Supplementary material for the paper entitled:

Vertical Collusion

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1 Contract deviation that does not stop collusion is unprofitable for R_i (heterogeneous case)

Following footnote 9, this section proves the following lemma:

Lemma: *Suppose that retailers coordinate on the collusive contract that maximizes their profits, such that $T^* = T_S(w^*, \delta)$. Then, R_i cannot profitably deviate to any contract offer $(w_i, T_i) \neq (w^*, T^*)$, where (w_i, T_i) is a deviation that maintains collusion.*

Proof:

Suppose that the supplier and R_i have the common beliefs that if the supplier accepts the deviation, the supplier also accepts R_j's offer and R_i maintains collusion. Whenever R_i makes this deviation, the supplier expects that R_i will set p_M in the current period and therefore R_j will not detect it. The supplier's profit from accepting the deviation depends on whether the supplier expects that in the next period R_i will offer the equilibrium contract or continue offering the deviating contract. We consider each possibility in turn.

Suppose first that the supplier expects that R_i offers a one-period deviation, (w_i, T_i), and will continue offering (w^*, T^*) in all future periods. The supplier anticipates that if he accepts this contract, the deviation will not be detected by R_j and therefore collusion is going to continue in future periods. Therefore, the supplier accepts the deviation iff:

$$w^* q_M + T^* + w_i q_M + T_i + \frac{\delta}{1-\delta} 2(w^* q_M + T^*) > w^* \widehat{q}(p_M) + T^*, \tag{1}$$

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where the left-hand-side is the supplier's profit from accepting a one-period deviation given that doing so maintains the collusive equilibrium in all future periods and the right-hand-side is the supplier's profit from accepting only R_j's contract and stopping collusion. Substituting $T^* = T_S(w^*, \delta)$ (from the proof of lemma 3) into (1) and solving for T_i, the supplier accepts the deviation if:

$$T_i > \frac{(1-\delta)\left(\widehat{q}(p_M) - q_M\right) w^*}{1+\delta} - q_M w_i. \tag{2}$$

R_i prefers making this one-period deviation if R_i earns a higher one-period profit than the equilibrium profit. However, R_i's profit from this deviation is:

$$\begin{aligned}(p_M - w_i)\, q_M - T_i &< p_M q_M - \frac{(1-\delta)\left(\widehat{q}(p_M) - q_M\right) w^*}{1+\delta} \qquad (3)\\ &= (p_M - w^*)\, q_M - T_S(w^*, \delta),\end{aligned}$$

where the inequality follows from substituting (2) into T_i in (3). Notice that we only need to look at the one-period profit, because the deviation in the current period does not affect the collusive profits in future periods. We therefore have that R_i cannot benefit from making the deviation.

Suppose now that the supplier expects that R_i's deviation is permanent. Now, the supplier agrees to the deviation if:

$$\frac{w^* q_M + T^* + w_i q_M + T_i}{1-\delta} > w^* \widehat{q}(p_M) + T^*, \tag{4}$$

where the left-hand-side is the supplier's profit from accepting the deviation given that the supplier expects that the deviation is permanent and the right-hand-side is the supplier's profit from accepting only R_j's offer and stopping collusion. Substituting $T^* = T_S(w^*, \delta)$ into (4) and solving for T_i, the supplier agrees to the deviation if:

$$T_i > \frac{(1-\delta)\left(\widehat{q}(p_M) - q_M\right) w^*}{1+\delta} - q_M w_i \tag{5}$$

(notice that (5) is identical to (2)). The profit of R_i from making this deviation in the current and all future periods is:

$$\frac{(p_M - w_i)q_M - T_i}{1-\delta} < \tag{6}$$

$$\frac{p_M q_M}{1-\delta} - \frac{(\widehat{q}(p_M) - q_M)\, w^*}{1+\delta} = \frac{(p_M - w^*)\, q_M - T_S(w^*, \delta)}{1-\delta},$$

where the inequality follows from substituting T_i from (5) into (6). We therefore have that R_i cannot profitably make a permanent deviation to (w_i, T_i) that motivates R_i to maintain collusion. ■

The reason that both a one-shot and a permanent contract deviation are unprofitable is that the supplier's incentive constraint binds in both cases, since we look at the collusive contract that maximizes the retailers' profits. When the deviation is a one-shot deviation, a retailer cannot offer a single period contract deviation that maintains collusion, because any profits from such a deviation would be at the supplier's expense. The supplier is already indifferent between agreeing to collude or not. When the deviation is permanent, a retailer cannot benefit from a multi-period contract deviation that maintains collusion, because the collusive contract already maximizes the retailers' profits.

2 Mixed strategy equilibrium following a deviation to a $(w_i, T_i) \neq (w^*, T^*)$ that stops collusion (homogeneous case)

Following footnote 12, this section shows that when R_i and the supplier believe the contract deviation, $(w_i, T_i) \neq (w^*, T^*)$, will cause collusion to stop, R_i can earn at most $p_M Q_M - (w^* Q_M + T^*)$. The reason is that R_i needs to compensate the supplier for his alternative profit from rejecting the deviation, accepting R_j's equilibrium contract and earning $w^* Q_M + T^*$. Rational beliefs following the contract deviation (if accepted by the supplier) cannot yield pure-strategies: If the supplier believes R_i will slightly undercut p_M and capture the whole market, he will reject R_j's offer. But if R_i anticipates this, he would rather charge a monopoly price and not price cut. However, there is a mixed-strategy equilibrium following the contract deviation, in which the supplier accepts R_j's offer with a very small probability and R_i mixes between charging $p_M - \varepsilon$ and charging R_i's monopoly price given w_i , $p(w_i)$. R_i's profit from the deviation is at most $p_M Q_M - \varepsilon - (w^* Q_M + T^*)$.

To see why, suppose that R_i deviated from collusion by offering a contract $(w_i, T_i) \neq (w^*, T^*)$ that makes both R_i and the supplier believe that collusion is going to stop, while R_j offered the supplier the equilibrium contract (w^*, T^*). We first show that the subgame induced by this deviation has a mixed-strategy equilibrium in which the supplier believes that in the end of the current period R_i sets the monopoly price given w_i, $p(w_i)$, with probability γ and sets $p_M - \varepsilon$ with probability $1-\gamma$, while R_i believes that the supplier accepts R_j's offer with probability θ and rejects R_j's offer with probability $1-\theta$. We then show that the highest expected profit that R_i can make in such a deviation is $p_M Q_M - \varepsilon - (w^* Q_M + T^*)$.

Suppose that the supplier accepted the deviating contract $(w_i, T_i) \neq (w^*, T^*)$. Consider first the case $w_i > 0$, such that $p(w_i) > p_M$. When the supplier rejects R_j's equilibrium contract offer, the supplier earns (gross of T_i) $w_i Q(p(w_i))$ if R_i sets $p(w_i)$ and conversely the supplier earns $w_i Q_M$ if R_i sets $p_M - \varepsilon$. Hence, the supplier's expected profit from rejecting R_j's offer is $\gamma w_i Q(p(w_i)) + (1-\gamma) w_i Q_M$. When the supplier accepts R_j's offer, the supplier earns $w^* Q_M + T^*$ if R_i sets $p(w_i)$ and earns $w_i Q_M + T^*$ if R_i sets $p_M - \varepsilon$. Hence, the supplier's expected profit from accepting R_j's offer is $\gamma(w^* Q_M + T^*) + (1-\gamma)(w_i Q_M + T^*)$. The equilibrium condition requires that:

$$\gamma w_i Q(p(w_i)) + (1-\gamma)\, w_i Q_M = \gamma(w^* Q_M + T^*) + (1-\gamma)(w_i Q_M + T^*). \tag{7}$$

Next, consider R_i's equilibrium strategy. When R_i sets $p(w_i)$, R_i earns 0 (gross of T_i) if the supplier accepts R_j's offer and earns $(p(w_i) - w_i)Q(p(w_i))$ if the supplier rejects R_j's offer. If R_i sets $p_M - \varepsilon$, R_i earns $(p_M - \varepsilon - w_i)Q_M$ regardless of whether the supplier accepts R_j's offer. Hence, the equilibrium condition requires that:

$$(1-\theta)(p(w_i) - w_i)Q(p(w_i)) = (p_M - \varepsilon - w_i)Q_M. \tag{8}$$

Notice that any $p_i \notin \{p(w_i), p_M - \varepsilon\}$ provides R_i with a lower expected profit than $(1-\theta)(p(w_i) - w_i)Q(p(w_i))$ and therefore R_i only mixes between playing $p(w_i)$ and

$p_M - \varepsilon$.

Solving (7) and (8) yields that the equilibrium values of θ and γ, given w_i, are:

$$\gamma(w_i) = \frac{T^*}{w_i Q(p(w_i)) - w^* Q_M}, \quad \theta(w_i) = 1 - \frac{(p_M - \varepsilon - w_i)\, Q_M}{(p(w_i) - w_i)\, Q\,(p\,(w_i))}.$$

We have that $0 < \theta(w_i) < 1$, because $p(w_i)$ maximizes $(p - w_i)Q(p)$, implying that $(p(w_i) - w_i)Q(p(w_i)) > (p_M - \varepsilon - w_i)Q_M > 0$. To see that $\gamma(w_i) > 0$, recall that $T^* < 0$ (this follows from the proof of lemma 8). Moreover,

$$w^* Q_M > \pi_S^C = w^C Q(w^C) > \max_w \{wQ(p(w))\} \geq w_i Q(p(w_i)).$$

where the first inequality follows because $w^* Q_M + 2T^* > \pi_S^C$ and $T^* < 0$ implies that $w^* Q_M > \pi_S^C$ and inequality follows from Lemma 5. We therefore have that both the nominator and the denominator of $\gamma(w_i)$ are negative and hence $\gamma(w_i) > 0$. To see that $\gamma(w_i) < 1$, we need to show that $w^* Q_M - w_i Q(p(w_i)) > -T^*$. This holds because

$$w^* Q_M - w_i Q(p(w_i)) > \pi_S^C - 2T^* - w_i Q(p(w_i)) > \pi_S^C - 2T^* - \pi_S^C = -2T^* > -T^*,$$

where the first inequality follows because $w^* Q_M + 2T^* > \pi_S^C$ implies that $w^* Q_M > \pi_S^C - 2T^*$, the second inequality follows because $\pi_S^C > w_i Q(p(w_i))$ and the third inequality follows because $T^* < 0$.

Suppose now that $w_i = 0$, such that R_i sets $p(w_i) = p_M$ with probability γ and $p_M - \varepsilon$ with probability $1 - \gamma$. We solve this special case because there is a discontinuity in the mixed strategy equilibrium between $w_i > 0$ and $w_i = 0$. When the supplier rejects R_j's equilibrium contract offer, the supplier earns 0 (gross of T_i) because $w_i = 0$. When the supplier accepts R_j's offer, the supplier earns $w^* \frac{Q_M}{2} + T^*$ if R_i sets p_M and earns T^* if R_i sets $p_M - \varepsilon$. Hence, the supplier's expected profit from accepting R_j's offer is $\gamma(w^* \frac{Q_M}{2} + T^*) + (1 - \gamma)T^*$. The equilibrium condition requires that:

$$\gamma \left(w^* \frac{Q_M}{2} + T^* \right) + (1 - \gamma)\, T^* = 0. \tag{9}$$

Next, consider R_i's equilibrium strategy. When R_i sets p_M, R_i earns $p_M \frac{Q_M}{2}$ (gross of T_i) if the supplier accepts R_j's offer, and earns $p_M Q_M$ if the supplier rejects R_j's offer.

If R_i sets $p_M - \varepsilon$, R_i earns $(p_M - \varepsilon)Q_M$ regardless of whether the supplier accepts R_j's contract offer. Hence, the equilibrium condition requires that:

$$\theta \frac{p_M Q_M}{2} + (1-\theta) p_M Q_M = p_M Q_M - \varepsilon. \tag{10}$$

Solving (9) and (10) yields that the equilibrium values of γ and θ given $w_i = 0$, are: $\gamma(0) = \frac{-2T^*}{w^* Q_M}, \theta(0) = \varepsilon$, where $0 \leq \gamma(0) \leq 1$ because $T^* < 0$ and $w^* Q_M + 2T^* > 0$ and $0 \leq \theta(0) \leq 1$ because ε is positive and small.

Next we turn to showing that R_i can earn at most $p_M Q_M - \varepsilon - (w^* Q_M + T^*)$. Given that the deviating contract $(w_i, T_i) \neq (w^*, T^*)$ where $w_i > 0$ induces the above-mentioned mixed strategy equilibrium, the supplier accepts R_i's offer if

$$\gamma(w_i) w_i Q(p(w_i)) + (1 - \gamma(w_i)) w_i Q_M + T_i > w^* Q_M + T^*,$$

implying that the best R_i can do is to offer:

$$T_i(w_i) = w^* Q_M + T^* - \left(\gamma(w_i) w_i Q(p(w_i)) + (1 - \gamma(w_i)) w_i Q_M \right).$$

Hence, R_i's expected profit as a function of w_i is:

$$\begin{aligned} E\pi_R(w_i; w^*, T^*) &= (1 - \theta(w_i)) \left((p(w_i) - w_i) Q(p(w_i)) \right) - T_i(w_i) \qquad (11) \\ &= Q_M \left(p_M - \varepsilon - w^* + \frac{T^* \left(w^* - w_i \right)}{w_i Q(p(w_i)) - w^* Q_M} \right). \end{aligned}$$

The derivative of $E\pi_R(w_i; w^*, T^*)$ with respect to w_i is:

$$\frac{\partial E\pi_R(w_i; w^*, T^*)}{\partial w_i} = Q_M T^* \frac{w^* \left[Q_M - Q\left(p\left(w_i\right)\right) \right] + w_i \left(w_i - w^* \right) \frac{dQ(p(w_i))}{dw_i}}{\left(w_i Q\left(p\left(w_i\right)\right) - w^* Q_M \right)^2}.$$

We have that $\frac{\partial E\pi_R(w_i; w^*, T^*)}{\partial w_i} = 0$ when $w_i \to 0$, because the term in the first squared brackets equals zero as $Q(p(0)) = Q_M$. The second derivative, evaluated at $w_i \to 0$, is:

$$\frac{d^2 E\pi_R(w_i)}{d^2 w_i} \mid_{w_i \to 0} = -T^* \frac{\frac{dQ(p(w_i))}{dw_i}}{Q_M w^*} < 0,$$

where the inequality follows because $T^* < 0$ and $Q(p(w_i))$ is decreasing with w_i. To see that $E\pi_R(w_i; w^*, T^*)$ is concave in w_i for all $0 \leq w_i \leq w^*$, notice that since $T^* < 0$,

$$sign\left(\frac{\partial E\pi_R(w_i; w^*, T^*)}{\partial w_i}\right) = sign\left(w^*\left[Q\left(p\left(w_i\right)\right) - Q_M\right] + w_i\left[\left(w^* - w_i\right)\frac{dQ\left(p\left(w_i\right)\right)}{dw_i}\right]\right).$$

The term in the first squared brackets is negative because $w_i \geq 0$ implies that $Q_M \geq Q(p(w_i))$, and the term in the second squared brackets is negative because $w_i \leq w^*$ and $dQ(p(w_i))/dw_i < 0$. This implies that $\frac{\partial E\pi_R(w_i;w^*,T^*)}{\partial w_i} < 0$ for all $0 \leq w_i \leq w^*$, and since $\frac{\partial E\pi_R(w_i;w^*,T^*)}{\partial w_i} = 0$ for $w_i = 0$, $w_i = 0$ maximizes $E\pi_R(w_i; w^*, T^*)$ among all $0 < w_i \leq w^*$. Notice that the term in the second squared brackets is positive if $w^* < w_i$, but since the term in the first squared brackets is still negative for $w^* < w_i$, $\frac{\partial E\pi_R(w_i;w^*,T^*)}{\partial w_i} < 0$ for $w^* < w_i$ as well, as long as w_i is not too high.

Finally, substituting $w_i \to 0$ into $E\pi_R(w_i; w^*, T^*)$ yields that at $w_i \to 0$, $E\pi_R(w_i, w^*, T^*) \to p_M Q_M - \varepsilon - (w^* Q_M + T^*)$. Evaluating $E\pi_R(w_i, w^*, T^*)$ at exactly $w_i = 0$ yields the same profit. When $w_i = 0$, equation (9) implies that the supplier's profit gross of T_i is 0, and therefore the supplier accepts the deviation as long as $w^* Q_M + T^* < T_i$. Therefore, given that R_i sets $w_i = 0$, R_i offer at least $T_i = w^* Q_M + T^*$ and earn $p_M Q_M - \varepsilon - T_i = p_M Q_M - \varepsilon - (w^* Q_M + T^*)$.

3 Competition among suppliers-exclusive dealing absent communication: description of mixed strategy equilibrium (footnote 14)

Suppose that R_i offered $S_k \neq S_1$ a contract $w_i = T_i = 0$, and did not make S_1 an offer while R_j offered S_1 the equilibrium contract (w^*, T^*). Consider a mixed-strategy equilibrium in which S_1 believes that in the end of the current period R_i sets p_M with probability γ and sets $p_M - \varepsilon$ with probability $1 - \gamma$ while R_i believes that S_1 accepts R_j's offer with probability θ and rejects R_j's offer with probability $1 - \theta$. If S_1 rejects R_j's offer, S_1 earns 0 regardless of R_i's actions. If S_1 accepts R_j's offer, his expected profits are:

$$\gamma\left(w^*\frac{Q_M}{2} + T^*\right) + (1 - \gamma)\, T^*.$$

The first term corresponds to the case where R_i sets p_M, in which case R_i and R_j

split the monopoly profit and so S_1 sells $\frac{Q_M}{2}$ units to R_j and earns $w^* \frac{Q_M}{2}$. The second term corresponds to the case where R_i sets $p_M - \varepsilon$, so that R_j makes no sales and hence pays nothing to S_1, who nevertheless pays R_j the equilibrium slotting allowance.

In a mixed strategy equilibrium, S_1's indifference dictates that:

$$\gamma \left(w^* \frac{Q_M}{2} + T^* \right) + (1 - \gamma) T^* = 0,$$

where the right-hand side is S_1's expected profit from rejecting R_j's offer. Hence it is straightforward to show that in a mixed strategy equilibrium:

$$\gamma = \frac{-2T^*}{w^* Q_M}.$$

Notice that indeed $\gamma > 0$. Recall that we are contemplating collusive equilibria for $\delta < \frac{1}{2}$, and according to Lemma 2, $T^* < 0$ in such cases. Note also that $\gamma \leq 1$. To see why, recall that S_1's one-period profit in a collusive equilibrium is $w^* Q_M + 2T^* > 0$, which requires that $w^* Q_M > -2T^*$.

As for R_i, when he sets p_M, his expected profits are:

$$\theta p_M \frac{Q_M}{2} + (1 - \theta) p_M Q_M. \tag{12}$$

The first term corresponds to the case where S_1 accepts R_j's contract, so that R_i splits the monopoly profits with R_j. The second term corresponds to the case where S_1 rejects R_j's contract, so that R_i earns the entire monopoly profit.

When R_i sets $p_M - \varepsilon$, he makes $p_M Q_M - \varepsilon$ regardless of whether S_1 accepts or rejects R_j's contract, since in both cases R_j makes no sales. In a mixed strategy equilibrium, R_i's indifference requires:

$$\theta p_M \frac{Q_M}{2} + (1 - \theta) p_M Q_M = p_M Q_M - \varepsilon.$$

Hence,

$$\theta = \frac{2\varepsilon}{p_M Q_M}.$$

For an arbitrarily small and positive ε, θ too is arbitrarily small and positive.