$, where D_A is the demand for platform A and[1]

$$\begin{aligned} \hat{q} + \beta(1 - 2F(\hat{q})) &= p_B - p_A, \\ \textit{or } \hat{q} &= \bar{q} \leq p_B - p_A + \beta, \\ \textit{or } \hat{q} &= \underline{q} \geq p_B - p_A - \beta. \end{aligned}$$

With a general distribution F, for some range of prices, there may be multiple allocations of consumers that constitute consumers' best responses to the prices and strategies of other consumers (henceforth "outcomes"). The concept of focality then implies that *consumers coordinate on the outcome that yields the largest demand for the focal platform.*

We may then extend our analysis by assuming that in any period t the platform that sells the most in the current period becomes focal in the next period.

To illustrate how focality shapes demand, suppose that the distribution of q has density $f(q)$ where f is continuous unimodal with a peak at $\mu > 0$. The slope of $q + \beta(1 - 2F(q))$ is $1 - 2\beta f(q)$. Assume network effects are strong enough that $2f(\mu)\beta > 1$. Then the function $q + \beta(1 - 2F(q))$ is not monotonic. More precisely, defining q_1 and q_2 as the smallest and the largest roots of

$$1 = 2\beta f(q_i),$$

the function $q + \beta(1 - 2F(q))$ is

$$\begin{array}{lr} \text{increasing on} & q < q_1, \\ \text{decreasing on} & q_1 < q < q_2, \\ \text{increasing on} & q > q_2. \end{array}$$

We conclude that if $\Delta_2 = q_2 + \beta(1 - 2F(q_2)) < p_B - p_A < \Delta_1 = q_1 + \beta(1 - 2F(q_1))$, there are three possible outcomes for the allocation of consumers. This is illustrated in Figure 1 which shows the function for a normal distribution and $\mu = 1$. On the range (Δ_2, Δ_1), the intermediate outcome is unstable. Then focality selects the allocation with $\hat{q} > q_2$ if platform A is focal, and the allocation with $\hat{q} < q_1$ if platform B is focal. The figure shows the respective marginal consumers.

Notice that when platform A is focal (thick red curve), there is a discontinuous

[1] Akerlof, Holden and Rayo (2018) analyze a similar demand system.

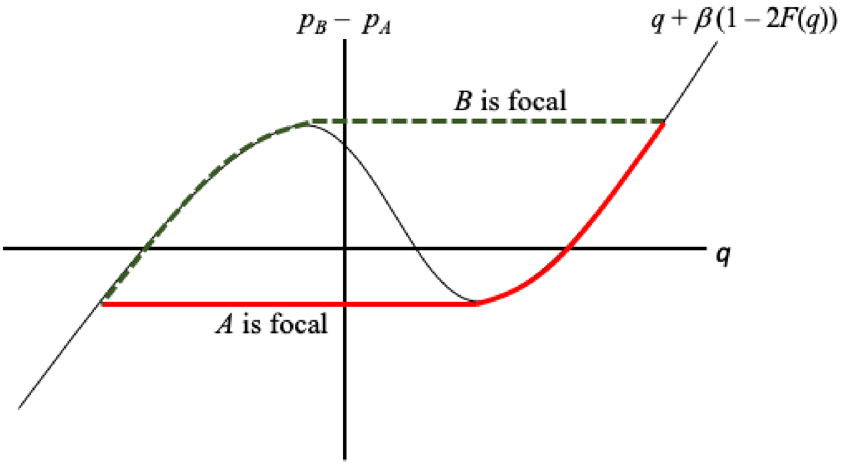

Figure 1: the function $q + \beta\left(1 - 2F\left(q\right)\right)$

jump in its demand at $p_B - p_A = \Delta_2$. Moreover, provided that β is large or μ is small, the value of Δ_2 is negative so that despite lower quality on average, a focal platform A may sell more that its competitor at higher prices, a feature that was key in our analysis.

1.2 Illustration

We now use an example to illustrate how the static and the dynamic analyses extend to heterogeneous consumers.

Consider a distribution F on $(\mu - 1, \mu + 1)$ that consists of a constant density $f < 1/2$ and a mass point $1 - 2f$ at μ, where we assume that $2\beta f < 1$.

This is a limit case of a unimodal distribution when the peak goes to infinity. Then $q + \beta\left(1 - 2F\left(q\right)\right)$ is increasing linearly except for a downward discontinuity at μ :

$$
\begin{aligned}
q + \beta\left(1 - 2F\left(q\right)\right) &= q + \beta\left(1 - 2f\right) + 2\beta f\left(\mu - q\right) \quad \text{if} \quad \mu - 1 \leq q < \mu, \\
q + \beta\left(1 - 2F\left(q\right)\right) &= q - \beta\left(1 - 2f\right) + 2\beta f\left(\mu - q\right) \quad \text{if} \quad \mu < q \leq \mu + 1.
\end{aligned}
$$

Let us first consider a one-period game where platform A is focal. Firms set prices

p_A and p_B, and consumers with $q < q^A$ join the focal platform, where:

$$\begin{aligned}
p_B - p_A =& q^A + \beta(1-2f) + 2\beta f\left(\mu - q^A\right) && \text{if } \ \mu - 1 + \beta < p_B - p_A < \mu - \beta(1-2f)),\\
q^A =& \mu - 1 && \text{if } \ p_B - p_A < \min(\mu - 1 + \beta, \mu - \beta(1-2f))\\
p_B - p_A =& q^A - \beta(1-2f) + 2\beta f\left(\mu - q^A\right) && \text{if } \mu - \beta(1-2f) < p_B - p_A < \mu + 1 - \beta\\
q^A =& \mu + 1 && \text{if } p_B - p_A > \max(\mu + 1 - \beta, \mu + \beta(1-2f)).
\end{aligned}$$

Let us consider a candidate outcome with $D_A = F\left(\hat{q}^A\right)$, $\mu < \hat{q}^A < \mu + 1$ and $\hat{p}_B^A - \hat{p}_A^A = \hat{q}^A - \beta(1-2f) + 2\beta f\left(\mu - \hat{q}^A\right)$. Consider the choice of price by platform A. Holding the price p_B constant, choosing p_A in the range $\left(\hat{p}_B^A - \mu - 1 + \beta, \hat{p}_B^A - \mu + \beta(1-2f)\right)$ amounts to choosing $\hat{q}$ in the range $(\mu, \mu+1)$ with the profit

$$\Pi^A = \left(\hat{p}_B^A - \hat{q} + \beta(1-2f) - 2\beta f(\mu - \hat{q})\right)(1 - f + f(\hat{q} - \mu)),$$

which is concave. The first order condition for the price of A is then

$$-(1 - \beta 2f)\left(1 - f + f\left(\hat{q}^A - \mu\right)\right) + \hat{p}_A^A f = 0.$$

Similarly, given $\hat{p}_A^A$, choosing a price p_B in the range $\left(\hat{p}_A^A + \mu - \beta(1-2f), \hat{p}_A^A + \mu + 1 - \beta\right)$ yields concave profit

$$\Pi^B = \left(\hat{p}_A^A + \hat{q} - \beta(1-2f) + 2\beta f(\mu - \hat{q})\right) f(1 + \mu - \hat{q})$$

leading to the first-order condition

$$(1 - \beta 2f) f\left(1 + \mu - \hat{q}^A\right) - \hat{p}_B^A f = 0.$$

Adding the two first-order conditions together yields

$$-(1 - \beta 2f)\left(1 - f + f\left(\hat{q}^A - \mu\right)\right) + (1 - \beta 2f) f\left(1 + \mu - \hat{q}^A\right) + \left(\hat{p}_A^A - \hat{p}_B^A\right) f = 0.$$

Using $\hat{p}_B^A - \hat{p}_A^A = \hat{q}^A - \beta(1-2f) + 2\beta f\left(\mu - \hat{q}^A\right)$, we obtain

$$\hat{q}^A = \mu + \frac{(1 - 2\beta f)(2f - 1) + \beta(1-2f) f - f\mu}{3(1 - 2\beta f) f}, \tag{1}$$

which lies between μ and $\mu + 1$ if

$$\frac{-f - 1 + 3\beta f}{f} < \mu < \frac{(1 - 2f)(3f\beta - 1)}{f}.$$

This holds for a small μ if

$$\frac{f + 1}{3} > \beta f > \frac{1}{3}. \tag{2}$$

Prices are then

$$\hat{p}_A^A = (1 - 2\beta f) \frac{F\left(\hat{q}^A\right)}{f} = \frac{2 - f(1 + 3\beta + \mu)}{3f} > 0, \tag{3}$$

$$\hat{p}_B^A = (1 - \beta 2f) \frac{1 - F\left(\hat{q}^A\right)}{f} = \frac{1 + f(1 - 3\beta + \mu)}{3f} > 0, \tag{4}$$

where the inequalities follow because whenever $\mu < \hat{q}^A < \mu + 1$, then $1 > F\left(\hat{q}^A\right) > 0$. Given these prices, platform A cannot profit from reducing the price below $\hat{p}_B^A - \mu - 1 + \beta$ as this would not raise demand above 1. Platform A cannot profit from increasing the price above $\hat{p}_B^A - \mu + \beta(1 - 2f)$, because doing so would result in a discontinuous decrease in platform A's demand.[2]

Consider now platform B. It has no profitable deviation for prices above $\hat{p}_A + \mu - \beta(1 - 2f)$, because profit is concave on the relevant range. Setting $p_B < \hat{p}_A + \mu - \beta(1 - 2f)$ is not profitable if $\hat{p}_A + \mu - \beta(1 - 2f) < 0$, which holds for μ small if

$$\frac{2 - f}{6(1 - f)} < \beta f. \tag{5}$$

As $\frac{2-f}{6(1-f)} > \frac{1}{3}$, we conclude that this an equilibrium for μ small if

$$\frac{f + 1}{3} > \beta f > \frac{2 - f}{6(1 - f)}. \tag{6}$$

Since $f < \frac{1}{2}$, (6) holds only when $\beta > 1$, which we assume in what follows.

To conclude, we find that when the conditions (6), $f < \frac{1}{2}$, $\beta > 1$ and $1 > 2\beta f$ hold, there is a static equilibrium in which platform A is focal. Prices are given by (3) and (4), and both platforms gain a positive market share. Notice that the range of parameters satisfying these conditions is nonempty.

[2]The optimal deviation on this range can be shown to be at $\hat{p}_B^A - \mu + \beta(1 - 2f)$ which can not be optimal as p_A slightly below this level induces an upward jump in demand.

1.3 Dynamic analysis

Still assuming a uniform distribution with a mass-point, we can now extend our analysis to a dynamic case. In particular the next result shows that with an infinite horizon and patient firms, there exists a Markov equilibrium where platform A stays focal in any state. In this equilibrium, when platform A is focal in a certain period, both platforms set the static Nash prices defined in the previous section:

$$p_A^A = \hat{p}_A^A, \quad p_B^A = \hat{p}_B^A, \quad \text{and} \quad q^A = \hat{q}^A.$$

And both platforms gain positive market share (i.e., $\mu - 1 < \hat{q}^A < \mu + 1$). When platform B is focal in a certain period, then in equilibrium

$$p_A^B = -\mu - \beta\left(1 - 2f\right), \quad p_B^B = 0, \quad \text{and} \quad \hat{q}^B = \mu + 1.$$

That is, the nonfocal platform A sets a negative price (recall that $f < \frac{1}{2}$), dominates the entire market, and becomes focal in the next period.

Notice that this equilibrium is qualitatively similar to the equilibrium in our base model. In both cases, platforms set the static prices when A is focal, and platform A sets a negative price when it is nonfocal. The main difference is that here, the losing platform B has an incentive to remain active. When it is nonfocal, platform B gains positive market share. When it is focal, platform B earns zero profits in the current period — followed by positive profits in all future periods — making it worthwhile for platform B to remain active.

The Markov equilibrium is characterized by p_i^j, D_i^j, q^j and V_i^j, where i is the platform and j the focal platform. The equilibrium profit is

$$V_i^j = p_i^j D_i^j + \delta V_i^A,$$

where $D_A^j = 1 - D_B^j = F\left(q^j\right)$. The equilibrium values are $V_A^j = p_A^j F\left(q^j\right) + \delta V_A^A$ and $V_B^j = p_B^j\left(1 - F\left(q^j\right)\right) + \delta V_B^A$.

To solve for this equilibrium, suppose first that platform A is focal. Platform B plays its short-term best-response because it expects that even if it gains the focal position, in the next period it will earn zero profits. Platform A plays its short-term best response as well, because doing so is enough to maintain its focal position. We therefore conclude that when platform A is focal, the equilibrium prices are the same as in the static case. The value functions are

$$V_A^A = \frac{\hat{p}_A^A F\left(\hat{q}^A\right)}{1-\delta} > 0 \quad \text{and} \quad V_B^A = \frac{\hat{p}_B^A \left(1 - F\left(\hat{q}^A\right)\right)}{1-\delta} > 0\,.$$

Notice that static Nash equilibrium conditions ensure that firm i would not deviate from such an equilibrium when A is focal if $V_i^A \geq V_i^B$, because the deviation profit gain would be smaller than in a static game. We will see below that this is the case for both platforms.

Suppose now that platform B is focal. By our assumptions that $2f\beta < 1$ and $\beta > 1$, we have $\mu + \beta(1-2f) > \mu + 1 - \beta$ and $\mu - 1 + \beta > \mu - \beta(1-2f)$. This implies that at the equilibrium prices $p_A^B = -\mu - \beta\,(1-2f)$ and $p_B^B = 0$, $p_B - p_A > \max\left(\mu + 1 - \beta, \mu + \beta(1-2f)\right)$, and therefore $\hat{q}^A = 1 + \mu$ and platform A dominates the market. Platform B would not deviate because winning the market in the current period would require a negative price and would delay by one period the time where it can sell at positive prices. Platform B cannot profitably deviate to a higher price, because it will not gain positive market share. Firm A could deviate by setting non-negative price and lose focality, but the profit would be δV_A^B which is less than V_A^B and thus not profitable if $V_A^B > 0$. Thus this is an equilibrium if

$$\mu + \beta\,(1-2f) < \delta \frac{\hat{p}_A^A F\left(\hat{q}^A\right)}{1-\delta} \quad \Longleftrightarrow \quad \frac{\mu + \beta\,(1-2f)}{\mu + \beta\,(1-2f) + \hat{p}_A^A F\left(\hat{q}^A\right)} < \delta < 1,$$

which holds for large δ.

This shows that excess inertia equilibria — where despite lower quality one platform would price aggressively and win back its focal position had it lost it — are robust to demand heterogeneity. It also shows that in this situation the nonfocal platform can survive with a positive market share.

We conclude this Appendix by pointing out that a similar reasoning would show that by contrast, for high discount factors, a Markov equilibrium where in any state the focal platform stays focal does not exist with demand heterogeneity. This confirms that this type of equilibrium should be expected only if firms are not too forward-looking.

2 Network Effects and Switching Costs

This extension shows that when platforms can price-discriminate between existing and switching consumers, the results of Section 4 (existence of Markov perfect equilibria under infinite time horizon where the same platform always wins, even if it is of lower quality) also hold in a setting with both network effects and switching costs. Moreover, this section highlights how network effects and switching costs differently affect the results.

Consider our base model and suppose that consumers experience both network effects, β, and switching costs, s. We maintain our assumptions that $q_A > s \geq 0$ and $\beta \geq q_B - q_A \geq 0$. We allow platforms to price discriminate between existing and new consumers. And we explore the existence of Markov equilibria where on the equilibrium path all consumers buy from the same firm (which becomes focal). For consistency with our base model, we distinguish between the equilibrium and out-of-equilibrium prices. Consider the out-of-equilibrium scenario in which all consumers are on a focal platform i, and there is one consumer on platform j. Define p_i^i as the price of the focal platform i to consumers on platform i, and $\tilde{p}_i^i$ as the price of the focal platform i to the consumer on platform j, if such a consumer exists. Likewise, define p_j^i as the price of the nonfocal platform j to consumers on the focal platform i, and $\tilde{p}_j^i$ as the price of the nonfocal platform j to the consumer on platform j, if such a consumer exists. Notice that p_j^i has the same interpretation as in our base model, while $\tilde{p}_j^i$ is the out-of-equilibrium price in the case where only one consumer switched.[3]

In what follows, we say that a Markov equilibrium is consistent with focality if at any date (a) the platform that wins the market becomes focal next period, and (b) the nonfocal platform cannot win the market if holding constant the Markov strategies of the platforms, there exists another outcome of the (dynamic) subgame where consumers (strictly) prefer to buy from the focal platform. Rephrasing, it means that consumers may buy from the nonfocal platform in equilibrium only if there is no other outcome where consumers prefer to buy from the focal platform. And thus, a nonfocal platform, in order to be active in the market needs to price in such a way that it eliminates all alternative outcomes in which consumers prefer to buy from the focal platform. This is a relatively strong notion of focality which aims at showing that even with patient firms, inefficiencies may prevail in the long-run due to network

[3] As one consumer is of mass 0, we assume that other consumers are not affected when a single consumer deviates from equilibrium path, but for consistency of prices we assume that each platform optimizes the prices set for this deviating consumer (as it would be the case with a finite but large set of consumers).

externalities and incumbency advantage.

We are interested in establishing the existence of an equilibrium where platform i wins the market in all periods irrespective of whether it is focal or not, and would find it optimal to win back a consumer in the event the consumer had switched to the competing platform.

Consider an equilibrium in which platform A wins when it is focal and when it is not. In this equilibrium, platform B charges $p_B^B = p_B^A = 0$ because as in our base model, platform B cannot hold on to consumers at time $t+1$ even if it were to attract them at time t. As for platform A, it needs to set prices that satisfy

$$q_A - p_A^A + \beta + \delta U^A \geq q_B - p_B^A - s + \delta(q_A - \tilde{p}_A^A - s + \beta + \delta U^A), \tag{7}$$

where

$$U^i = \frac{q_i - p_i^i + \beta}{1 - \delta}.$$

The left-hand side is the consumer's utility from staying in platform A, given that all other consumers stay with A. If a consumer switches to B, the consumer expects to be alone in the current period. Then, in the next period, the focal platform A charges this consumer $\tilde{p}_A^A$ which convinces this consumer to switch back to A. Then, once back on A, the consumer stays with A in all future periods.

To ensure that these expectations are rational, a focal platform A should be able to attract the deviating consumer back from the nonfocal platform B. That is:

$$q_A - \tilde{p}_A^A + \beta - s + \delta U^A \geq q_B - \tilde{p}_B^A + \delta(q_A - \tilde{p}_A^A - s + \beta + \delta U^A), \tag{8}$$

and $\tilde{p}_B^A = 0$. That is, if a consumer switched in period $t-1$ from a focal platform A to B, while A remains focal at the beginning of period t, platform B cannot hold on to this consumer; even at $\tilde{p}_B^A = 0$, the consumer prefers to switch back to the focal platform A, over waiting another period and only then switch. Notice that platform A's profit from attracting back this marginal consumer is negligible, because of our assumption of price discrimination between existing and switching consumers and because consumers have continuum mass. Yet, to ensure that beliefs are consistent, the equilibrium requires that this marginal change in platform A's profit should be positive.

Suppose now that platform A is nonfocal. A consumer joins the nonfocal platform A if it is worthwhile to do so given the beliefs that all other consumers stay with platform B in the current period, and that this consumer will switch back in the next period to the focal platform B:

$$q_A - p_A^B - s + \delta(q_B - \tilde{p}_B^B + \beta - s + \delta U^B) \geq q_B - p_B^B + \beta + \delta U^B. \tag{9}$$

Notice that such a one-period deviation is the least beneficial deviation. Whatever benefit the consumer gets from being on A is short-lived and she quickly needs to incur another switching cost. Note that if it is beneficial for a customer to deviate for one period, it will be even more beneficial to deviate and stay with A for longer. That is, we demand from the nonfocal platform to be so attractive that users still want to incur switching costs even if they can benefit only for one period. We impose such a strong condition in the spirit of part (b) of the focality definition as stated above. If condition (9) holds there does not exist any equilibrium in which consumers want to stay with B this period, even if they expect other consumers to stay with B.

Finally, consider the out-of-equilibrium outcome in which platform B is focal at period t, and a consumer switched to A at period $t-1$ and is alone in A. Notice that when conditions (1) to (3) hold, the deviating consumer knows that all customers of platform B are switching to platform A and will stay there forever. Hence despite B being focal, this consumer stays with A if it is worthwhile to do so given the beliefs that everybody else joins platform A (this is in the spirit of divide and conquer strategies). We therefore need that

$$q_A - \tilde{p}_A^B + \beta + \delta U^A \geq q_B - \tilde{p}_B^B - s + \delta(q_A - \tilde{p}_A^A + \beta - s + \delta U^A), \tag{10}$$

and $\tilde{p}_B^B = 0$.

Binding conditions (7) - (10) along with the 4 conditions $p_B^B = p_B^A = \tilde{p}_B^B = \tilde{p}_B^A = 0$ define the 8 equilibrium prices. Solving, we have

$$p_A^A = \beta - (q_B - q_A) + s(1-\delta), \quad \tilde{p}_A^A = \beta - (q_B - q_A) - s(1+\delta), \tag{11}$$

$$p_A^B = -\beta - (q_B - q_A) - s(1+\delta), \quad \tilde{p}_A^B = \beta - (q_B - q_A) + s(1-\delta).$$

This equilibrium exists when the following conditions hold. First, a focal platform A earns positive value from wining the market; that is, A profitably wins the competition described by equation (7), i.e.,

$$V_A^A(s) \equiv \frac{p_A^A}{1-\delta} = \frac{\beta - (q_B - q_A)}{1-\delta} + s \geq 0. \tag{12}$$

Second, a focal platform A earns positive profit from attracting a consumer that moved to a nonfocal B in the previous period, and then keeps this consumer in all future periods; that is, A profitably wins the competition described by equation (8):

$$\tilde{p}_A^A + \frac{\delta}{1-\delta} p_A^A = \frac{\beta - (q_B - q_A)}{1-\delta} - s \geq 0. \tag{13}$$

Third, a nonfocal platform A earns positive profit from winning the market at period t, and then becoming focal in period $t+1$ onward; that is, A profitably wins the competition described by equation (9):

$$V_A^B(s) \equiv p_A^B + \delta V_A^A = \frac{\beta(2\delta - 1) - (q_B - q_A)}{1-\delta} - s \geq 0. \tag{14}$$

Fourth, a nonfocal A earns positive profit from keeping a consumer that switched from B to A in the previous period, given that A becomes focal in the next period onwards; that is, A profitably wins the competition described by equation (10):

$$\tilde{p}_A^B + \frac{\delta}{1-\delta} p_A^A = \frac{\beta - (q_B - q_A)}{1-\delta} + s \geq 0. \tag{15}$$

Notice that as $s \to 0$, then $V_A^A(s)$ and $V_A^B(s)$ converge to V_A^A and V_A^B in our base model without switching costs. Also, as in our base model, the binding condition from among (12), (13), (14) and (15) is $V_A^B(s) \geq 0$. Hence, such an equilibrium holds iff

$$V_A^B(s) \geq 0 \iff q_B - q_A < \beta(2\delta - 1) - s(1-\delta). \tag{16}$$

Recall that in our base model with only network effects, when $q_B > q_A$, this inefficient equilibrium exists iff $\delta > \frac{1}{2}$ and $q_B - q_A < \beta(2\delta - 1)$ (cf. Section 4 in our model). Yet, this equilibrium vanishes in a model with only switching costs (Section 5 in our model). Condition (16), which takes into account both network effects and switching costs, is a combination of these two polar cases. As $s \to 0$, condition (16) converges to $q_B - q_A < \beta(2\delta - 1)$. Moreover, as s increases, this condition becomes tighter, implying that the region in which platform A wins whether it's focal or not shrinks. This result is consistent with the findings of our paper — that the inefficient equilibrium emerges at high values of δ because of network effects and not switching costs. In a model with both $\beta > 0$ and $s > 0$, β has a positive effect on condition (16) while s has a negative effect.

Consider now an equilibrium in which platform B wins whether it's focal or not. The analysis is symmetric to the analysis above, and we find that such an equilibrium exists iff

$$V_B^A(s) \geq 0 \iff q_B - q_A > \beta(1-2\delta) + s(1-\delta). \tag{17}$$

Recall that in our base model with only network effects (Section 4), this equilibrium holds when $q_B - q_A > \beta(1-2\delta)$. When the model only has switching costs (Section 5), this equilibrium holds when $q_B - q_A > s(1-\delta)$. Condition (17) is a combination of these two conditions. Intuitively, switching costs make it more difficult for a high-quality but a nonfocal platform to win the market. This result is again consistent with the findings of our base model.

3 Stochastic Qualities with a Uniform Distribution: Welfare and Consumer Surplus

This appendix extends Section 6. We study how δ affects welfare and consumers' surplus. The following corollary shows that when the stochastic qualities are distributed according to a uniform distribution, average per-period social welfare is lower when $\delta = 1$ than when $\delta = 0$. In contrast, consumer surplus is increasing with δ at least when μ is sufficiently close to 0.

Let $\bar{S}^i$ $(i = A, B)$ denote the expected consumer surplus when platform i is focal in period t, and let $S^i = (1-\delta)\bar{S}^i$ denote the average per-period expected consumer surplus. Recalling the definitions of W^A and W^B in the paper, we obtain the following result:

Corollary 1 (Welfare and consumer surplus under uniform distribution) *Let Q be uniformly distributed along the interval $[\mu-\sigma, \mu+\sigma]$, and suppose that $\sigma > 2\beta$ and $0 \geq \mu < \sigma + 2\frac{\beta^2}{\sigma} - 3\beta$. Then $W^A > W^B$ for all $0 < \delta < 1$ and $S^A > S^B$ for all $0 \leq \delta < 1$. Moreover, $W^A|_{\delta=0} = W^B|_{\delta=0} > W^A|_{\delta=1} = W^B|_{\delta=1}$. Yet, S^A and S^B are increasing with δ when $\mu \to 0$.*

Proof:

Substituting $F(Q) = \frac{Q-\mu+\sigma}{2\sigma}$ into equation (11) from the proof to Proposition 3 yields:

$$\bar{Q}^A = \beta - \frac{2\delta\mu\beta}{\sigma - 2\delta\beta} \quad \text{and} \quad \bar{Q}^B = -\beta - \frac{2\delta\mu\beta}{\sigma - 2\delta\beta}.$$

To ensure that $\bar{Q}^B > \mu - \sigma$, we need σ to be large enough that $\sigma > 2\beta$ and $0 \geq \mu <$

$\sigma + 2\frac{\beta^2}{\sigma} - 3\beta$. The recursive expected social welfare functions are then

$$\bar{W}^A = \int_{\mu-\sigma}^{\beta - \frac{2\delta\mu\beta}{\sigma - 2\delta\beta}} (\beta + \delta\bar{W}^A)\frac{1}{2\sigma}\,dq + \int_{\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}}^{\mu+\sigma} (\beta + q + \delta\bar{W}^B)\frac{1}{2\sigma}\,dq,$$
$$\bar{W}^B = \int_{-\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}}^{\mu+\sigma} (\beta + q + \delta\bar{W}^B)\frac{1}{2\sigma}\,dq + \int_{\mu-\sigma}^{-\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}} (\beta + \delta\bar{W}^A)\frac{1}{2\sigma}\,dq.$$

Therefore,

$$\begin{aligned}
W^A &= (1-\delta)\bar{W}^A \\
&= \frac{1}{4}\Bigg(4\beta - \frac{\beta^2}{\sigma} + \sigma + \frac{\mu(4\delta^2\beta^2(2\beta - 3\sigma) - \sigma^2(\mu + 2\sigma) + \delta\beta\sigma(5\mu - 4\beta + 10\sigma)))}{(\delta\beta - \sigma)(\sigma - 2\delta\beta)^2}\Bigg), \\
W^B &= (1-\delta)\bar{W}^B \\
&= \frac{1}{4}\Bigg(4\beta - \frac{\beta^2}{\sigma} + \sigma + 2\mu + \frac{(\mu(8(-1+\delta)\delta^2\beta^3 + \delta\beta(5\mu - 4(-1+\delta)\beta)\sigma - \mu\sigma^2))}{(\delta\beta - \sigma)(\sigma - 2\delta\beta)^2}\Bigg).
\end{aligned}$$

The gap $W^A - W^B$ can now be written as

$$W^A - W^B = \frac{2(1-\delta)\delta\mu\beta^2}{(\sigma - \delta\beta)(\sigma - 2\delta\beta)}.$$

As $\sigma > 2\beta$ (by assumption), $W^A - W^B > 0$ for all $0 < \delta < 1$, and $W^A - W^B = 0$ for $\delta = 0$ and $\delta = 1$. Moreover,

$$W^A|_{\delta=0} - W^A|_{\delta=1} = \frac{\mu^2\beta^2(2\sigma - \beta)}{\sigma(\sigma - \beta)(\sigma - 2\beta)^2} > 0$$

where the inequality follows because, by assumption, $\sigma > 2\beta$ and $\mu > 0$.

Turning to consumer surplus, we have

$$\bar{S}^A = \int_{-\infty}^{\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}} (\beta - p_A^A + \delta\bar{S}^A)\frac{1}{2\sigma}dq + \int_{\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}}^{\infty} (\beta + q - p_B^A + \delta\bar{S}^B)\frac{1}{2\sigma}dq,$$

$$\bar{S}^B = \int_{-\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}}^{\infty} (\beta + q - p_B^B + \delta\bar{S}^B)\frac{1}{2\sigma}dq + \int_{-\infty}^{-\beta - \frac{2\delta\mu\beta}{\sigma-2\delta\beta}} (\beta - p_A^B + \delta\bar{S}^A)\frac{1}{2\sigma}dq.$$

Substituting the prices from Section 6,

$$\begin{aligned}
S^A &= (1-\delta)\bar{S}^A \\
&= \frac{1}{4}\left(2\mu + 4\beta(1+\delta) - \frac{4\mu(\mu+\beta(1-\delta))}{\sigma-\delta\beta} - \frac{3\beta^2}{\sigma} - \sigma - \frac{10\delta\mu^2\beta}{(\sigma-2\delta\beta)^2} + \frac{\mu(3\mu+8(1-\delta)\beta)}{\sigma-2\delta\beta}\right), \\
S^B &= (1-\delta)\bar{S}^B \\
&= \frac{1}{4}\left(2\mu + 4\beta(1+\delta) - \frac{4\mu(\mu-\beta(1-\delta))}{\sigma-\delta\beta} - \frac{3\beta^2}{\sigma} - \sigma - \frac{10\delta\mu^2\beta}{(\sigma-2\delta\beta)^2} + \frac{\mu(3\mu-8(1-\delta)\beta)}{\sigma-2\delta\beta}\right).
\end{aligned}$$

We have

$$S^A - S^B = \frac{2(1-\delta)\mu\beta\sigma}{(\sigma-\delta\beta)(\sigma-2\delta\beta)}\,.$$

Hence, $S^A > S^B$ for all $\delta < 1$. Moreover,

$$\left.\frac{\partial S^A}{\partial \delta}\right|_{\mu=0} = \left.\frac{\partial S^B}{\partial \delta}\right|_{\mu=0} = \beta > 0.$$

Notice that S^A and S^B are continuous in μ. Hence, they are increasing with δ as long as μ is not too high.

This completes the proof of Corollary 1.