Abstract

We study a stochastic economy where both employed and unemployed workers search randomly for labor contracts posted by ex ante heterogeneous firms, while aggregate productivity is subject to shocks. A firm can commit to a (Markov) contract, which specifies a wage contingent on all payoff-relevant states, but must pay equally all of its workers, who have limited commitment and are free to quit at any time. Our exercise provides the first dynamic stochastic general equilibrium analysis of a popular class of search wage-posting models, drawing in part from the literature on recursive contracts under moral hazard. An equilibrium of the contract-posting game is Rank-Preserving [RP] if larger firms offer a larger value to their workers in all states of the world. We find two sufficient (but not necessary) conditions for every equilibrium to be RP: either firms only differ in their initial size, or they also differ in their fixed idiosyncratic productivity but more productive firms are also initially weakly larger. In the latter case, turnover is always efficient, as workers always move from less to more productive firms. In both cases, the ranking of firm sizes never changes on the RP equilibrium path, a property that has two useful implications. First, the stochastic dynamics of firm size provide an intuitive and robust explanation for the empirical finding that large employers in the US are more cyclically sensitive (Moscarini and Postel-Vinay, 2009). Second, RP equilibrium computation is fairly tractable, and we construct and simulate calibrated examples.
1 Introduction

We study the aggregate equilibrium dynamics of a frictional labor market where workers search randomly on and off the job for employment contracts posted by firms. Our exercise provides the first analysis of aggregate dynamics of a popular class of search wage-posting models, originating with Burdett and Mortensen (1998, henceforth BM). They provide the first successful formalization of the hypothesis that cross-sectional wage dispersion is largely a consequence of labor market frictions. In so doing the BM model has started what has now established itself as the most promising line of research in the analysis of wage inequality, as the vibrant and empirically very successful literature organized around that hypothesis continues to show.

That literature, however, is invariably cast in deterministic steady state. Ever since the inception of the BM model, job search scholars have regarded the characterization of its out-of-steady-state behavior as a daunting problem, essentially because one of the model’s state variables, which is also the main object of interest, is the endogenous distribution of wage (or job value) offers. This is an infinite-dimensional object, endogenously determined in equilibrium as the distribution across a continuum of firms of strategies that are all best responses to one another.

We find a way around this problem by considering a class of equilibria satisfying what we call the Rank-Preserving property, i.e. equilibria in which the workers’ ranking of firms is time-invariant. We show that this class of equilibria is generic if all firms are equally productive and the environment is not subject to aggregate stochastic shocks. We further show that the same property holds in equilibrium when firms have heterogeneous productivity, where more productive firms offer a larger value and employ more workers at all points in time, if (but not only if) they have more employees to begin with. We view the fact that the workers’ ranking of firms also reflects the hierarchy of productivity in a Rank-Preserving Equilibrium in the presence of productive heterogeneity across firms as a very appealing property of the model. It parallels a similar property of BM’s static equilibrium, and in ensures constrained-efficient labor reallocation at all dates. Finally, we investigate existence of Rank-Preserving Equilibria in a stochastic environment.

Besides being of intrinsic theoretical interest, our characterization of the dynamics of the BM model opens the analysis of aggregate labor market dynamics as a whole potential new field of application of search/wage-posting models. Unlike the typical representative-agent model, the BM model makes predictions about the business cycle behavior of wage distributions, firm size distributions, or patterns of labor reallocation across firms. In Moscarini and Postel-Vinay (2009, MPV09), we quantitatively gauge those predictions against facts
documented using various (often new) data sets. Moreover, by advocating the BM model as a potentially useful tool for the study of aggregate labor market dynamics, we hope to contribute to a synthesis between the BM approach and the “other”, equally successful side of the search literature, organized around the matching framework (Pissarides, 1990; Mortensen and Pissarides, 1994), initially designed for the understanding of labor market flows and equilibrium unemployment.\(^1\)

The rest of the paper is organized as follows. In Section 2 we lay out the basic environment. In Section 3 we describe and formally define an equilibrium of our model economy. We then introduce the notion of Rank Preserving Equilibria in Section 4, where we also state our main result about the generality of RPE and give a characterization of RPE. Then Section 5 shows how to practically simulate equilibrium paths for wages and employment in a RPE. Finally, Section 6 concludes.

## 2 The economy

We study a stochastic economy where firms commit to employment contracts and workers search randomly for those contracts. The special case of a stationary and deterministic economy where contracts are restricted to a constant wage is the BM wage posting model with heterogeneous firm types. We present our model in discrete time, as it affords more clarity in the presentation of the contract posting problem under one-sided commitment as a recursive problem, following the seminal insights of Spear and Srivastava (1987).

The labor market is populated by a unit-mass of workers, who can be either employed or unemployed, and by a unit measure of firms\(^2\) Workers and firms are risk neutral, infinitely lived, and maximize payoffs discounted with factor $\beta \in (0, 1)$. Firms operate constant-return technologies with heterogeneous productivity levels $\omega \theta$, where $\omega$ is an aggregate component, evolving within some bounded set of values $\Omega \subset \mathbb{R}_+$ according to a discrete-time stationary

\(^1\)Rudanko (2008) and Menzio and Shi (2008) formulate and solve wage contract-posting models with aggregate productivity shocks, where job search is directed, rather than random in the spirit of BM. This assumption greatly simplifies the analysis, by severing the link between the individual firm’s contract-posting problem and the distribution of contract offers. This is the main hurdle that we face, and that we resolve by exploiting the idea and emergence of Rank-Preserving Equilibrium, while maintaining BM’s assumption of random search common to the vast majority of the search literature. From a theoretical viewpoint, we see both programs as fruitful directions of exploration. From a quantitative viewpoint, the directed search approach is focused on the response of the job-finding rate to aggregate shocks. This approach does not generate a well-defined notion of employer size. Hence, it is silent on the wealth of new evidence on cyclical patterns of the size distribution and employer size/growth relationship that we offer in MPV09, and that we envision as central to our understanding of the propagation of aggregate shocks in labor markets.

\(^2\)We implicitly fix the measure of active firms, thus remaining mostly silent on the question of entry and exit. A simple extension of the model to make it capture entry and exit of firms over the business cycle is illustrated in our companion paper Moscarini and Postel-Vinay (2008, MPV08).
first-order Markov process \( H (d\omega' \mid \omega) \), and \( \theta \) is a fixed, idiosyncratic heterogeneity component, distributed across firms \( \theta \sim \Gamma (\cdot) \) with density \( \gamma = \Gamma' \) over \([\theta, \bar{\theta}]\).

The labor market is affected by search frictions in that unemployed workers can only sample job offers sequentially with some probability \( \lambda_0 \in (0, 1) \) each period. Employed workers earn a wage, are allowed to search on the job, and face a per-period sampling chance of job offers of \( \lambda_1 \in (0, 1) \). For notational simplicity we will assume uniform sampling of firms by workers, in that any worker receiving a job offer draws the type of the firm from which the offer emanates from the distribution \( \Gamma (\cdot) \). Firm-worker matches are dissolved with chance \( \delta \in (0, 1) \). Upon match dissolution, the worker becomes unemployed. Note that all these transition probabilities, although exogenous, are allowed to depend on the aggregate state \( \omega \). Workers attach a common lifetime value of \( U \) to being unemployed.

In each period, the timing is as follows. Given a current state \( \omega \) of aggregate labor productivity and size (measure of workers employed) \( L \):

1. a firm of type \( \theta \) produces output and pays wages in state \( \omega \). The flow benefit \( b_\omega \) accrues to unemployed workers;
2. the new state \( \omega' \) of aggregate labor productivity is realized;
3. new workers are hired from unemployment, each with probability \( \lambda_0' \);
4. employed workers can quit to unemployment;
5. jobs are destroyed exogenously with chance \( \delta' \);
6. the remaining employed workers receive an outside offer with chance \( \lambda_1' \) and decide whether to accept it or to stay with the current employer.

To close the model, we need to specify how are wages set. A firm of idiosyncratic productivity \( \theta \) chooses and commits to an employment contract, namely a state-contingent wage depending on some state variable \( \zeta \), to maximize the present discounted value of profits, given other firms’ contract offers. The firm is further subjected to an equal treatment constraint, whereby it must pay the same wage to all its workers. This is the sense in which we generalize the BM restrictions placed on the set of feasible wage contracts to a non-steady-state environment. Under commitment, such a wage function implies a value \( V \) for any

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3 Later, we extend the model to allow for non-uniform sampling, in that different types-\( \theta \) firms have different chances of being sampled by job searchers. This extension is theoretically straightforward and useful in quantitative applications.

4 We thus rule out, among other things, wage-tenure contracts (Stevens, 2004; Burdett and Coles, 2003), offer-matching or individual bargaining (Postel-Vinay and Robin, 2002; Dey and Flinn, 2005; Cahuc, Postel-Vinay and Robin, 2006), contracts conditioned on employment status (Carrillo-Tudela, 2009). Note, however,
worker to work for that firm, which is also a function of the state $\zeta$. For reasons that will become clear shortly, we assume that a contract offered by a firm to its workers is observable only by the parties involved.

3 Equilibrium

3.1 Definition

Let $Z$ be a (Borel-)measurable set and $V_Z$ the set of measurable functions $[\theta, \overline{\theta}] \times Z \rightarrow \mathbb{R}$. The state space $Z$ is the set of all histories of play in the game. A behavioral strategy of the contract-posting game is a function $V \in V_Z$ such that, when the state of the game is $\zeta \in Z$, each firm $\theta \in [\theta, \overline{\theta}]$ offers value $V(\theta, \zeta)$ to all of its workers.

As $V$ is measurable, the c.d.f.

$$F(W | \zeta, V) := \int_{\theta}^{\overline{\theta}} \mathbb{I}\{V(\theta, \zeta) \leq W\} d\Gamma(\theta) \tag{1}$$

is well-defined for every $\zeta \in Z$, $W \in \mathbb{R}$ and $\mathbb{I}$ an indicator function. This is the probability that a randomly drawn firm offers value no greater than $W$, given history $\zeta$ and given that all firms follow strategy $V$. Let $F = 1 - F$ denote the survival function.

Let $\Lambda(\theta)$ be the measure of cumulated employment at all firms of productivity up to $\theta$, so $\Lambda(\overline{\theta})$ is total employment and $1 - \Lambda(\overline{\theta})$ is unemployment. For any increasing $\Lambda : [\theta, \overline{\theta}] \rightarrow [0, 1]$, $\zeta \in Z$, $W \in \mathbb{R}$, the following c.d.f.

$$G(W | \zeta, \Lambda, V) := \frac{1}{\Lambda(\overline{\theta})} \cdot \int_{\theta}^{\overline{\theta}} \mathbb{I}\{V(\theta, \zeta) \leq W\} d\Lambda(\theta) \tag{2}$$

is also well-defined. This is the probability that a randomly drawn worker is currently earning value no greater than $W$ after history $\zeta$.

Let primes denote next period values. Given a strategy $V \in V_Z$ followed by all firms and the resulting $F$ and $\Lambda$, an unemployed earns a value solving:

$$U(\zeta | V) = b^\omega + \beta \mathbb{E}_{\zeta'|\zeta} \left[ (1 - \lambda_0^\omega') U(\zeta' | V) + \lambda_0^\omega' \int \max \langle v, U(\zeta' | V) \rangle dF(v | \zeta', V) \right], \tag{3}$$

because she collects a flow value $b^\omega$ and, one period later, when the aggregate state becomes $\omega'$, she draws with chance $\lambda_0^\omega'$ a job offer from the equilibrium distribution of offered values $F$, which she accepts if the associated value exceeds that of staying unemployed.

that the model can be generalized to allow for time-varying individual heterogeneity under the assumption that firms offer the type of piece-rate contracts described in Barlevy (2008). In that sense experience and/or tenure effects can be introduced into the model. Shimer (2008) proposes an alternative formulation, which maintains BM’s restriction of a constant posted wage, even out of steady state, and delivers a few of the same results.
Invoking a large numbers approximation, a firm of current size $L$ which posts a value $W$ in state $\zeta$ has size zero next period if $W < U$, otherwise new firm size is:

$$L' = L(\zeta, W \mid V) := L \left(1 - \delta^{\omega'}\right) \left(1 - \lambda^{\omega'}_V F(W \mid \zeta, V)\right) + \lambda^{\omega'}_0 \left[1 - \Lambda(\bar{\theta})\right] \mathbb{I}\{W \geq U(\zeta \mid V)\} + \lambda^{\omega'}_0 \left(1 - \delta^{\omega'}\right) \Lambda(\bar{\theta}) G(W \mid \zeta, V). \tag{4}$$

After the new aggregate state $\omega'$ is drawn, of the measure $L$ of workers currently employed by this firm, a fraction $(1 - \delta^{\omega'})$ are not separated exogenously into unemployment. Of these survivors, a fraction $\lambda^{\omega'}_V F(W \mid \zeta, V)$ quit because they draw from $F$ an outside offer which gives them a value larger than $W$. The currently unemployed $1 - \Lambda(\bar{\theta})$ find jobs with chance $\lambda^{\omega'}_0$, and accept an offer from a firm offering $W$ if this is better than unemployment.

By random matching, each firm offering more than $U$ receives the same inflow from unemployment. The employed who have not lost their jobs $(1 - \delta^{\omega'}) \Lambda(\bar{\theta})$ receive an offer with chance $\lambda^{\omega'}_0$, and accept it if the value $W$ they draw is larger than what they were earning before (probability $G(W \mid \zeta, V)$), in which case they quit to this firm offering $W$.

A consistency condition requires the cumulated firm size to evolve as the sum of individual firm sizes on the equilibrium path. For any $\theta \in [\underline{\theta}, \bar{\theta}]$:

$$\Lambda(\theta \mid \zeta, V)' = \int_{\underline{\theta}}^{\theta} \mathcal{L}(\zeta, V(x, \zeta) \mid V) d\Gamma(x) := T(\zeta \mid V). \tag{5}$$

The support of $\Lambda$ is contained in that of $\Gamma$, because there cannot be a firm $\theta$ with employees if there exists no such firm of type $\theta$. By induction, starting from the initial distribution of employment and for every history of the game, $\Lambda$ has a (possibly nil) Radon-Nikodym derivative $d\Lambda(\theta \mid \zeta, V) / d\Gamma(\theta)$ everywhere in $\theta$, and the consistency condition requires this derivative to be $\mathcal{L}(\zeta, V(\theta, \zeta) \mid V)$. Therefore, $F$ and $G$ also exist at all nodes of the game when firms play the strategy $V$.

A value strategy $W \in \mathcal{V}_Z$ can also be represented by a wage strategy $w \in \mathcal{V}_Z$ such that the worker’s Bellman equation is solved by $W$, in which case we say that $w$ “implements” $W$ given that all other firms play $V$:

$$W(\theta, \zeta) = w(\theta, \zeta) + \beta \mathbb{E}_{\zeta' \mid \zeta} \left[\delta^{\omega'} U(\zeta' \mid V) \right.$$

$$\left. + \left(1 - \delta^{\omega'}\right) \left(W(\theta, \zeta') + \lambda^{\omega'}_V \int_{V(\theta, \zeta')}^{+\infty} [v - W(\theta, \zeta')] dF(v \mid \zeta', V)\right)\right] \tag{6}$$

We are now going to define an equilibrium of the contract-posting game. Each firm plays a game against other firms as well as vis-a-vis its current and prospective workers. Workers act sequentially, as they are always free to quit. Firms follow a behavioral strategy $V$ (a
value policy) that must be a best-response against other firms at any node \( \zeta \) of the game, including those reached with probability zero on the equilibrium path. For example, a firm may find itself losing more workers than predicted by current equilibrium play. This requires specifying a consistent belief assessment. In this sense, firms also act sequentially, optimizing at any game node, but subject to the constraint of delivering the value to the workers once hired. This constraint is binding because, after hiring a worker with a promise of \( W \), the firm would like to renege and to squeeze the worker against the participation constraint \( W = U \).

The reputational underpinnings of the firm’s commitment power have been widely explored in the wage-posting literature. As is standard, behavioral strategies in the extensive form dynamic game generate strategy profiles of the equivalent static, strategic form game where each firm chooses a map once and for all at time 0.

**Definition 1** A sequential equilibrium of the contract posting game is a measurable function \( V \in \mathcal{V}_Z \) of, and a set of consistent beliefs over, the histories of the game \( Z \), such that \( V \) maximizes the present discounted value of profits, given that all other firms also play \( V \) and given beliefs: at least one solution \( w^* \in \mathcal{V}_Z \) to (6) with \( W = V \) also solves:

\[
w^*(\theta, \zeta) = \arg \max_w \mathbb{E} \left[ \sum_{t=0}^{+\infty} \beta^t (\omega_t \theta - w(\theta, \zeta_t)) \mathcal{L}(\zeta_{t-1}, W \mid V) \mid \zeta_0 = \zeta \right],
\]

where \( \Lambda_t(\theta) = \int_\theta^\theta \mathcal{L}(\zeta_{t-1}, V(x, \zeta_{t-1}) \mid V) d\Gamma(x) \), and \( F, G, U, W \) are defined by (1), (2), (3), (6) with \( \zeta = \zeta_t \).

The equilibrium strategy \( V \) is a fixed point in the usual game-theoretic sense: if all firms follow \( V \) and workers act optimally, then given the implied evolution of the cross-section distributions of values offered \( F \) and earned \( G \) and of the value of unemployment \( U \), each firm \( \theta \)'s best response is to follow the same strategy \( W = V \), or \( w = w^* \).

### 3.2 Markov perfect equilibrium

Our first task is to find the state space \( Z \) on which equilibrium strategies can be conditioned. By assumption, past play by other firms is unobservable, hence cannot be part of \( Z \). For the same reason, and because it is small, a firm takes its competitors’ behavior (the distributions \( F \) and \( G \)) as given when choosing a strategy, because its own deviations cannot be detected and be subject to retaliation, so its actions cannot affect the distribution of offers in the economy. Each firm can only observe the public histories of \( \{\omega, F, G, L, \Lambda, U\} \), which form a set \( Q \). Hence \( Z \subseteq Q \).

We look for the smallest subset \( Z \subseteq Q \) which is sufficient for \( Q \). For every \( \zeta \in Z \) and \( V(\cdot, \zeta) \), the current offer distribution \( F(\cdot \mid \zeta, V) \) is uniquely determined from (1), so it
contains no independent information about \( \zeta \). Similarly, given \( \zeta \), \( V(\cdot, \zeta) \) and \( F(\cdot | \zeta, V) \), the value of unemployment \( U(\zeta | V) \) is uniquely determined recursively forward by (3), so it contains no independent information about the past history \( \zeta \). The same is true, from (2), of \( G(\cdot | \zeta, V) \), given \( \zeta \), \( V \) and \( \Lambda \). Next, each individual firm takes the strategy \( V \) chosen by others as given, whether or not this firm is maximizing, given \( \zeta \). Therefore, for every \( V \), a firm can calculate the history of \( \Lambda \) based only on the history of \( \omega \). That is, each firm takes the path of employment at other firms \( \Lambda \) as an exogenous stochastic process. Hence, for every value-offering strategy defined on the history of \( \omega \) and \( \Lambda \), there exists an equivalent value-offering strategy defined on the history of \( \omega \) only, which produces the same payoff relevant variables for firm \( \theta \). For the purpose of calculating firm \( \theta \)'s best response, the history of \( \omega \) is sufficient for the history of \( \Lambda \).

The only other independent piece of information that is relevant to a firm’s profit maximization is own size \( L \), that is directly and covertly controlled by the firm and has a direct impact on the firm’s continuation payoffs. Because the history of own size \( \{L_s\}_{s=1}^t \) is private information, it cannot affect values offered by other firms. Hence only current size \( L_t \) can affect the firm’s best response, because of its direct impact on profits. We conclude that the only strategically relevant history for a firm can be \( \zeta_t = \{\omega_1, \ldots, \omega_t, L_t\} \). Clearly, past values of \( \omega \) cannot be ruled out of the state space \( Z \), as they are exogenous and public events that firms can use to coordinate actions. Given the strategic complementarity of these values, past states of aggregate productivity could always be strategically relevant as a public randomization device, although they are no longer payoff-relevant given the Markov evolution of \( \omega \).

Because the history of aggregate productivity is too large a space to be tractable, we look for equilibria in strategies that depend only on current values of payoff-relevant variables. From our discussion, it is clear that

\[
\hat{\zeta} = \{\theta, L, \omega', \Lambda\}
\] (7)

is both the smallest and largest such state vector on which equilibrium strategies can depend. If all firms condition their current offers on these four objects in \( \hat{\zeta} \), then from Definition 1 of equilibrium so should each firm in its best response. Let \( \mathcal{V}_Z \) be the space of measurable functions \( \hat{Z} \to \mathbb{R} \). Then we focus on:

**Definition 2** A MARKOV PERFECT EQUILIBRIUM of the contract posting game is a sequential equilibrium \( V \) in the set \( \mathcal{V}_Z \subset \mathcal{V}_Z \), a measurable function of \( \hat{\zeta} \) defined in (7).

Making strategies independent of past values of aggregate productivity comes at the cost of introducing in the state the current distribution of employment \( \Lambda \). This is also an infinitely dimensional object, but it turns out to be much more tractable, as we will see next.
From now on, we let the new value distributions $F (\cdot \mid \omega', \Lambda)$ and $G (\cdot \mid \omega', \Lambda)$, firm size $L (L, \omega', \Lambda, W)$, employment distribution $T (\omega', \Lambda)$, and value of unemployment $U (\omega', \Lambda)$ be defined as in (1) - (5), with the Markov state in (7) replacing $\zeta$. Notice that only new firm size $L$ depends on $L$; the other objects only depend on the aggregate components of the state, $\omega'$ and $\Lambda$, that each firm takes as given stochastic processes on and off the equilibrium path. That is, $\hat{\zeta}$ contains only one endogenous (to the firm) state variable, $L$.

4 Equilibrium characterization

4.1 The firm’s contract-posting problem: recursive formulation

We look for a Markov perfect equilibrium of the contract-posting game. Suppose all other firms offer a value $V (\theta, L, \omega', \Lambda)$ which depends on own productivity $\theta$, beginning-of-period own size $L$ and distribution of employment $\Lambda$, and new state of aggregate productivity $\omega'$. Then, by inspection of the firm’s sequential profit maximization problem, these four objects are sufficient to pin down the firm’s best response and evolve according to a Markov process. Therefore, it is natural to seek a recursive formulation of the firm’s problem. As is standard in the contracting literature (Spear and Srivastava, 1987), the firm’s sequential contracting problem is equivalent to a recursive problem, in which the firm takes the value currently promised to its workers as a state variable, and faces a promise-keeping constraint. Therefore, we focus on the following recursive problem.

We fix the strategy of other firms $V$ and omit it from the notation for simplicity. The firm can always guarantee itself zero flow profits by making the participation constraint $W (\omega') \geq U$ bind and dismissing all workers, so offering any value lower than $U$ is equivalent to an offer $W (\omega') = U$. The firm solves:

$$\Pi (\theta, L, \omega, \Lambda, V) = \sup_{w, W (\omega') \geq U (\omega', T (\omega', \Lambda))} \left\{ (\omega \theta - w) L \right.$$  
$$+ \beta \int_{\Omega} \Pi [\theta, L (L, \omega, \Lambda, W (\omega')), \omega', T (\omega', \Lambda), W (\omega')] H (d\omega' \mid \omega) \right\} (8)$$

subject to a Promise-Keeping (PK) constraint to deliver at least the promised $\bar{V}$:

$$V \leq w + \beta \cdot \int_{\Omega} \left\{ \delta \omega' U (\omega', T (\omega', \Lambda)) \right.$$  
$$+ (1 - \delta \omega') \cdot \left\{ (1 - \lambda_1^f F (W (\omega') \mid \omega', \Lambda)) W (\omega') + \lambda_1^f \int_{W (\omega')}^{+\infty} v dF (v \mid \omega', \Lambda) \right\} H (d\omega' \mid \omega) \right\}. (9)$$
Given the timing of events, the firm collects flow revenues, equal to per worker productivity $\omega \theta$ times firm size $L$, then pays the flow wage $w$ to each worker, then observes the new state of aggregate productivity $\omega'$, and finally chooses the continuation contract, so that wage and continuation values deliver at least the current expected value $\bar{V}$ to the workers.

Notice that at time 0 the firm could extract full rents by offering $w = -\infty$, because it is “too late” for the initial workers to quit. To avoid this pathological outcome, we let the initial wage be chosen according to some bargaining procedure that splits rents from the contract and leaves the firm a non-negative cut, so the recursive formulation (8)-(9) only applies from period $t = 1$ on. Therefore, $\bar{V} \geq U(\omega, \Lambda)$ is always guaranteed, because $\bar{V}$ is the value promised a period before under the worker participation constraint $W(\omega') \geq U(\omega, \Lambda)$.

4.2 An equivalent unconstrained recursive formulation

The constraints PC $W(\omega') \geq U(\omega', T(\omega', \Lambda))$ and PK in (9) are appended to the firm’s Bellman equation with their associated Lagrange multipliers, $m\omega'$ for PC and $\pi\omega'$ for PK, respectively, to form a Lagrangian. Since the maximand is smooth in the wage $w$, we can take a derivative w.r.t. to $w$ to see that optimality requires $-L + \pi\omega' \leq 0$. In other words, PK (9) must bind: if it did not, then $\pi\omega' = 0$, $-L < 0$, so the optimal wage would be as low as it can be, making (9) bind, a contradiction. So $\pi\omega' > 0$, and we can solve for the wage from (9) and replace it into the firm’s Bellman equation. When we do this, we see that $\Pi$ is differentiable w.r.t. the promised value, with $\Pi_{\bar{V}}(\theta, L, \omega, \Lambda) = -L$. This fact has two implications. First, firm profits are linear in promised value, and we can define the joint value of the firm-worker collective as a function independent of the value promised to the worker:

$$S = \Pi + \bar{V}L.$$  

Second, we can replace this expression into the Bellman equation to obtain an equivalent DP problem in the surplus function $S$, rather than in profits $\Pi$, without the PK constraint:

$$S(\theta, L, \omega, \Lambda) = \omega \theta L + \beta \int_\Omega \left\{ \delta^{\omega'} U(\omega', T(\omega', \Lambda)) L \right\}$$

$$+ \max_{W(\omega') \geq U(\omega', T(\omega', \Lambda))} \left\{ S(\theta, L, \omega', \Lambda, W(\omega')) + \omega', T(\omega', \Lambda)L (1 - \delta^{\omega'}) \lambda^{\omega'} \int_{W(\omega')}^{+\infty} v dF(v | \omega', \Lambda) - W(\omega') \left( \lambda^{\omega'}_0 (1 - \Lambda(\bar{\theta})) + \lambda^{\omega'}_1 (1 - \delta^{\omega'}) \Lambda(\bar{\theta}) G(W(\omega') | \omega', \Lambda) \right) \right\} H(d\omega' | \omega). \quad (10)$$

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5. Except possibly for start-up firms that have $L = 0$ initially, but then for those the concept of a promised value inherited from the past is moot and the choice of $w$ is irrelevant.
An equilibrium is a solution \( V \) that coincides with the one followed by the other firms. To solve for equilibrium, we proceed as follows. We assume that an equilibrium strategy \( V \), a value offered by each firm to workers which is also a best response to itself, exists, and we show which properties \( V \) must have to be an equilibrium. This allows us to restrict the set of possible equilibrium functions \( V \), in particular, we prove that under certain sufficient conditions an equilibrium strategy must be strictly increasing in own productivity \( \theta \). In that smaller set, we will construct an equilibrium.

### 4.3 The Rank-Preserving Property

While solving for equilibrium directly is an intractable problem because the size distribution of firms \( \Lambda \) is an infinitely-dimensional state variable, we can still define a tractable and natural class of equilibria, which have the following property. Let \( L^* (\theta) \) denote employment size of a type-\( \theta \) firm along the equilibrium path, i.e. the size attained by that firm given the initial size distribution at date 0 and given that all firms have implemented the equilibrium strategy from date 0 up to the current date. Then:

**Definition 3** An equilibrium is **Rank-Preserving (RP)** if a more productive firm always pays its workers more: \( \theta \mapsto V(\theta, L^* (\theta), \omega, \Lambda) \) is increasing in \( \theta \).

A direct consequence of the above definition is that in a Rank preserving Equilibrium (RPE) workers rank their preferences to work for different firms according to firm productivity at all dates. The following two properties thus hold true at all dates under the RP assumption: the proportion of firms that offer less than \( \theta \) is simply that proportion of firms that are less productive than \( \theta \)

\[
F (V(\theta, L^* (\theta), \omega', \Lambda) \mid \omega', \Lambda) \equiv \Gamma (\theta),
\]

and the number of employed workers who earn a value that is lower than that offered by \( \theta \) equals employment at firms less productive than \( \theta \):

\[
\Lambda (\theta) G (V(\theta, L^* (\theta), \omega', \Lambda) \mid \omega', \Lambda) = \Lambda (\theta).
\]

As we will see those restrictions will decisively simplify the calculations involved in solving for equilibrium in the stochastic model. Moreover, the RP property is theoretically appealing for at least two more reasons. First, it parallels a well-known property of the static equilibrium characterized by BM, which is to have a unique equilibrium where workers rank firms according to productivity. Second, RPE feature constrained-efficient labor reallocation at all dates: if workers consistently rank more productive firms higher than less productive ones, then job-to-job moves will always be up the productivity ladder. It is therefore natural to ask
how general Rank-Preserving Equilibria are. We now show that under some weak sufficient conditions on the initial size distribution of employment, all Markov equilibria must have this property. This is the central result of the paper. We assume that \( \Omega \) is finite only for simplicity of exposition and proof, to avoid dealing with measurability issues, but nothing conceptually depends on this restriction.

**Proposition 1 (Ranked Initial Firm Size Implies Rank-Preserving Equilibrium)**

If \( \Omega \) is finite and at the initial date 0 the initial state of the economy is such that \( L_0 \) is non-decreasing in \( \theta \) (i.e. higher-\( \theta \) firms start out no smaller), then any symmetric equilibrium of the dynamic value-posting game is necessarily Rank-Preserving, and the initial ranking of firms’ relative sizes is maintained on the equilibrium path. If \( \Gamma \) is degenerate and firms are equally productive, then the same conclusion holds and initially larger firms offer more and remain larger on any equilibrium path.

The proof is in Appendix A. Although that proof is technically quite involved, the proposition has a simple economic intuition. In BM’s steady-state model, more productive firms offer higher wages due to a single-crossing property of their steady state profits, which in turn reflects two very basic economic forces. First, a higher wage implies a larger firm size, as a more generous offer makes it easier to poach workers and to fend off competition. Second, a larger firm size is more valuable to a more productive firm, because each worker produces more. Therefore, by a simple monotone comparative statics argument, it must be the case that more productive firms offer more, employ more workers, and earn higher profits. Simply put, a productive firm can afford paying more, and is willing to do so to attract workers, because its opportunity cost of not producing is higher. Key to this argument is the fact that firm size is an endogenous object, and BM look for an appropriate firm size distribution which guarantees a stationary allocation.

In our dynamic model, firm size is a state variable, and its initial value is a parameter of the model, arbitrarily fixed, not an endogenous object. Therefore, in order to get a start on monotone comparative statics, it is sufficient (but not necessary) that the initial size distribution shares the key property of BM’s steady state distribution; namely, it is increasing in productivity. In the proof, we essentially invoke a single-crossing property of the maximand in the Bellman equation of the modified but equivalent value-posting problem (10).\(^6\) A more productive firm still wants and can afford to pay more, now in terms of values accruing to workers. If initially (or once) larger, this firm has a further motive to offer more, namely more workers to retain, independently of its productivity. In contrast, the effect of a

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\(^6\)In a way similar to that in which Caputo (2003) appeals to single-crossing properties of the Hamiltonian in his analysis of comparative dynamics for infinite-horizon optimal control problems.
higher offer on successful poaching from other firms is independent of current size, because of CRS in production. Therefore, the initial ranking of sizes by productivity is preserved throughout, and values offered to workers remain ranked by firm productivity at all points in the future. This condition is only sufficient. We conjecture that it is not necessary. It aligns two separate motives to pay workers more, firm productivity and size, so clearly there is some slack. If firms are equally productive and only differ in their initial size, then only the size motive operates and all equilibria are RP, with no additional conditions.

We stress that this is a characterization result, which neither establishes nor requires existence, let alone uniqueness, of a RPE. Our main result says that, if a Markov Perfect Equilibrium $V$ exists, then $V$ can be a best-response to itself if and only if it is increasing in $\theta$, including the effect of endogenous size on the posted value. So ours is a general monotonicity result, which does not require to either propose or calculate a particular value-offer strategy. In the next section, we show by construction existence and uniqueness of a RPE, which must then be the unique Markov Perfect Equilibrium of the contract-posting game.

4.4 Evolution of the firm size distribution in RPE

In a given aggregate state $\omega$, firm sizes evolve following (4). In a RPE, in which firm size equals

$$L^\ast_t (\theta) = \Lambda^t (\theta) / \gamma (\theta),$$

(4) reads as:

$$L^\ast_{t+1} (\theta) = L^\ast_t (\theta) (1 - \delta^\omega) \left( 1 - \lambda^\omega_t \Gamma (\theta) \right) + \lambda^\omega_t u_t + \lambda^\omega_t (1 - \delta^\omega) \Lambda_t (\theta)$$

$$= L^\ast_t (\theta) (1 - \delta^\omega) \left( 1 - \lambda^\omega_t \Gamma (\theta) \right) + \lambda^\omega_t u_t + \lambda^\omega_t (1 - \delta^\omega) \int_{\theta}^{\theta} L^\ast_t (x) d\Gamma (x),$$

(13)

where a discrete time index $t$ was introduced for convenience, together with the notation $u_t := 1 - \Lambda_t (\theta)$ to designate the economy’s unemployment rate. Equation (13) combines an ordinary differential equation and a first-order difference equation in $\Lambda$, a function of time and $\theta$. We can solve it forward from any initial condition for any realization of the history of $\omega$, independently of wages and offered values. Indeed multiplying through by $\gamma (\theta)$ in (13) and integrating with respect to $\theta$ yields:

$$\Lambda_{t+1} (\theta) = \lambda^\omega_0 u_t \Gamma (\theta) + (1 - \delta^\omega) \left( 1 - \lambda^\omega_t \Gamma (\theta) \right) \Lambda_t (\theta).$$

For any initial condition $\Lambda_0 (\theta)$ at some (renormalized) initial date 0 such that the aggregate state remained at $\omega$ between 0 and $t$, the last law of motion is a first-order difference equation which solves as:

$$\Lambda_t (\theta) = \left[ (1 - \delta^\omega) \left( 1 - \lambda^\omega_t \Gamma (\theta) \right) \right]^t \Lambda_0 (\theta) + \lambda_0 \Gamma (\theta) \sum_{s=1}^{t} \left[ (1 - \delta^\omega) \left( 1 - \lambda^\omega_s \Gamma (\theta) \right) \right]^{t-s-1} u_{t-s},$$

(14)
where the unemployment rate $u_s$ solves the simple first-order, constant-coefficient difference equation $u_{s+1} = \delta^s (1 - u_s) + (1 - \lambda^s_0) u_s$ over $[0, t]$ with initial condition $u_0 = 1 - \Lambda_0 (\tilde{\theta})$. Next differentiating with respect to $\theta$, one obtains a closed-form expression for the workforce of any type-$\theta$ firm:

$$L^*_t (\theta) = \frac{\Lambda^*_t (\theta)}{\gamma (\theta)} = (1 - \delta^s)^t (1 - \lambda_t^s \Gamma (\theta))^{t-1} \left[ (1 - \lambda_t^s \Gamma (\theta)) L^*_0 (\theta) + t \lambda_t^s \Lambda_0 (\theta) \right] + \lambda_0^s \left\{ u_{t-1} + \sum_{s=2}^{t} (1 - \delta^s)^{s-1} (1 - \lambda_t^s \Gamma (\theta))^{s-2} [1 - \lambda_t^s + \lambda_t^s s \Gamma (\theta)] u_{t-s} \right\}, \quad (15)$$

where $L^*_0 (\theta)$ was the value of this solution under state $\omega' \time$ of the last state switch from $\omega$. The steady-state versions of (14) and (15) (assuming the aggregate state forever stays at $\omega$) are:

$$L^*_\infty (\theta) = \frac{\delta^s \lambda_0^s}{\delta^s + \lambda_0^s} \cdot \frac{1 - (1 - \delta^s) (1 - \lambda_t^s)}{[1 - (1 - \delta^s) (1 - \lambda_t^s \Gamma (\theta))]}^2 \quad \text{and}$$

$$\Lambda^*_\infty (\theta) = \frac{\delta^s \lambda_0^s}{\delta^s + \lambda_0^s} \cdot \frac{\Gamma (\theta)}{1 - (1 - \delta^s) (1 - \lambda_t^s \Gamma (\theta))},$$

which are the familiar steady-state expressions found in the BM model.\(^7\)

Before going any further into characterizing Rank-Preserving Equilibria, we should notice that the analysis of firm size and employment dynamics carried out in this paragraph would apply to any job ladder model in which a similar concept of RPE can be defined. Indeed nothing in the dynamics of $L^*_t$ or $\Lambda_t$ depends on the particulars of the wage setting mechanism, so long as this is such that employed jobseekers move from lower-ranking into higher-ranking jobs in the sense of a time-invariant ranking. Therefore, this model’s predictions about everything relating to firm sizes are in fact much more general than the wage-(or value-) posting assumption retained in the BM model.

### 4.5 A RPE with differentiable contracts

We now look for a solution to the firm’s problem (10) such that the value function $S$ is differentiable w.r.t the state variable $L$ and the policy function is differentiable w.r.t. $\theta$.\(^7\)

\(^7\)Incidentally, this is the point at which the necessity for sampling weights appears. Note from equation (16) that the steady-state size ratio of the largest to the smallest firm in the market in units of (non-normalized) employment is $L^*_\infty (\tilde{\theta}) / L^*_\infty (\tilde{\theta}) = (1 - \lambda_1 + \lambda_1 / \delta)^2$, a ratio which is in the order of 25-30 given standard estimates of $\lambda_1$ and $\delta$. Now of course the data counterpart of that size ratio is virtually infinite. It appears that the BM model requires a sampling distribution that is very heavily skewed toward high-productivity firms in order to replicate the observed distribution of firm sizes.
along the equilibrium path.\(^8\) Those differentiability properties allow the use of first-order conditions, which, for each state \(\omega\), write down as (using the definition of \(L(\cdot)\) again and using subscripts to denote partial derivatives):

\[
\lambda_0 \omega' (1 - \Lambda (\bar{\theta})) + \lambda_1 \omega' (1 - \delta^\omega) \Lambda (\bar{\theta}) G (W (\omega') | \omega', \Lambda) \\
= [S_L (\theta, L (\omega', \Lambda, W (\omega'))), \omega', \tau (\omega', \Lambda)) - W (\omega')] \\
\times (1 - \delta^\omega) \lambda_1 \omega' [L f (W (\omega') | \omega', \Lambda) + \Lambda (\bar{\theta}) g (W (\omega') | \omega', \Lambda)] - m^\omega'
\]

(17)

with complementary slackness \(m^\omega' [W (\omega') - U (\omega', \tau (\omega', \Lambda))] = 0\). In a RPE, (17) is solved by \(W = V (\theta, L^* (\theta), \omega, \Lambda)\) which also satisfies, among other restrictions, (11) and (12). Differentiating those two restrictions w.r.t. \(\theta\) yields:

\[
f (V | \omega', \Lambda) \cdot \frac{dV}{d\theta} = \gamma (\theta) \quad \text{and} \quad g (V | \omega', \Lambda) \cdot \frac{dV}{d\theta} = L^* (\theta) \gamma (\theta).
\]

We now introduce a time index \(t\) again. With a slight notational abuse, we denote:

\[
V_{t+1} (\theta | \omega) := V (\theta, L^*_t (\theta), \omega, \Lambda_t)
\]

and further define the costate variable:

\[
\mu_{t+1} (\theta | \omega) := S_L (\theta, L (L^*_t (\theta), \omega, \Lambda_t, V_{t+1} (\theta | \omega)), \omega, \tau (\omega, \Lambda_t)),
\]

which measures the shadow value to the worker-firm collective of the marginal worker, given the aggregate state, along the equilibrium path. Note that the dependence of \(V\) and \(\mu\) on the aggregate (uncontrolled) state variable \(\Lambda\) is subsumed into the time index in the above notation, which is licit as \(\Lambda\) evolves deterministically conditional on \(\omega\). Combining (17) and the various restrictions (11), (12), and (18) that hold in a RPE, we obtain the RPE version of the FOC:

\[
\lambda_0 u + \lambda_1 T (\theta) = \lambda_0 (\mu_{t+1} - V_{t+1}) [L^*_t (\theta) f (V_{t+1} | \omega, \Lambda_t) + (1 - u_t) g (V_{t+1} | \omega, \Lambda_t)] - m^\omega_t \\
= 2 \lambda_1 \omega \frac{L^*_t (\theta) \gamma (\theta)}{dV_{t+1} / d\theta} (\mu_{t+1} - V_{t+1}) - m^\omega_t.
\]

---

\(^8\)Lemma 1 in the Appendix ensures that \(S\) is convex in \(L\), so a.e. differentiable and with left and right derivatives at all points. Differentiability of \(S\) is a problem only at points where \(V\) is not continuous w.r.t. \(L\). Now Lemma 1 also establishes that \(V\) is strictly increasing in \(\theta\) and \(L\), so it is a.e. differentiable in \(\theta\) and \(L\) and can have at most countably many discontinuities, all of the jump type, which must then be jumps up. Thus \(V\) has everywhere a right and left derivative in \(\theta\) and \(L\). Finally, from equation (15), \(L\) is differentiable in \(\theta\) if and only if \(L_0\) is. So at this stage we cannot rule out existence of equilibria that are RP, but such that \(V\) has countably many jump discontinuities in \(L\), implying non-differentiability of \(S\) at those points.
Next, going back to the firm’s problem (10), the Envelope condition w.r.t. firm size writes down as:

\[
S_L (\theta, L, \omega, \Lambda) = \omega \theta + \beta \int_\Omega \left\{ \delta \omega' U (\omega', T (\omega', \Lambda)) + \left( 1 - \delta \omega' \right) \lambda_1 \int_{W(\omega')}^{+\infty} v dF (v | \omega', \Lambda) \right\} H (d\omega' | \omega) + S_L (\theta, L, \omega', \Lambda) \left( 1 - \delta \omega' \right) \left( 1 - \lambda_1 \bar{T} (W (\omega') | \omega', \Lambda) \right) H (d\omega' | \omega).
\]  

(18)

In a RPE, this becomes:

\[
\mu_t (\theta | \omega) = \omega \theta + \beta \int_\Omega \left\{ \delta \omega' U (\omega', T (\omega', \Lambda)) + \left( 1 - \delta \omega' \right) \lambda_1 \int_{\theta}^{+\infty} V_{t+1} (x | \omega') d\Gamma (x) \right\} H (d\omega' | \omega) + \mu_{t+1} (\theta | \omega') \left( 1 - \delta \omega' \right) \left( 1 - \lambda_1 \bar{T} (\theta) \right) H (d\omega' | \omega),
\]  

(19)

Note that now the shadow marginal value \( \mu \) only depends on the distribution of employment \( \Lambda \) through the distribution of employment across firms up to \( \theta, \Lambda (\theta) \) and the corresponding density \( L^* (\theta) \gamma (\theta) \). Both are scalars, and the state reduces from \( \zeta = (\theta, L, \omega', \Lambda) \), which is infinite-dimensional due to the relevance of the entire firm size distribution \( \Lambda \), to the four-dimensional vector \( z = (\theta, L, \omega', \Lambda (\theta)) \): in order to make its decisions, the firm only needs to know the mass of employment at less productive firms and not the entire size distribution \( \Lambda \).

Finally, a Transversality Condition (TVC) requires that the discounted value of the collective vanishes in expectation w.r. to the stochastic path of \( \omega \)

\[
\lim_{t \to \infty} E \left[ \beta^t \mu_t (\theta | \omega) L^*_t (\theta) | \omega_0 \right] = 0.
\]  

(20)

5 Practical implementation of RPE

5.1 A strategy to solve for stochastic RPE

We now show how to practically “solve” for the RPE, by which we mean simulate the dynamic paths of the distributions of employment and wages across firms, given an initial state and a subsequent realization of a sequence of aggregate shocks. For the sake of illustration, we focus on the case where the aggregate state can take on two values, \( \Omega = \{ \omega, \omega' \} \), with conditional switching probabilities \( \sigma^\omega \) and \( \sigma^{\omega'} \). Generalization to any first-order Markov process over a finite set is conceptually trivial.
The Euler equation — or Envelope condition — (19) now becomes:

\[
\begin{align*}
\mu_t (\theta | \omega) &= \omega \theta + \beta \sigma^\omega \left\{ \delta^\omega U (\omega', T (\omega', \Lambda)) + \left( 1 - \delta^\omega \right) \lambda_1^\omega \int_\theta^{+\infty} V_{t+1} (x | \omega') \, d\Gamma (x) \\
&\quad + \mu_{t+1} (\theta | \omega') \left( 1 - \delta^\omega \right) \left( 1 - \lambda_1^\omega \Gamma (\theta) \right) \right\} \\
&\quad + \beta (1 - \sigma^\omega) \left\{ \delta^\omega U (\omega, T (\omega, \Lambda)) + \left( 1 - \delta^\omega \right) \lambda_1^\omega \int_\theta^{+\infty} V_{t+1} (x | \omega) \, d\Gamma (x) \\
&\quad + \mu_{t+1} (\theta | \omega) (1 - \delta^\omega) \left( 1 - \lambda_1^\omega \Gamma (\theta) \right) \right\}.
\end{align*}
\]

Taking derivatives w.r. to \( \theta \) on both sides:

\[
\begin{align*}
\frac{\partial \mu_t}{\partial \theta} (\theta | \omega) &= \omega \\
&\quad + \beta \sigma^\omega \left( 1 - \delta^\omega \right) \left\{ \lambda_1^\omega \gamma (\theta) \pi_{t+1} (\theta | \omega') + \frac{\partial \mu_{t+1}}{\partial \theta} (\theta | \omega') \left( 1 - \lambda_1^\omega \Gamma (\theta) \right) \right\} \\
&\quad + \beta (1 - \sigma^\omega) (1 - \delta^\omega) \left\{ \lambda_1^\omega \gamma (\theta) \pi_{t+1} (\theta | \omega) + \frac{\partial \mu_{t+1}}{\partial \theta} (\theta | \omega) \left( 1 - \lambda_1^\omega \Gamma (\theta) \right) \right\},
\end{align*}
\]

where \( \pi_t (\theta | \omega) := \mu_t (\theta | \omega) - V_t (\theta | \omega) \) denotes the shadow value to the firm of the marginal worker. Together with the FOC for an interior solution of the promised value:

\[
\frac{\partial \pi_t}{\partial \theta} (\theta | \omega) = \frac{\partial \mu_t}{\partial \theta} (\theta | \omega) - \frac{2 \lambda_1^\omega \Lambda' (\theta)}{\lambda_0^\omega u_t + \lambda_1^\omega \Lambda (\theta)} \pi_t (\theta | \omega)
\]

this gives a system of four PDEs in \( \pi_t (\theta | \omega), \partial \mu_t (\theta | \omega) / \partial \theta \), all functions of \( \theta \) and \( t \), a pair for each value of \( \omega \).

The main difficulty in solving this system lies in the interdependence of \( \partial \mu_t (\theta | \omega) / \partial \theta \) and \( \partial \mu_t (\theta | \omega') / \partial \theta \), that is, on the jump in the shadow marginal value of one worker, differentiated across firms, when the aggregate state switches. To get around this problem, we can approximate that “jump term” by a known function \( J \) (e.g. polynomials) of the state variables \( \Lambda (\theta) \) and \( L^* (\theta) \), depending on a finite vector of unknown coefficients, \( a \) in the following fashion:

\[
\left[ \frac{\partial \mu_t}{\partial \theta} (\theta | \omega') - \frac{\partial \mu_t}{\partial \theta} (\theta | \omega) \right] \approx J (\Lambda (\theta), L^* (\theta) | a).
\]

Given a specific vector of coefficients \( a \), system (21,22), together with the transversality condition (20), becomes a pair of independent systems of PDEs, one for each aggregate state, which can be separately numerically solved over the infinite future for any initial value of \( (\Lambda (\cdot), L^* (\cdot)) \) using the algorithm described in MPV08.

We thus proceed in the following steps:
0. Pick an initial state of the economy \((\omega_0, \Lambda_0(\cdot), L_0^*(\cdot))\) and simulate a path of \(\omega\). Denote switching dates as \((s_1, s_2, \ldots)\).

1. Fix a parameter \(a\).

2. Given the choice of \(a\) made at step 1 and the implied \(J\)-function, solve (21,22,20) using the appropriate initial conditions. More specifically:

   (a) Solve (21,22,20) with initial condition \((\Lambda_0(\cdot), L_0^*(\cdot))\) as if state \(\omega_0\) prevailed forever. This implies certain values for \(\partial \mu_0 (\theta, \omega_0) / \partial \theta\), \(\partial \mu_{s_1} (\theta, \omega_0) / \partial \theta\) and \((\Lambda_{s_1} (\cdot), L_{s_1}^*(\cdot))\) at date \(s_1\) when the next aggregate shock occurs.

   (b) Solve (21,22,20) with initial condition \((\Lambda_{s_1}(\cdot), L_{s_1}^*(\cdot))\) as if state \(\omega_1\) prevailed over \(t \in [s_1, +\infty)\). This implies certain values for \(\partial \mu_{s_1} (\theta, \omega_1) / \partial \theta\), \(\partial \mu_{s_2} (\theta, \omega_1) / \partial \theta\) and \((\Lambda_{s_2} (\cdot), L_{s_2}^*(\cdot))\).

   (c) Solve (21,22,20) with initial condition \((\Lambda_{s_2}(\cdot), L_{s_2}^*(\cdot))\) as if state \(\omega_2\) prevailed over \(t \in [s_2, +\infty)\), etc. That is, repeat step 2, mutatis mutandis, for the first \(K\) jumps in \(\omega\) (how many jumps depends on the dimension of the vector \(a\)).

3. The simulations performed at stage 3 provide a vector of jumps in \(\partial \mu / \partial \theta\):

   \[
   \left[ \frac{\partial \mu_{s_k} (\theta, \omega')}{\partial \theta} - \frac{\partial \mu_{s_k} (\theta, \omega)}{\partial \theta} \right], \quad k = 1, \ldots, K.
   \]

   Compare those with the jumps predicted from the initially chosen function \(J (\cdot | a)\) and the simulated path of \((\Lambda (\cdot), L^*(\cdot))\). If different, update \(a\) and start over at step 1.\(^9\)

The proposed algorithm is akin to a projection method to solve the system of PDEs that characterize equilibrium. Its specific feature is that projection is only used to approximate the jumps in \(\partial \mu / \partial \theta\) caused by aggregate shocks, the rest of the system being solved “exactly”.

Once the algorithm has converged, it produces a solution for \(\{\Lambda, L^*, \partial \mu_{t+1} / \partial \theta, \pi_t\}\) over \(t \in [0, +\infty)\) given the initially simulated sequence of aggregate states. Wages are finally retrieved from:

\[
\begin{align*}
   w_t (\theta | \omega) &= \omega \theta - \pi_t (\theta | \omega) + \beta \sigma^\omega \left( 1 - \delta^\omega \right) \left( 1 - \lambda_t^\omega \Gamma (\theta) \right) \pi_{t+1} (\theta | \omega') \\
   &\quad + \beta (1 - \sigma^\omega) (1 - \delta^\omega) (1 - \lambda_t^\omega \Gamma (\theta)) \pi_{t+1} (\theta | \omega)
\end{align*}
\]

\(^9\)Exactly how \(a\) is updated depends on the chosen functional form for the approximate jump function \(J\). In practice we use a projection on a base of polynomials, and the updated vector of coefficients \(a\) is obtained by regression of the “simulated jumps” \(\partial \mu_{s_k} (\theta, \omega') / \partial \theta - \partial \mu_{s_k} (\theta, \omega) / \partial \theta\) on the monomial elements of \(J\).
5.2 Simulations: preliminary results [in progress]

We calibrate parameters as in our MPV08 companion paper, and the aggregate productivity process to reflect the average duration of booms and recessions in the US. Figures 1-5 illustrate the results of some representative simulations. The results do not differ qualitatively from those of the transitional dynamics illustrated in MPV08 and MPV09. The model predicts procyclical wages and EE rate, although the wage jumps in a direction opposite to labor productivity when aggregate shocks hit. As documented in MPV08, the growth rate differential of employment at (initially) large minus small firms collapses upon a recession and rises slowly through an expansion, reflecting the slow upgrading of workers to better jobs through on the job search. This upgrading also strongly smoothes and propagates the effects of the aggregate labor productivity shock on measured average labor productivity. Reallocation across firms is quantitatively important.

6 Conclusion [in progress]
References


Appendix

A Proof of Proposition 1

For convenience, we repeat the firm’s DP problem (10):

\[
S(\theta, L, \omega, \Lambda) = \omega \theta L + \beta \int_{\Omega} \delta \omega' U(\omega', T(\omega', \Lambda)) L \\
+ \sup_{W(\omega') \geq U(\omega', T(\omega', \Lambda))} \left\{ S(\theta, L(\omega', \Lambda, W(\omega')) , \omega', T(\omega', \Lambda)) + L \left( 1 - \delta \omega' \right) \lambda^{\omega'}_{1} \int_{W(\omega')}^{+\infty} v dF(v | \omega', \Lambda) \\
- W(\omega') \left( \lambda^{\omega'}_{0} \left( 1 - \Lambda(\overline{\theta}) \right) + \lambda^{\omega'}_{1} \left( 1 - \delta \omega' \right) \Lambda(\overline{\theta}) G(W(\omega') | \omega', \Lambda) \right) \right\} H(d\omega' | \omega),
\]

and the claim: if this problem has a solution \( S \), then \( S(\theta, L^*(\theta), \omega, \Lambda) \) is increasing in \( \theta \). We introduce the following notation:

\[
A(\theta, L, \omega, \Lambda) := \omega \theta L + \beta \int_{\Omega} \delta \omega' U(\omega', T(\omega', \Lambda)) L H(d\omega' | \omega),
\]

\[
B(L, \omega', \Lambda; W(\omega')) := L \left( 1 - \delta \omega' \right) \lambda^{\omega'}_{1} \int_{W(\omega')}^{+\infty} v dF(v | \omega', \Lambda) \\
- W(\omega') \left( \lambda^{\omega'}_{0} \left( 1 - \Lambda(\overline{\theta}) \right) + \lambda^{\omega'}_{1} \left( 1 - \delta \omega' \right) \Lambda(\overline{\theta}) G(W(\omega') | \omega', \Lambda) \right).
\]

Our proof strategy is as follows. We define certain supermodularity properties SM of a value function that imply that the maximizer \( V \) in (10) is increasing in \( \theta \). Then, we fix an arbitrary \( \Lambda \) and show that the Bellman operator in (10) for the restricted problem with fixed \( \Lambda \) is a contraction mapping from the space of SM functions into itself, and that this space is Banach and closed under the sup norm. Therefore, for any fixed \( \Lambda \) (10) has a unique solution. Finally, if there exists a solution \( S \) to (10) when \( \Lambda \) is not fixed, then \( S \) must also solve the restricted problem (10) for any fixed \( \Lambda \). By uniqueness and SM of the solution to the restricted problem any solution to the unrestricted problem must also have the SM properties.

So fix \( \Lambda \) to be some given CDF over \([\theta, \overline{\theta}]\). Then, for any function \( F(\theta, L, \omega) \), we define the following operator \( M^{\Lambda} \):

\[
M^{\Lambda} F(\theta, L, \omega) := A(\theta, L, \omega, \Lambda) \\
+ \beta \int_{\Omega} \max_{W(\omega')} \left\{ F(\theta, L(\omega', \Lambda, W(\omega')) , \omega') + B(L, \omega', \Lambda; W(\omega')) \right\} H(d\omega' | \omega).
\]

Two considerations simplify the proof: \( F \) and \( G \) must be atomless in equilibrium, and in our proof we can ignore the worker participation constraint \( W \geq U \). To see why there cannot be an atom in the offer distribution, except an atom at \( W = U \) of firms that shut down this period, observe the following. By the equal payment constraint, if \( G \) had an atom at some value \( W \), then
so would $F$. But an atom in $F$ would open the way to a profitable deviation, as in BM. A firm that is part of the atom that offers the same $W$ in some state could deviate, offer an epsilon more, win the competition for employed workers against all other competitors offering $W$, and poach all of their workers whenever they match, a positive measure, at a negligible cost. To see why we can only focus on firms, if any, that do not choose a corner solution $W = U$, observe the following. Once we establish that an interior solution is increasing in $\theta$, we can conclude that any set of firms that offers a corner solution $W = U$ and shuts down must be the set of the least productive firms. But then, the global solution, including the corner, is weakly increasing in $\theta$ as claimed. Incidentally, if all firms offered $U$, from the previous reasoning (and barring the trivial case where all firms are too unproductive to operate) the most productive firms would deviate and profitably offer more, so there exist always some firms that have an interior solution where PC does not bind.

Lemma 1 Let $F(\theta, L, \omega)$ be bounded, continuous in $\theta$ and $L$, convex in $L$ and with increasing differences in $(\theta, L)$ over $(\theta, \bar{\theta}) \times (0, 1)$. Then:

1. $M^A F$ is bounded and continuous in $\theta$ and $L$;
2. There exists a measurable selection $V(\theta, L, \omega, \Lambda)$ from the maximizing correspondence associated with $M^A F$;
3. Any such measurable selection $V$ is increasing in $\theta$ and $L$;
4. $M^A F$ is convex in $L$ and with increasing differences in $(\theta, L)$ over $(\theta, \bar{\theta}) \times (0, 1)$.

Proof. In this proof, wherever possible without causing confusion, we will make the dependence of all functions on aggregate state variables $\omega$ and $\Lambda$ implicit to streamline the notation a little bit.

First, it is easy to compactify the choice set of the firm. $\bar{W} = \max \omega \theta / r$ is a natural upper bound to the offered value: the firm can always do weakly better by offering less than $\bar{W}$, as it can hope to make some profits. So $W \in \left[ U(\omega', T(\omega', \Lambda)), \bar{W} \right]$.

Points 1 and 2 of this lemma are immediate: continuity of $M^A F$ is a direct consequence of Berge’s Theorem. Boundedness of $M^A F$ is obvious by construction. Existence of a measurable selection from the maximizing correspondence associated with $M^A F$ is a direct consequence of the Measurable Selection Theorem.

To prove point 3, we first establish that the maximand in (23) has increasing differences in $(\theta, W)$ and $(L, W)$. Monotonicity of $V$ in $\theta$ and $L$ will then follow from standard monotone comparative statics arguments. First note that, since $F$ is assumed to be continuous and convex in $L$, it has left and right derivatives everywhere (and those two can at most differ at countably many points). Furthermore, as $F$ and $G$
are atomless but continuously strictly increasing, both $B$ and $L$ (the law of motion of employment) are left- and right-differentiable everywhere w.r.t. $W$. Finally, $L$ is continuous and strictly increasing w.r.t. $W$. Thus the maximand in (23) has a left derivative w.r.t. $W$ everywhere, which is given by:\footnote{This uses $B_{W,\ell}(L; W) = -W \cdot L_{W,\ell}(L; W) - \left[ \lambda^\circ_0 (1 - \Lambda(\overline{\theta})) + \lambda^\circ_1 \left(1 - \delta^\circ\right) \Lambda(\overline{\theta}) G(W \mid \omega', \Lambda) \right]$.}

$$\partial_{W,\ell} [\mathcal{F}(\theta, L(L, W)) + B(L; W)] = [\mathcal{F}_{L,\ell}(\theta, L(L, W)) - W] \cdot \mathcal{L}_{W,\ell}(L, W)$$

$$- \left[ \lambda^\circ_0 (1 - \Lambda(\overline{\theta})) + \lambda^\circ_1 \left(1 - \delta^\circ\right) \Lambda(\overline{\theta}) G(W \mid \omega', \Lambda) \right],$$

(24)

where $f_{x,\ell}$ is used to designate the left [right] partial derivative of $f$ w.r.t. $x$. Note that the above left derivative is necessarily non-negative when evaluated at any selection $V$ from the optimal policy correspondence for (23) (again, we can focus only on interior solutions), otherwise the firm would do strictly better by offering a slightly lower value. Hence:

$$[\mathcal{F}_{L,\ell}(\theta, L(L, V)) - V] \cdot \mathcal{L}_{W,\ell}(L, V)$$

$$\geq \lambda^\circ_0 (1 - \Lambda(\overline{\theta})) + \lambda^\circ_1 \left(1 - \delta^\circ\right) \Lambda(\overline{\theta}) G(V \mid \omega', \Lambda) > 0.$$  \hspace{1cm} (25)

Our next task is to establish that the r.h.s. of (24) is increasing in $L$ and $\theta$. Because the second term in that r.h.s. is independent of those two variables, this amounts to showing that the function $(\theta, L) \mapsto [\mathcal{F}_{L,\ell}(\theta, L(L, W)) - W] \cdot \mathcal{L}_{W,\ell}(L, W)$ is increasing in $L$ and $\theta$ for all $W$. This is trivially true for $\theta$ from the assumption that $\mathcal{F}$ has increasing differences in $(\theta, L)$. For $L$, using (25), it suffices to remark that both $L$ and $\mathcal{L}_W$ are strictly increasing in $L$, and $\mathcal{F}_{L,\ell}$ is nondecreasing in $L$ by convexity of $\mathcal{F}$. Note that the latter train of arguments in fact establishes that the r.h.s. of (24) is strictly increasing in $L$. The maximand in (23) is therefore a continuous function of $\theta$, $L$ and $W$ whose left derivative w.r.t. $W$ exists everywhere and is increasing in $\theta$ and $L$ (strictly so for $L$), which implies that is has increasing differences in $(\theta, W)$ and $(L, W)$,\footnote{Consider a continuous function $f(x, y)$ such that $f_{x,\ell}(x, y)$ exists everywhere and is increasing in $y$. Fix $y_2 > y_1$. Define $g(x) = f(x, y_2) - f(x, y_1)$. $g$ is continuous and everywhere left-differentiable with $g'(x) = f_x(x, y_2) - f_x(x, y_1) > 0$ (as $f_x(x, y)$ is increasing in $y$), so $g$ is an increasing function which proves that $f$ has increasing differences.} and proves point 1 of the lemma.

Now on to point 4. Take $(\theta_0, L_0) \in (\theta, \overline{\theta}) \times (0, 1)$ and $h > 0$ such that $(\theta_0 + h, L_0 + h)$ are still in $(\theta, \overline{\theta}) \times (0, 1)$. We first consider right-differentiability of $\mathbf{M}^\Lambda \mathcal{F}$ w.r.t. $L$ at $L_0$. Again fixing an arbitrary selection $V$ from the optimal policy correspondence, we note that, while $V$ may have a discontinuity at $(\theta_0, L_0)$, the fact that it is increasing in $L$ ensures that $V(\theta_0, L_0^+, \omega') := \lim_{h \to 0^+} V(\theta_0, L_0 + h, \omega')$ exists everywhere (and
likewise for \( V (\theta_0^+, L_0, \omega') \). By point 3, \( V (\theta_0, L_0^+, \omega') \) is increasing in \( L_0 \). Then:

\[
\begin{align*}
\mathbf{M}^A \mathbf{F} (\theta_0, L_0 + h) - \mathbf{M}^A \mathbf{F} (\theta_0, L_0^+) & = A (\theta_0, L_0 + h) - A (\theta_0, L_0) \\
& + \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0 + h, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0^+, \omega'))] \\
& + B (L_0 + h; V (\theta_0, L_0 + h, \omega')) - B (L_0; V (\theta_0, L_0^+, \omega')) \right) H (d\omega' | \omega)
\end{align*}
\]

\[
\geq A (\theta_0, L_0 + h) - A (\theta_0, L_0) \\
+ \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0^+, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0^+, \omega'))] \\
+ B (L_0 + h; V (\theta_0, L_0^+, \omega')) - B (L_0; V (\theta_0, L_0^+, \omega')) \right) H (d\omega' | \omega)
\]

\[
= \left( \omega \theta_0 + \beta \int_{\Omega} \delta \omega' U (\omega') H (d\omega' | \omega) \right) \cdot h \\
+ \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0^+, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0^+, \omega'))] \\
+ h \cdot \left( 1 - \delta \omega' \right) \lambda_1^F \int_{V(\theta_0,L_0^+,\omega')}^{+\infty} v dF (v | \omega') \right) H (d\omega' | \omega) ,
\]

(26)

where the last equality follows from the definitions of \( A \) and \( B \). Then again:

\[
\begin{align*}
\mathbf{M}^A \mathbf{F} (\theta_0, L_0 + h) - \mathbf{M}^A \mathbf{F} (\theta_0, L_0^+) & = A (\theta_0, L_0 + h) - A (\theta_0, L_0) \\
& + \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0 + h, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0^+, \omega'))] \\
& + B (L_0 + h; V (\theta_0, L_0 + h, \omega')) - B (L_0; V (\theta_0, L_0^+, \omega')) \right) H (d\omega' | \omega)
\end{align*}
\]

\[
\leq A (\theta_0, L_0 + h) - A (\theta_0, L_0) \\
+ \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0 + h, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0^+, \omega'))] \\
+ B (L_0 + h; V (\theta_0, L_0 + h, \omega')) - B (L_0; V (\theta_0, L_0^+, \omega')) \right) H (d\omega' | \omega)
\]

\[
= \left( \omega \theta_0 + \beta \int_{\Omega} \delta \omega' U (\omega') H (d\omega' | \omega) \right) \cdot h \\
+ \beta \int_{\Omega} \left( \mathbf{F} [\theta_0, \mathcal{L} (L_0 + h, V (\theta_0, L_0 + h, \omega'))] - \mathbf{F} [\theta_0, \mathcal{L} (L_0, V (\theta_0, L_0 + h, \omega'))] \\
+ h \cdot \left( 1 - \delta \omega' \right) \lambda_1^F \int_{V(\theta_0,L_0^+,\omega')}^{+\infty} v dF (v | \omega') \right) H (d\omega' | \omega) ,
\]

(27)

Now dividing through by \( h \) in (26) and (27), and invoking continuity w.r.t. \( V \) of \( \mathcal{L}_L (L, V) = \left( 1 - \delta \omega' \right) \left( 1 - \lambda_1^F T (V) \right) \) (by continuity of \( F \)), everywhere right-differentiability of \( \mathbf{F} \) w.r.t. \( L \) (by convexity of \( F \)), and existence of a right limit of \( V \) at any \( L_0 \) (by monotonicity of \( V \) established in point 1 of this lemma), we see that the lower
and upper bounds of \( \frac{1}{h} \left[ \mathbf{M}^\Lambda \mathbf{F} (\theta_0, L_0 + h) - \mathbf{M}^\Lambda \mathbf{F} (\theta_0, L_0^+) \right] \) exhibited in (26) and (27) both converge to the same limit as \( h \to 0^+ \), which establishes right-differentiability of \( \mathbf{M}^\Lambda \mathbf{F} \) w.r.t \( L \) with the following expression for \( \left[ \mathbf{M}^\Lambda \mathbf{F} \right]_{L,r} (\theta,L) \)

\[
\left[ \mathbf{M}^\Lambda \mathbf{F} \right]_{L,r} (\theta,L) = \omega \theta + \beta \int_{\Omega} \delta \omega' U(\omega') H(\delta \omega' | \omega) + \beta \int_{\Omega} \left( \mathcal{F}_{L,r} [\theta, \mathcal{L} (L, V(\theta, L^+, \omega'))] : \mathcal{L}_L (L, V(\theta, L^+, \omega')) \right)
+ \left(1 - \delta \omega'\right) \lambda^\theta_1 \int_{V(\theta, L^+, \omega')}^{\infty} v dF(v \mid \omega') \right) H(\delta \omega' | \omega).
\]

We now show that \( \left[ \mathbf{M}^\Lambda \mathbf{F} \right]_{L,r} (\theta,L) \) is increasing in \( L \) and \( \theta \). It is sufficient to show that the term under the \( \int \) in (28) is increasing in \( L \) and \( \theta \) for all \( \omega' \in \Omega \). We begin with \( L \). Let \( L_1 < L_2 \in [0,1]^2 \). To lighten the notation, let \( V_k = V(\theta, L^+_k, \omega') \) for \( k = 1,2 \). Because \( V \) is increasing in \( L \), \( V_2 \geq V_1 \). Then:

\[
\mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] - \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_1, V_1)] = \mathcal{L}_L (L_2, V_2) - \mathcal{L}_L (L_1, V_1) \\
= \mathcal{L}_L (L_2, V_2 - L_1, V_1) \cdot \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] \\
+ \mathcal{L}_L (L_1, V_1) \cdot (\mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] - \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_1, V_1)]) \\
= \mathcal{L}_L (L_1, V_1) \cdot (\mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] - \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_1, V_1)])
\]

where the last equality stems from the definition of \( \mathcal{L}_L \). Because \( \mathcal{F}_{L,r} \) and \( \mathcal{L} \) are both increasing in \( L \), and because \( \mathcal{L} \) is also increasing in \( V \), the first term in the r.h.s. of the last equality above is positive. Finally, convexity of \( \mathcal{F} \) combined with the first-order condition (25) implies that \( \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] \geq \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] \geq V_2 \), so that \( \mathcal{F}_{L,r} [\theta, \mathcal{L} (L_2, V_2)] \geq v \) for all \( v \leq V_2 \), implying that the integral term is nonnegative.

This shows that \( \left[ \mathbf{M}^\Lambda \mathbf{F} \right]_{L,r} \) is (strictly) increasing in \( L \). The proof that \( \left[ \mathbf{M}^\Lambda \mathbf{F} \right]_{L,r} \) is strictly increasing in \( \theta \) proceeds along similar lines (details available upon request).

Thus \( \mathbf{M}^\Lambda \mathbf{F} \) is a continuous function whose right partial derivative w.r.t. \( L \) exists everywhere, is increasing in \( L \) — which proves convexity w.r.t. \( L \) —, and increasing in \( \theta \) — which proves increasing differences in \( (\theta,L) \).

Now consider the set of functions defined over \( [\underline{\theta}, \bar{\theta}] \times [0,1] \times \Omega \) that are continuous in \( (\theta,L) \) and call it \( C_{[\underline{\theta}, \bar{\theta}] \times [0,1] \times \Omega} \). That set is a Banach space when endowed with the sup norm. As Lemma 1
suggests we will be interested in the properties a subset $C'_\theta \times [0,1] \times \Omega \subset C_{\theta, \Omega}$ of functions that are convex in $L$ and have increasing differences in $(\theta, L)$. We next prove two ancillary lemmas, which will establish as a corollary (Corollary 1) that $C'_\theta \times [0,1] \times \Omega$ is closed in $C_{\theta, \Omega}$ under the sup norm.

**Lemma 2** Let $X$ be a convex set and $f_n : X \to \mathbb{R}$, $N \in \mathbb{N}$ be convex functions such that $\{f_n\}$ converges uniformly to $f$. Then $f$ is convex.

**Proof.** Let $(x_1, x_2) \in X^2$ and $\alpha \in [0, 1]$. Fix $k \in \mathbb{N}$. By uniform convergence, $\exists n_k \in \mathbb{N}$: $\forall n \geq n_k, \forall x \in X$, $|f_n(x) - f(x)| < \frac{1}{2k}$. Then:

$$f[\alpha x_1 + (1 - \alpha) x_2] - [\alpha f(x_1) + (1 - \alpha) f(x_2)]$$

$$= f[\alpha x_1 + (1 - \alpha) x_2] - f_{n_k}[\alpha x_1 + (1 - \alpha) x_2]$$

$$+ f_{n_k}[\alpha x_1 + (1 - \alpha) x_2] - [\alpha f_{n_k}(x_1) + (1 - \alpha) f_{n_k}(x_2)]$$

$$< 0 \text{ by convexity of } f_{n_k}$$

$$+ \alpha[f_{n_k}(x_1) - f(x_1)] + (1 - \alpha)[f_{n_k}(x_2) - f(x_2)].$$

This shows that $\forall k \in \mathbb{N}, \forall (x_1, x_2) \in X^2, \forall \alpha \in [0, 1]$: $f[\alpha x_1 + (1 - \alpha) x_2] - [\alpha f(x_1) + (1 - \alpha) f(x_2)] < \frac{1}{k}$, which proves convexity of $f$. \qed

**Lemma 3** Let $f_n : \mathbb{R}^2 \to \mathbb{R}$, $N \in \mathbb{N}$ be functions with increasing differences such that $\{f_n\}$ converges uniformly to $f$. Then $f$ has increasing differences.

**Proof.** Let $\{(x_1, y_1), (x_2, y_2)\} \in X^2$ such that $x_2 > x_1$ and $y_2 > y_1$. Fix $k \in \mathbb{N}$. By uniform convergence, $\exists n_k \in \mathbb{N}$: $\forall n \geq n_k, \forall (x,y) \in X$, $|f_n(x, y) - f(x, y)| < \frac{1}{4k}$. Then:

$$f(x_2, y_2) - f(x_1, y_2)$$

$$= f(x_2, y_2) - f_{n_k}(x_2, y_2) + f_{n_k}(x_2, y_2) - f_{n_k}(x_1, y_2) + f_{n_k}(x_1, y_2) - f(x_1, y_2)$$

$$> -\frac{1}{4k} + \{f_{n_k}(x_2, y_1) - f_{n_k}(x_1, y_1)\}$$

$$= -\frac{1}{2k} + f_{n_k}(x_2, y_1) - f(x_2, y_1) + f(x_2, y_1) - f(x_1, y_1) + f(x_1, y_1) - f_{n_k}(x_1, y_1)$$

$$> -\frac{1}{4k} + f(x_2, y_1) - f(x_1, y_1).$$

As the above is valid for an arbitrary choice of $k \in \mathbb{N}$ and $\{(x_1, y_1), (x_2, y_2)\} \in X^2$, it establishes that $f$ has increasing differences. \qed
Corollary 1 The set $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ of functions defined over $[\theta,\overline{\theta}] \times [0,1] \times \Omega$ that are convex in $L$ and have increasing differences in $(\theta, L)$ is a closed subset of $C_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ under the sup norm.

The latter corollary establishes that, given a fixed $\Lambda$, the set of functions that are relevant to Lemma 1 is a closed subset of a Banach space of functions under the sup norm. The following lemma shows that the operator considered in Lemma 1 is a contraction under that same norm.

Lemma 4 The operator $M^\Lambda$ defined in (23) maps $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ into itself and is a contraction of modulus $\beta$ under the sup norm.

Proof. That $M^\Lambda$ maps $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ into itself flows directly from a subset of the proof of Lemma 1. To prove that $M$ is a contraction, it is straightforward to check using (23) that $M^\Lambda$ satisfies Blackwell’s sufficient conditions with modulus $\beta$. □

We are now in a position to prove the proposition. Given the initially fixed $\Lambda$, the operator $M^\Lambda$, which by Lemma 4 is a contraction from $C_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ into itself, and has a unique fixed point $F_\Lambda$ in that set (by the Contraction Mapping Theorem). Moreover, since $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ is a closed subset of $C_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ (Lemma 2) and since $M^\Lambda$ also maps $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$ into itself (Lemma 1), that fixed point $F_\Lambda$ belongs to $C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$.

Summing up, what we have established thus far is that for any fixed $\Lambda \in C_{[\theta,\overline{\theta}]}$, the operator $M^\Lambda$ over functions of $(\theta, L, \omega)$ has a unique, bounded and continuous fixed point $F_\Lambda = M^\Lambda F_\Lambda \in C_{[\theta,\overline{\theta}]\times[0,1]\times\Omega} \subset C_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$.

We finally turn to the Bellman operator $M$ which is relevant to the firm’s problem. That operator $M$ applies to functions $G$ defined on $[\theta,\overline{\theta}] \times [0,1] \times \Omega \times C_{[\theta,\overline{\theta}]}$ and is defined as the following “extension” of $M^\Lambda$:

$$M^\Lambda (\theta, L, \omega, \Lambda) := A (\theta, L, \omega, \Lambda) + \beta \int_{\Omega} \max_{\omega'} \langle G [\theta, L (L, \omega', \Lambda, W (\omega')) \pm \omega', T (\omega, \Lambda)] + B (L, \omega', \Lambda; W (\omega')) \rangle H (d\omega' | \omega).$$

If an equilibrium exists, then a firm has a best response and a value $S$ which solves $S = MS$. For every $\Lambda \in C_{[\theta,\overline{\theta}]}$, by definition of $M$ and $M^\Lambda$ this implies $S = M^\Lambda S$. Since the fixed point of $M^\Lambda$ is unique, if $S = MS$ exists then for every fixed $\Lambda \in C_{[\theta,\overline{\theta}]}$ we have for all $(\theta, L, \omega) \in [\theta,\overline{\theta}] \times [0,1] \times \Omega$: $S (\theta, L, \omega, \Lambda) = F_\Lambda (\theta, L, \omega)$. Therefore, if the value function $S$ and an equilibrium of the contract-posting game exist, then $S \in C'_{[\theta,\overline{\theta}]\times[0,1]\times\Omega}$: the typical firm’s value function is continuous in $\theta$ and $L$, convex in $L$ and has increasing differences in $(\theta, L)$. By the same standard monotone comparative statics arguments that we invoked in the proof of Lemma 1, the maximizing correspondence is increasing in $\theta$ and $L$ in the strong set sense, hence all of its measurable selections are weakly increasing in $\theta$ and $L$.  

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The proposition is finally established by the following simple induction. Consider two firms with \( \theta_2 > \theta_1 \). By assumption, at date 0, \( L_2 \geq L_1 \). Because any selection \( V(\theta, L, \omega, \Lambda) \) from the maximizing correspondence of the typical firm’s problem is increasing in \( \theta \) and \( L \), the values posted by those two firms at date 0 are such that \( V_2 \geq V_1 \). Then because \( L \) is strictly increasing in both \( L \) and \( V \), firm 2 is again larger than firm 1 at date 1. The same reasoning applies again at date 1, and at all subsequent dates, so that \( V_2 \geq V_1 \) holds true at all dates.

Finally, if firms are equally productive the RP property follows as a simple corollary of the convexity of \( S \) in \( L \), by the assumption that the initial \( \Lambda \) is continuous.

We conclude with a remark on atoms in the initial size distribution, including the symmetric case of identical firms that are equally productive and start out with the same size. That case would require mixed strategies in the first period. After the mixing plays out, in the second period of play firms would differ in size, and the previous case would apply from then on. We leave the computation of the equilibrium mixed strategies to future research.
Fig. 1: Mean output per worker
Fig. 2: Mean wage

Fig. 3: EE rate
Fig. 4: Firm size growth differential and unemployment

Fig. 5: Firm size growth differential and productivity growth