Vertical Collusion
Note on differentiated retailers

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This appendix extends the basic model to horizontally differentiated retailers. The main conclusion of this appendix is that vertical collusion is easier to maintain than horizontal collusion (i.e., vertical collusion holds for lower values of δ than horizontal collusion). However, unlike the case of homogeneous retails, differentiated retailers and their joint supplier cannot maintain collusion for all values of δ. We obtain a closed-form solution only for a high degree of differentiation that ensures that each retailer earns positive markets share following any potential deviation and the supplier always deals with both retailers. We do not consider cases of low degree of differentiation that result in corner solutions because it makes the analysis substantially cumbersome. Our base model is a special case of a substantially low degree of product differentiation. We therefore expect that our results should follow to cases in which retailers are only slightly differentiated. The add value of this appendix is in showing that the results are qualitatively the same in the other extreme in which retailers are highly differentiated.

Consider a representative consumer with the utility function:

$$U(q_1,q_2)=\sum_{i=1}^{2}\left(q_i-\tfrac{1}{2}q_i^2\right)-\sigma q_1 q_2-\sum_{i=1}^{2}\left(q_i p_i\right), \tag{1}$$

where q_i and p_i are the price and quantity of R_i, and σ, $0 < \sigma < 1$, measures the degree of horizontal differentiation between the two retailers. When $\sigma = 0$, the two retailers are monopolies and they become closer substitutes as σ increases. To avoid corner solutions in which one of the retailers dominates the market, suppose that $\sigma > 0$ but is sufficiently close to 0.[1]

Differentiating (1) with respect to q_1 and q_2 yields the demand functions facing R_i:

$$q_i(p_i,p_j)=\tfrac{1}{1+\sigma}-\tfrac{1}{1-\sigma^2}p_i+\tfrac{\sigma}{1-\sigma^2}p_j\,.$$

Consider first the benchmark case of horizontal collusion between retailers that behave as two competing firms that can obtain the input at marginal costs 0. We ask under which values

[1] Our analysis below holds for all values of δ is $\sigma < 0.2$, and can hold for some values of δ even for $\sigma > 0.2$.

of δ such collusion is sustainable. The collusive prices that maximize the monopoly profit, $p_1q_1(q_1,q_2) + p_2q_2(q_2,q_1)$ are $p_1 = p_2 = p^* = 1/2$ which yields the monopoly quantity and profit of:

$$q^* \equiv q_i(p^*, p^*) = \tfrac{1}{2(1+\sigma)}, \quad \pi^* \equiv p^* q^* = \tfrac{1}{4(1+\sigma)}.$$

The competitive price of firm i maximizes $p_iq_i(p_i, p_j)$ given p_j. Hence, the competitive price, quantity and profit are

$$p^C = 1 - \tfrac{1}{2-\sigma}, \quad q^C \equiv q_i(p^C, p^C) = \tfrac{1}{2+\sigma-\sigma^2}, \quad \pi_R^C \equiv p^C q^C = \tfrac{1-\sigma}{(2-\sigma^2)(1+\sigma)}.$$

Notice that p^C and $\pi_R{}^C$ are decreasing with σ.

When R_j sets the collusive price, p^*, while R_i deviates from collusion, then R_i sets p_i as to maximize $p_iq_i(p_i,p^*)$. Let $p_i(p_j)$ denote R_i's best-response to p_j. Hence, the deviating price, quantity and profit are:

$$p_i(p^*) = \tfrac{2-\sigma}{4}, \quad q_i(p_i(p^*), p^*) = \tfrac{2-\sigma}{4(1-\sigma^2)}, \quad \pi^D \equiv p_i(p^*) q_i(p_i(p^*), p^*) = \tfrac{(2-\sigma)^2}{16(1-\sigma^2)}.$$

The competing retailer, R_j, sells

$$q_j(p^*, p_i(p^*)) = \tfrac{2-\sigma(2+\sigma)}{4(1-\sigma^2)},$$

which is positive if σ is sufficiently small. Horizontal collusion, without vertical contracts, is therefore possible if:

$$\frac{\pi^*}{1-\delta} > \pi_R^D + \frac{\delta}{1-\delta}\pi_R^C \quad \Leftrightarrow \quad \delta > \delta^C = \frac{(2-\sigma)^2}{8-\sigma(8-\sigma)}. \tag{2}$$

It is straightforward to show that $1/2 < \delta^C < 1$ and δ^C is increasing with σ.

Next, we turn to the case where retailers have secret vertical contracts with the supplier and we ask whether the parties can maintain vertical collusion for $\delta < \delta^C$. We construct a collusive equilibrium in which in every period the two retailers offer the supplier in the first stage the secret contract (w^*, T^*) that the supplier accepts, and then in the second stage the two retailers set the collusive price p^*. As in our base model, suppose that any observable deviation at period t triggers the perfectly competitive equilibrium from period t +1 onwards. For simplicity, we focus on the competitive equilibrium in which each retailer offers the supplier $w_i = T_i = 0$ and then the two retailers charge p^C and earn $\pi_R{}^C$ while the supplier earns $\pi^C = 0$.[2]

The collusive contract has to satisfy two conditions. The first condition is that once retailers offered a contract (w^*, T^*) that the supplier accepted, R_i indeed plays in stage 2 the collusive price, p^*, instead of deviating to R_i's short-run best response to p^*. Let $p_i(p^*;\, w^*)$ denote the p_i that maximizes R_i's profit given $p_j = p^*$ and given w^*, $(p_i - w^*)q_i(p_i, p^*)$, where:[3]

$$p_i(p^*;w^*) = \tfrac{1}{4}(2 - \sigma + 2w^*).$$

The first necessary condition is therefore:

$$\begin{aligned}&(p^* - w^*)q^* + \frac{\delta}{1-\delta}\big((p^* - w^*)q^* - T^*\big) \geq \\ &\big(p_i(p^*;w^*) - w^*\big)q_i(p_i(p^*;w^*), p^*) + \frac{\delta}{1-\delta}\pi_R^C,\end{aligned} \tag{3}$$

where the left hand side is R_i's profit from maintaining collusion and the right hand side is R_i's profit from deviating. Notice that (3) is the equivalent of condition (2) in the paper for the case where retailers are differentiated.

Next, we move to the second condition regarding the collusive contract, which involves the supplier's participation constraint. Suppose that at the beginning of a certain period, both retailers offered the supplier the collusive contract (w^*, T^*). If the supplier accepts both offers, the supplier earns in the current period $2(w^*q^* + T^*)$ and collusion continues to the next period. Suppose, however that the supplier decides to deviate from collusion by rejecting one of the offers, say, the offer of R_2. R_1 cannot observe this deviation in the second stage of the period, and will therefore set $p_1 = p^*$. Let $q(p^*,\infty)$ denote the quantity that R_1 sells when it charges $p_1 = p^*$ and consumers cannot buy from R_2. We can solve for $q(p^*,\infty)$ by substituting $q_2 = 0$ and $p_1 = p^*$ into the utility of the representative consumer in (1) and differentiating with respect to q_1. Hence, we obtain that $q(p^*,\infty) = 1/2$. The supplier earns $w^*q(p^*,\infty) + T^*$ from this deviation in the current period, but then collusion stops in all future periods. The supplier's participation constraint is therefore:

$$\frac{2(w^*q^* + T^*)}{1-\delta} = w^*q(p^*,\infty) + T^*. \tag{4}$$

[2] O'Brien and Shaffer (1992) find that when two differentiated retailers sign secret vertical contracts with a joint supplier, there is a unique "negotiation proof" contract equilibrium in which retailers set $w = T = 0$ and then charge the competitive prices.

[3] By our assumption that σ is close to 0, when R_i deviates from the collusive price, R_i does not fully monopolize the market.

Condition (4) is the equivalent of condition (3) in the paper for the case where retailers are differentiated.[4] Solving (4) for T^* yields:

$$T^*(w^*) = -\left[\tfrac{1+\delta-\sigma(1-\delta)}{2(1+\delta)(1+\sigma)}\right] w^*. \qquad (5)$$

The term in the squared brackets in (5) is positive, implying that $T^*(w^*) < 0$ whenever $w^* > 0$. Substituting $T^*(w^*)$ from (5) into $\pi_R(w^*, T^*) = (p^* - w^*)q^* - T^*$ and $\pi_S(w^*, T^*) = 2(w^*q^* + T^*)$ yields :

$$\pi_R(w^*, T^*(w^*)) = \tfrac{1}{4(1+\sigma)} - \tfrac{(1-\delta)\sigma}{2(1+\sigma)(1+\delta)} w^*, \quad \pi_S(w^*, T^*(w^*)) = \tfrac{(1-\delta)\sigma}{1+\delta+\sigma+\delta\sigma} w^*. \qquad (6)$$

As in the base model, $\pi_R(w^*,T^*(w^*))$ is decreasing in w^* while $\pi_S(w^*,T^*(w^*))$ is increasing in w^* and $\pi_S(w^*) > 0$ if and only if $w^* > 0$.

Next, we turn to solve the lowest w^* that satisfies conditions (3) and (5). Substituting (5) into (5) and rearranging, (5) becomes:

$$\Phi + \left[w^*(\Omega - \gamma w^*)\right] > 0, \qquad (7)$$

where

$$\Phi \equiv \frac{\pi^*}{1-\delta} - \pi_R^D - \frac{\delta}{1-\delta}\pi_R^C, \quad \Omega \equiv \tfrac{\sigma(1-\delta+2\delta\sigma)}{4(1+\delta)(1-\sigma^2)} > 0, \quad \gamma \equiv \tfrac{1}{4(1-\sigma^2)} > 0$$

If $\delta > \delta^C$, then from (7), $\Phi > 0$ and therefore (3) holds for $w^* = 0$. This implies that for $\delta > \delta^C$, there is a collusive equilibrium with $(w^*, T^*(w^*)) = (0, 0)$ which is identical to horizontal collusion.

Next suppose that $\delta < \delta^C$ such that $\Phi < 0$. Then, (7) is positive for:

$$\frac{1}{2\gamma}\left(\Omega - \sqrt{4\gamma\Phi + \Omega^2}\right) < w^* < \frac{1}{2\gamma}\left(\Omega + \sqrt{4\gamma\Phi + \Omega^2}\right),$$

$$\text{and } 4\gamma\Phi + \Omega^2 > 0 \quad \Leftrightarrow \quad \delta > \delta^*,$$

where

$$\delta^* = \tfrac{1}{2} - \tfrac{7-(4-\sigma)\sigma}{2+2(2-\sigma)^2\sigma} + \tfrac{2-\sigma}{2+2(2-\sigma)^2\sigma}\sqrt{(3-\sigma)(4-\sigma^2(1+\sigma)}.$$

We therefore have that a collusive equilibrium exists for $\delta > \delta^*$ and includes:

$$w^* = \frac{1}{2\gamma}\left(\Omega - \sqrt{4\gamma\Phi + \Omega^2}\right). \qquad (8)$$

[4] Notice that the right-hand-side of (4) is positive, because $q(p^*,\infty) > q^*$ and $w^*q^* + T^* > 0$.

Both δ^C, δ^* are complex polynomial functions of σ. Plotting δ^C and δ^*, we obtain that $0 < \delta^* < \delta^C < 1$. This implies that firms can sustain vertical collusion for $\delta \in [\delta^*, \delta^C]$, in which horizontal collusion is impossible. This result is qualitatively similar to the main result in our base model. However, recall that in our base model, vertical collusion is sustainable for all positive values of δ. In contrast, with differentiated retailers, vertical collusion is not sustainable for $\delta \in [0, \delta^*]$.[5] The intuition for this result is that retailers' differentiation makes it costly for the supplier to reject a deviating contract offer from a retailer, because by doing so the supplier does not gain access to certain consumers. This feature decreases the supplier's market power, which in turn decreases its ability to police the two retailers' adherence to the collusive equilibrium.

As for the equilibrium w*, the term in (8) is a complex polynomial function of σ and δ. Using numerical method, we find that $w^* > 0$ for $\delta^* < \delta < \delta^C$, w^* is decreasing with δ and $w^* = 0$ for $\delta = \delta^C$. Since $w^* > 0$, we have that in equilibrium $T^*(w^*) < 0$.

Next, we turn to show that R_i will not deviate in the first stage of every period to any other $(w_i, T_i) \neq (w^*, T^*)$. As in our base model, we assume that any such deviation induces the joint beliefs by R_i and the supplier that either R_i will stop collusion or maintain collusion. We analyze each of these beliefs it turn.

Suppose first that R_i deviates to a contract $(w_i, T_i) \neq (w^*, T^*)$ that induces the joint beliefs that R_i will deviate from collusion. Given that the supplier expects that collusion is going to stop in all future periods, the supplier will find it optimal to accept the collusive contract of R_j, if the two retailers are sufficiently differentiated. This is because $w^*q^* + T^* > 0$ implies that if σ is sufficiently low, $w^*q_j(p^*, p_i) + T^* > 0$ even though $p_i < p^*$. Intuitively, even when the supplier expects that R_i plans to undercut R_j, if the two retailers are sufficiently differentiated, R_i will not steal substantial sales from R_j and therefore it is worthwhile for the supplier to accept the contract of R_j.

Given these beliefs, the supplier accepts the contract (w_i, T_i) if:

$$w_i q_i\left(p(p^*;w_i), p^*\right) + w^* q_j\left(p^*, p(p^*;w_i)\right) + T^* + T_i \geq w^* q_j\left(p^*, \infty\right) + T^*, \tag{9}$$

where $q_j(p_j, \infty) = 1 - p_j$ is the demand facing R_j when R_j is a monopoly.[6] To find the highest possible profit that R_i can earn from making this deviation, we can solve (9) for T_i and substitute into R_i's profit from the deviation:

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[5] We find that there if $\delta < \delta^*$, there is no collusive equilibrium even if we do not impose equality on condition (4), and instead allow for collusive equilibria in which the supplier earns strictly higher profits from maintaining collusion (the left hand side of condition (4)) than from deviating (the right hand side of (4)).

[6] We obtain $q_j(p_j, \infty)$ by maximizing (1) with respect to q_j, given $q_i = 0$.

$$
\begin{aligned}
&\big(p(p^*;w_i)-w_i\big)q_i\big(p(p^*;w_i),p^*\big)-T_i+\frac{\delta}{1-\delta}\pi_R^C \\
&=p(p^*;w_i)q_i\big(p(p^*;w_i),p^*\big) \\
&+w^*q_j\big(p^*,p(p^*;w_i)\big)-w^*q_j\big(p^*,\infty\big)+\frac{\delta}{1-\delta}\pi_R^C.
\end{aligned} \tag{10}
$$

Maximizing (10) with respect to w_i, we obtain $w_i = \sigma w^*$. Substituting $w_i = \sigma w^*$ into (10), R_i can earn from such a deviation a one-period profit of at most $\pi_R^D(w^*)$, followed by the profit of π_R^C in all future periods, where

$$\pi_R^D(w^*)=\frac{(2-\sigma-2\sigma w^*)^2}{16(1-\sigma^2)}. \tag{11}$$

Hence, R_i will not make this deviation if:

$$\frac{\pi_R(w^*,T^*(w^*))}{1-\delta}>\pi_R^D(w^*)+\frac{\delta}{1-\delta}\pi_R^C, \tag{12}$$

where $\pi_R(w^*, T(w^*))$ and $\pi_R^D(w^*)$ are defined in (6) and (11) respectively. Rearranging (12), the deviation is unprofitable if

$$w^*>\frac{1}{2\gamma_1}\left(\Omega_1-\sqrt{4\gamma_1\Phi+\Omega_1{}^2}\right), \tag{13}$$

Where Φ is the same as in (7) and:

$$\Omega_1\equiv\frac{\sigma(\delta(2-\sigma)+\sigma)}{4(1+\delta)(1-\sigma^2)}>0,\quad \gamma_1\equiv\frac{\sigma^2}{4(1-\sigma^2)}>0.$$

Comparing (13) with (8) reveals that (13) always holds. This implies that R_i will not find it profitable to deviate to any other $(w_i, T_i) \neq (w^*, T^*)$ that stops collusion.

Suppose now that R_i deviates to a contract $(w_i, T_i) \neq (w^*, T^*)$ that induces the joint beliefs that R_i will maintain collusion. We can trivially extend the result of Lemma 6 in the paper and show that such a deviation is not profitable for R_i. The intuition is that since the collusive contract maximizes the retailers' profit subject to the supplier's participation constraint, a retailer cannot offer the supplier an even more profitable contract that maintains collusion.

To summarize, we find that for $\delta\in[0, \delta^*]$ there is no collusive equilibrium. For $\delta\in[\delta^*,\delta^C]$, there is an equilibrium with vertical collusion that involves $w^* > 0$ and $T^* < 0$. For $\delta\in[\delta^C,1]$, retailers can implement the standard horizontal collusion with $w^* = 0$ and $T^* = 0$.