29 November 1999

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 263 (1999) 137–146

www.elsevier.nl/locate/physleta

Protective measurements and Bohm trajectories

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Received 21 May 1999; accepted 31 August 1999
Communicated by P.R. Holland

Abstract

We consider a protective measurement on a particle in a box and find that the particle participates in a local interaction although its Bohm trajectory never comes near the interaction region. This confirms earlier results to the same extent for von Neumann measurements and weak measurements, and challenges any realistic interpretation of Bohm trajectories.

Keywords: Quantum mechanics; Protective measurements; Bohm trajectories

1. Introduction

One can conceive situations in which a quantum particle interacts locally with other degrees of freedom, but the trajectory that Bohmian mechanics [1,2] attributes to the particle never gets near the interaction region. In view of the realistic interpretation that is given to this trajectory – allegedly, it states the 'actual position' of the particle as a function of time – the discovery of this possibility a few years ago [3] in the context of which-way detection in interferometers [4], and its subsequent confirmation [5] in the context of weak measurements [6], should have been quite disturbing to adherents of Bohmian mechanics because it implies that the Bohm trajectories are forever hidden. If you cannot rely on local interactions to determine the 'actual position' of the particle, then you cannot determine it at all. The concept of position itself becomes shaky.

Here is a drastic example illustrating what is at stake. A track of bubbles in a bubble chamber comes about as the result of a succession of local interactions. It is common sense, common quantum sense, to conclude that the electron, say, that produced the bubbles went through the bubble chamber. Inasmuch as the electron's Bohm trajectory could have never entered the chamber, Bohmian mechanics defies common sense when it insists on the said realistic interpretation of its trajectories.

Thus, as attractive the fully deterministic view offered by Bohmian mechanics may be to its proponents, the prize one must pay for it is very high. But perhaps some are willing to pay it. It seems, however, that they must live with a great problem: On the one hand the particle's position plays a central

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PII: S0375-9601(99)00628-3

role, on the other hand it is unobservable in important situations.

The advocates of Bohmian mechanics have responded to the challenge raised by the observation of Refs. [3,5], but their answers are hardly satisfactory. [1] Perhaps the new example presented in this paper will be helpful in going beyond this stage and toward a resolution of the issue, possibly in the spirit expressed by one supporter of Bohmian mechanics who believes that Ref. [3], and by implication also Ref. [5], demonstrate ''the need for a theory of detection [...] which tells us when it is legitimate to infer from the firing of a detector that another system has 'passed through it.' '' [12]. In standard quantum mechanics, of course, the knowledge that short-range forces were at work in a well localized interaction is all one needs to infer that this passage – or should we say: this trespassing? – did occur, but Bohmian mechanics apparently needs additional criteria.

The new example concerns a protective measurement [13,14] rather than a von Neumann measurement as in Ref. [3] or a weak measurement as in Ref. [5]. But the findings are essentially the same: Also under the circumstances of a protective measurement does the particle interact locally with another object (the meter) although its Bohm trajectory does not come anywhere near the interaction region.

The protective measurement probes a particle in a box for its presence at the center of the box. A meter experiences a momentum transfer and a consequent displacement proportional to the probability for finding the particle at the box center. In Section 2 we give the solution of the Schrödinger equation for this situation in the necessary detail and state the conditions for the measurement to be a protective one indeed. The Bohm trajectories of both the particle and the meter are the subject matter of Section 3. The paper ends with a brief summary and the conclusions we draw.

2. A protective measurement

The protective measurement is performed on a particle of mass m that moves along the x axis and is confined to the interior of the box $-\ell < x < \ell$. Another mass M serves as the meter; it moves freely along the X axis and is coupled to the boxed-in particle through the interaction

$$H_{\text{int}} = \epsilon \frac{\hbar}{T} f(t/T)\, \delta(x)\, X \,. \tag{1}$$

The strength of this interaction is proportional to the dimensionless parameter ϵ (of order unity, say) and inversely proportional to its duration T. The parametric time dependence of H_{int} is specified by the function $f()$ whose essential properties are

$$f(t/T) \text{ is a smooth function of } t; \tag{2a}$$

$$f(t/T) = 0 \text{ for } t \le -T \text{ and } t \ge T,$$
$$f(t/T) \ge 0 \text{ for } -T < t < T; \tag{2b}$$

$$\int_{-\infty}^{\infty} \frac{dt}{T} f(t/T) = \int_{-T}^{T} \frac{dt}{T} f(t/T) = 1; \tag{2c}$$

$$f(t/T) = f(-t/T) \text{ is assumed for simplicity.} \tag{2d}$$

As a consequence of the normalization (2c), the t-integral of H_{int},

$$\int dt\, H_{\text{int}} = \epsilon\hbar\, \delta(x)\, X \,, \tag{3}$$

is independent of the duration T. [2] Examples that exhibit extreme smoothness (all derivatives are continuous) are shown in Fig. 1.

For the particle, the interaction of (1) is a highly local coupling to the meter, sensitive solely to the particle's presence in the immediate vicinity of $x = 0$. For the meter, however, H_{int} amounts to an external force whose strength is proportional to the probability density for finding the particle at the center of the box. Therefore, one should expect that the net effect on the meter is a momentum change ΔP whose size is determined by the time integral (3), roughly: $\Delta P \cong \epsilon\hbar \langle \delta(x) \rangle \sim \epsilon\hbar/\ell$.

If the duration T of the interaction is short on all relevant time scales, so that a very strong interaction

[1] See Refs. [7–10] for a selection and Ref. [11] for additional material.

[2] Only the parametric t dependence is integrated over, not the dynamical t dependence that might be hidden in the Heisenberg operators x and X.

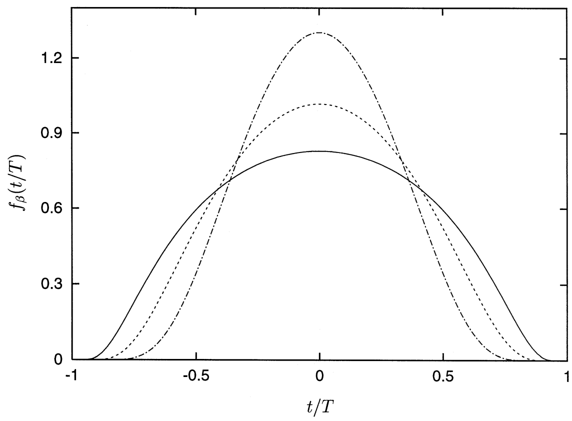

Fig. 1. A family of extremely smooth functions meeting the criteria in (2). The plot shows $f_\beta(t/T)=(2\beta)^{-1}[K_1(\beta)-K_0(\beta)]^{-1}\exp(-\beta[1+(t/T)^2]/[1-(t/T)^2])$ for $\beta=0.5$ (——), $\beta=1$ (– – –), and $\beta=2$ (– · – · –).

acts for a very short time, it is justified to replace $H_{\rm int}$ by its $T=+0$ version,

$$H_{\rm int}\to \epsilon\hbar\,\delta(t)\,\delta(x)\,X\,, \tag{4}$$

and the limit of an impulsive *von Neumann measurement* is realized. [3] By contrast, the *protective measurement*, which obtains when T is large on all relevant time scales, takes advantage of the adiabaticity that is characteristic of the temporal evolution in the opposite limit of a very weak interaction lasting for a very long time.

Prior to the interaction – that is: $t<-T$ – the particle is in its ground state, and the meter has a gaussian momentum distribution centered at $P=0$ and a gaussian position distribution centered at $X=0$, so that the initial wave function of the whole system is of the form

$$\begin{aligned}\Psi_{\rm ini}(t,x,X)=&\,\ell^{-1/2}\cos\left(\tfrac{1}{2}\pi x/\ell\right)(2/\pi)^{1/4}\\ &\times\left[\alpha(t)\,\delta P/\hbar\right]^{1/2}{\rm e}^{-\alpha(t)(X\delta P/\hbar)^2}\,.\end{aligned} \tag{5}$$

The first-line factor is the particle's ground-state wave function. In the gaussian of the second line we measure X conveniently in units of the meter's coherence length $\hbar/\delta P$. The complex function

$$\alpha(t)=\left[1+\frac{2i}{\hbar M}(\delta P)^2(t-t_0)\right]^{-1} \tag{6}$$

relates the time dependent spread $\delta X(t)$ in position to the time independent spread δP in momentum,

$$\delta X(t)=|\alpha(t)|^{-1}\frac{\hbar/2}{\delta P}\,, \tag{7}$$

and thus accounts for the dispersive spreading of the gaussian. [4] The parameter t_0 in (6) specifies the instant at which the meter is in a minimal uncertainty state, $\delta X(t_0)\,\delta P=\hbar/2$, typically when the initial state $\Psi_{\rm ini}$ is prepared, so that $t_0\le -T$.

One verifies easily that $\Psi_{\rm ini}$ obeys the Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\Psi(t,x,X)
=&\left\{-\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\pi}{2\ell}\right)^2\right]-\frac{\hbar^2}{2M}\left(\frac{\partial}{\partial X}\right)^2\right.\\
&\left.+\epsilon\frac{\hbar}{T}f(t/T)\,\delta(x)\,X\right\}\Psi(t,x,X)\end{aligned} \tag{8}$$

for $t<-T$, when $f(t/T)=0$ holds. The boundary conditions of the box, viz. $\Psi(t,x=\pm l,X)=0$, are respected as well, of course.

We now turn to the situation of a protective measurement and construct the corresponding approximate solution of (8), given in Eq. (18) below. This is done in a few steps. First we find the time dependent ground-state wave function of the particle. It depends parametrically on the strength of the perturbation $H_{\rm int}$ and thus on the product $\epsilon X f(t/T)$. As a consequence, the resulting energy shift is X dependent and gives rise to a force on the meter. This in turn leads to a change in the meter's momentum and, in the course of time, also of its position. The wave function (18) takes all of this into account, and we verify that this approximate solution of the Schrödinger Eq. (8) is indeed valid under the circumstances of a protective measurement.

[3] The product $\delta(t)\delta(x)$ must not be understood literally, but rather as indicating an interaction that is very well localized in t and in x.

[4] The identity ${\rm Re}\,\alpha(t)=|\alpha(t)|^2$ enters here.

If T is so large that a protective measurement is being performed, then the perturbation by $H_{\rm int}$ is turned on and off so slowly (and smoothly) that the particle undergoes an adiabatic transition to the time-dependent ground state of the perturbed Hamilton operator. In view of the identity

$$\left[-\left(\frac{\partial}{\partial x}\right)^2 - 2\gamma\cot\gamma\,\delta(x)\right]\sin(\gamma-\gamma|x/\ell|) = \left(\frac{\gamma}{\ell}\right)^2\sin(\gamma-\gamma|x/\ell|)\,, \tag{9}$$

the properly-normalized wave function of this ground state is (see Fig. 2)

$$\psi_\gamma(x) = \left[\ell\left(1-\frac{\sin(2\gamma)}{2\gamma}\right)\right]^{-1/2}\sin(\gamma-\gamma|x/\ell|)\,, \tag{10}$$

where $\gamma(t)$ is that solution of

$$-\gamma\cot\gamma = \epsilon\frac{m\ell X}{\hbar T}f(t/T)\,, \tag{11}$$

for which $\gamma\to\pi/2$ as $\epsilon\to 0$. Accordingly, $\gamma=\pi/2$ holds not only for $t<-T$ (‘before’) but also for $t>T$ (‘after’), and to ensure the desired adiabaticity of the evolution we must choose T so large that the right-hand side of (11) is small for all X values that are relevant. With ϵ of order unity, this says essentially that T has to be large compared with $m\ell X/\hbar$ for all X values in question, which are thus the ‘relevant time scales’ referred to above (more about this below).

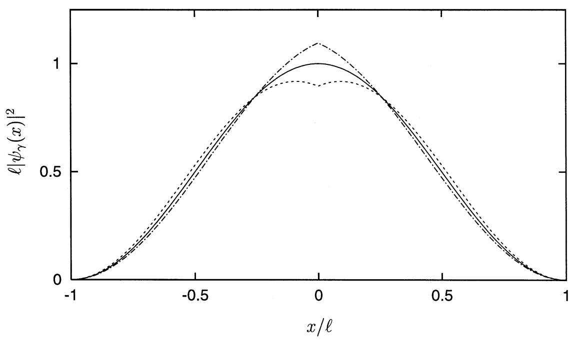

Fig. 2. Adiabatic ground-state probability densities of the particle in the box. The square of the wave function (10) is shown for $\gamma=\frac{1}{2}\pi$ (——), $\gamma=\frac{11}{20}\pi$ (– – –), and $\gamma=\frac{9}{20}\pi$ (– · – · –). During the protective measurement, the actual γ values are very close to $\gamma=\frac{1}{2}\pi$ all the time.

To first order in the perturbation – formally: to first order in ϵ (which is not necessarily a small number itself) – we then have

$$\gamma\cong\frac{\pi}{2}+\frac{2}{\pi}\epsilon\frac{m\ell X}{\hbar T}f(t/T)\,, \tag{12}$$

and the resulting time-dependent energy shift is

$$E_{\rm int}(t) = \frac{\hbar^2}{2m\ell^2}\left[\gamma^2-\left(\tfrac{1}{2}\pi\right)^2\right]\cong\epsilon\frac{\hbar/\ell}{T}f(t/T)\,X\,. \tag{13}$$

Since

$$\int_{-\ell}^{\ell}\mathrm{d}x\left|\psi_\gamma(x)\right|^2\delta(x) = \frac{1}{\ell}\left[1+O(\epsilon)\right]\,, \tag{14}$$

this shift is equal to the expectation value of $H_{\rm int}$ in the unperturbed state, as it must be to first order in the perturbation.

As a consequence of the energy shift $E_{\rm int}(t)$, the meter evolves under the influence of the force

$$F(t) = -\frac{\partial}{\partial X}E_{\rm int}(t) = -\epsilon\frac{\hbar/\ell}{T}f(t/T)\,, \tag{15}$$

and the resulting momentum change $\Delta P(t)$ and displacement $\Delta X(t)$ are given by

$$\Delta P(t) = \int_{-T}^{t}\mathrm{d}t'\,F(t')\,,$$

$$\Delta X(t) = \int_{-T}^{t}\mathrm{d}t'\,(t-t')F(t')/M\,. \tag{16}$$

We recall (2c) and (2d) and note the respective $t>T$ values,

$$\Delta P_{\rm net}\equiv\Delta P(t>T) = -\epsilon\hbar/\ell\,,$$

$$\Delta X_{\rm net}(t)\equiv\Delta X(t>T) = -\epsilon\hbar t/(M\ell)\,. \tag{17}$$

In summary, during the protective measurement the approximate, adiabatic solution of the Schrödinger Eq. (8) is given by

$$\Psi(t,x,X) = \psi_\gamma(x)\,(2/\pi)^{1/4}(\alpha\delta P/\hbar)^{1/2}\times\mathrm{e}^{-\alpha[(X-\Delta X)\delta P/\hbar]^2}\mathrm{e}^{i(X-\Delta X)\Delta P/\hbar+i\phi}\,, \tag{18}$$

where

$$\phi(t) = \frac{1}{\hbar}\int_{-T}^{t} \mathrm{d}t' \left(\frac{1}{2M} \left[\Delta P(t') \right]^2 + F(t')\, \Delta X(t') \right) \tag{19}$$

is a time-dependent phase of no further consequence.

When $\Psi(t,x,X)$ is inserted into (8), the difference of the two sides consists of essentially three terms. One of them contains

$$2\gamma \cot\gamma + \gamma^2 - (\pi/2)^2 \tag{20}$$

as a factor which is of order ϵ^2 and therefore negligible small. Another term features products such as

$$\frac{\partial\gamma}{\partial X}\,\frac{\partial\psi_\gamma}{\partial\gamma}\,. \tag{21}$$

Formally this term is of first order in ϵ, but the γ derivative of ψ_γ is orthogonal to ψ_γ and therefore a superposition of excited states of the particle in the box. The probability for exciting the particle is thus of order ϵ^2, and we can neglect the consequent corrections to Ψ in the adiabatic regime. The third term is proportional to

$$\frac{\partial\gamma}{\partial t}\,\frac{\partial\psi_\gamma}{\partial\gamma}\,. \tag{22}$$

Again, we meet the γ derivative of ψ_γ and the previous argument applies as well; in view of the smoothness of $f(t/T)$, the time derivative of γ is not problematic either.

Since γ of (11) and (12) depends on the meter position X, the particle and the meter are entangled at the intermediate times $-T<t<T$ to which (18) applies. But when the protective measurement is over, we have $\gamma(t>T)=\pi/2=\gamma(t<-T)$ and they are disentangled anew (at least in the adiabatic approximation). Indeed, the final wave function that obtains after the interaction is over,

$$\begin{aligned}\Psi_{\mathrm{fin}}(t,x,X) =& \ell^{-1/2}\cos\left(\tfrac{1}{2}\pi x/\ell\right)(2/\pi)^{1/4} \\ &\times\left[\alpha(t)\,\delta P/\hbar\right]^{1/2} \\ &\times \mathrm{e}^{-\alpha(t)\{[X-\Delta X_{\mathrm{net}}(t)]\delta P/\hbar\}^2} \\ &\times \mathrm{e}^{i[X-\Delta X_{\mathrm{net}}(t)]\Delta P_{\mathrm{net}}/\hbar + i\phi(t)}\,,\end{aligned} \tag{23}$$

is a product of a particle wave function and a meter wave function. The particle's final wave function is identical with its initial one, which is the defining property of a protective measurement, of course. The final wave function of the meter differs from the initial one by the momentum transfer ΔP_{net} and the displacement $\Delta X_{\mathrm{net}}(t)$ of (17), which come on top of the force-free evolution.

The $t>T$ values of ΔP and ΔX are independent of the duration T (in the adiabatic limit considered) and they are the same for any repetition of the protective measurement on equally prepared particles and meters. One could – at least in principle if not in practice – perform a yes/no measurement that determines whether the meter is indeed in the state specified by the X part of $\Psi_{\mathrm{fin}}(t,x,X)$, and the answer would invariably be 'yes.'

Since ΔP_{net} and ΔX_{net} are proportional to the expectation value of Eq. (14), we must conclude therefore that the particle wave function $\psi(x)$ possesses a physical significance for a single particle, not only for an ensemble of particles. This observation is a very important, perhaps the most important lesson of protective measurements.

In a final step, a standard von Neumann measurement is done to extract the result of the protective measurement from the $t>T$ state of the meter. Since both ΔP_{net} and $\Delta X_{\mathrm{net}}(t)$ contain this information, one could measure either the momentum P or the position X of the meter. Their probability distributions are, of course, the gaussians

$$\frac{1}{\sqrt{2\pi}\,\delta P}\exp\left(-\frac{1}{2}\left[\frac{P-\Delta P_{\mathrm{net}}}{\delta P}\right]^2\right) \tag{24}$$

and

$$\frac{1}{\sqrt{2\pi}\,\delta X(t)}\exp\left(-\frac{1}{2}\left[\frac{X-\Delta X_{\mathrm{net}}(t)}{\delta X(t)}\right]^2\right), \tag{25}$$

where the time dependent spread $\delta X(T)$ is as given in (7). As would be required of a good measurement, they can be well distinguished from the unshifted distributions obtained for $\Delta P_{\mathrm{net}}=0$ and $\Delta X_{\mathrm{net}}=0$, provided that

$$|\Delta P_{\mathrm{net}}| \gg \delta P \quad \text{or} \quad \ell\,\delta P \ll \hbar \tag{26}$$

holds. This condition is obvious for the momentum distribution (24), and for the position distribution (25) it follows from (6) and (17), since they imply that $\Delta X_{\mathrm{net}}(t)/\delta X(t) \cong \Delta P_{\mathrm{net}}/\delta P$ for sufficiently late times t.

We can now be more specific about the 'relevant time scales' that were left in limbo at Eq. (11). Initially X is in the $\delta X(-T)$ vicinity of $X=0$, finally in the $\delta X(T)$ vicinity of $X=\Delta X(T)$. Accordingly, relevant X scales are $\hbar/\delta P$ and $\hbar T/(M\ell)$, and $\hbar T/(m\ell)$ must be large compared with either one. This amounts to the requirements

$$T \gg m\ell/\delta P \quad \text{and} \quad m \ll M\,; \tag{27}$$

both are easily ensured.

3. Bohm trajectories

In Bohmian mechanics the Schrödinger wave function $\Psi(t,x,X)$ of (8) and (18) is supplemented by trajectories $x_{\mathrm{BM}}(t)$ and $X_{\mathrm{BM}}(t)$ that are determined by a pair of differential equations,

$$\frac{\mathrm{d}x_{\mathrm{BM}}}{\mathrm{d}t} = \upsilon(t,x_{\mathrm{BM}},X_{\mathrm{BM}})\,,$$
$$\frac{\mathrm{d}X_{\mathrm{BM}}}{\mathrm{d}t} = V(t,x_{\mathrm{BM}},X_{\mathrm{BM}})\,. \tag{28}$$

The velocity fields $\upsilon(t,x,X)$ and $V(t,x,X)$ are related to the probability currents associated with $\Psi(t,x,X)$, and we could find them in the usual manner by differentiating $\Psi(t,x,X)$ with respect to x and X. A consistent result would, however, require the inclusion of the corrections that go with the terms of (21) and (22). The book keeping is much simpler (and safer), when one identifies the velocity fields with the aid of the continuity equation

$$\frac{\partial}{\partial t}|\Psi|^2 + \frac{\partial}{\partial x}\left(\upsilon|\Psi|^2\right) + \frac{\partial}{\partial X}\left(V|\Psi|^2\right) = 0\,. \tag{29}$$

The squared modulus of the adiabatic solution (18) is given by the product of $|\psi_\gamma|^2$ and the gaussian of (25) [with $\Delta X(t)$, rather than $\Delta X_{\mathrm{net}}(t)$], so that the time derivative of $|\Psi|^2$ has a term proportional to $\partial\gamma/\partial t$, and two terms proportional to the time derivatives of ΔX and δX. The first one must be compensated for by the $\partial/\partial x$ contribution in (29), the other two by the $\partial/\partial X$ one.

To first order in the perturbation, we thus find

$$\upsilon(t,x,X) = \epsilon\frac{m\ell X}{\hbar T}\ell\frac{\partial f(t/T)}{\partial t}\frac{8}{\pi^3}\tan\left(\tfrac{1}{2}\pi x/\ell\right) \times\left[1-\tfrac{1}{2}\pi(1-|x|/\ell)\tan\left(\tfrac{1}{2}\pi|x|/\ell\right)\right] \tag{30}$$

for the particle's Bohm velocity field and

$$V(t,x,X) = \frac{\partial\delta X(t)}{\partial t}\frac{X-\Delta X(t)}{\delta X(t)} + \frac{\partial\Delta X(t)}{\partial t} + \epsilon\frac{m\ell}{\hbar T}f(t/T)\frac{\partial[\delta X(t)]^2}{\partial t}\frac{2}{\pi^2} \times\left[\pi(1-|x|/\ell)\tan\left(\tfrac{1}{2}\pi|x|/\ell\right)-1\right] \tag{31}$$

for the velocity field of the meter. Relevant steps on the way from (18) and (29) to (30) and (31) are given in the Appendix.

The Bohm equation of motion for the meter position $X_{\mathrm{BM}}(t)$ can be written in the compact form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{X_{\mathrm{BM}}(t)-\Delta X(t)}{\delta X(t)} = \epsilon\frac{m\ell}{\hbar T}f(t/T)\frac{\partial\delta X(t)}{\partial t}\frac{4}{\pi^2} \times\left[\pi(1-|x_{\mathrm{BM}}(t)|/\ell)\right. \left.\times\tan\left(\tfrac{1}{2}\pi|x_{\mathrm{BM}}(t)|/\ell\right)-1\right]. \tag{32}$$

Since the values of the x_{BM} dependent factor in the second line are from the range $-4/\pi^2\ldots4/\pi^2$ and since $|\partial\delta X/\partial t|<2\delta P/M$, the time integral of the right-hand side cannot exceed $(8/\pi^2)\epsilon(m/M)$-$(\ell\delta P/\hbar)$. Recalling the inequalities in (26) and (27) we note that this is a tiny number, so that the net effect of the right-hand side of (32) is an additional displacement by a very small fraction of $\delta X(t)$. Rare exceptions put aside, the trajectory $X_{\mathrm{BM}}(t)$ of the meter is, therefore, such that $X_{\mathrm{BM}}(-T)$ is in the $\delta X(-T)$ vicinity of $X=0$ and $X_{\mathrm{BM}}(T)$ in the $\delta X(T)$ vicinity of $X=\Delta X(T)$.

Before the interaction begins and after it is over, the velocity υ vanishes, so that the Bohm particle is at rest then. During the interaction it couples to the meter at the center of the box *and nowhere else*, and therefore one could imagine that a $x_{\mathrm{BM}}(t)$ trajectory that starts at $x_{\mathrm{BM}}(t<-T)=\ell/2$, say, will touch base at $x=0$ in the course of time and account in this way for the momentum transfer to the meter.

Such an *imagined* trajectory is sketched in Fig. 3. In fact, the *actual* trajectory is not like this because

no Bohm trajectory reaches $x = 0$ (if it didn't start there); (33a)

the vast majority of trajectories never come close to $x = 0$. (33b)

Concerning (33a) we note that the velocity field $v(t,x,X)$ vanishes linearly at $x = 0$ (see Fig. 4), so that the approach to $x = 0$ is exponential and $x = 0$ cannot be reached. This is not just a property of the adiabatic solution (18) but a more generally true consequence of the invariance under the $x \to -x$ inflection.

To demonstrate (33b) we estimate – very conservatively, of course – the maximal distance that the Bohm particle can move. The modulus of the x dependent factor in the second line of (30) never exceeds 0.06 (cf. Fig. 4), so that

$$|v(t,x_{\rm BM},X_{\rm BM})| < 0.06 \left|\epsilon \frac{m\ell X_{\rm BM}}{\hbar T}\right| \left|\frac{\rm d}{{\rm d}t} f(t/T)\right| \ell . \qquad (34)$$

If $f(t/T)$ first grows monotonically and then decreases, as is the typical situation and is the case for the examples of Fig. 1, one has

$$\int_{-T}^{T} {\rm d}t \left|\frac{\rm d}{{\rm d}t} f(t/T)\right| = 2 f(0) . \qquad (35)$$

As a consequence of what is said above about typical meter trajectories $X_{\rm BM}(t)$ [see after Eq. (32)] and in

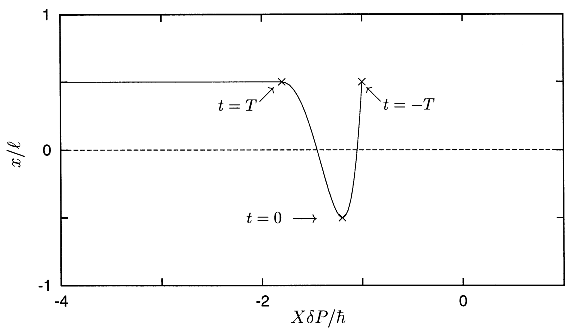

Fig. 3. An imagined Bohm trajectory. The meter coordinate X (abscissa) is given in units of the coherence length $\hbar/\delta P$, and the particle coordinate x in units of the box size ℓ. Crosses indicate the instants $t = -T$ (interaction begins), $t = 0$ (interaction is strongest), $t = T$ (interaction ends). This imagined trajectory crosses the interaction region at $x = 0$, but all actual trajectories stay on one side of the $x = 0$ line all the time (see Fig. 5).

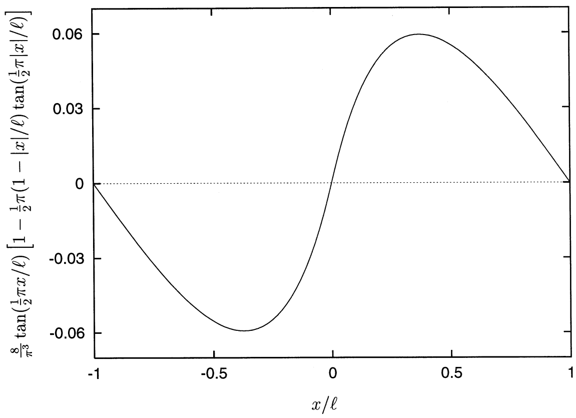

Fig. 4. The x dependent factor in the second line of Eq. (30).

view of the remarks at the end Section 2, the upper bound of the $X_{\rm BM}$ factor in (34) is a small number,

$$\left|\epsilon \frac{m\ell X_{\rm BM}}{\hbar T}\right| \le \left|\epsilon \frac{m\ell X_{\rm BM}}{\hbar T}\right|_{\max} \ll 1 , \qquad (36)$$

and thus we find

$$\begin{aligned} |x_{\rm BM}(T) - x_{\rm BM}(-T)| &= \left|\int_{-T}^{T} {\rm d}t\, v(t,x_{\rm BM}(t),X_{\rm BM}(t))\right| \\ &\times \left|0.06\, f(0)\right| \left|\epsilon \frac{m\ell X_{\rm BM}}{\hbar T}\right|_{\max} \times 2\ell \end{aligned} \qquad (37)$$

for the distance that the Bohm particle can move at most from its initial position. Since this maximal distance is tiny compared with the box width of 2ℓ, only a small fraction of all possible Bohm trajectories start sufficiently close to $x = 0$, and the statement of (33b) follows indeed.

4. Summary and conclusions

Our analysis of the protective measurement on the boxed-in particle has established (i) that the final state of the meter is independent of the duration of the interaction (in the relevant limit of adiabaticity); (ii) that this final state is invariably the same in each repetition of the protective measurement on identically prepared particles and meters; (iii) that the momentum transfer to the meter and its consequent displacement are proportional to the probability for finding the particle near the center of the box; and (iv) that the vast majority of Bohm trajectories of the particle never come close to the box center where the interaction with the meter happens. Fig. 5 illustrates the latter point.

In conjunction, observations (iii) and (iv) state that, in a typical case, the succession of 'actual positions' that are ascribed to the particle in Bohmian mechanics do not include the interaction region at the box center. Nevertheless, an interaction between the particle and the meter occurs undoubtedly, and its net effect is predictable.

Accordingly, one has to concede either that the particle's Bohm trajectory and its position are unrelated, or that the particle's position is irrelevant for its participation in local interactions. The second concession cannot be considered seriously because it would put away with the phenomenological meaning of position altogether. Therefore we can hardly avoid the conclusion that the formally introduced Bohm

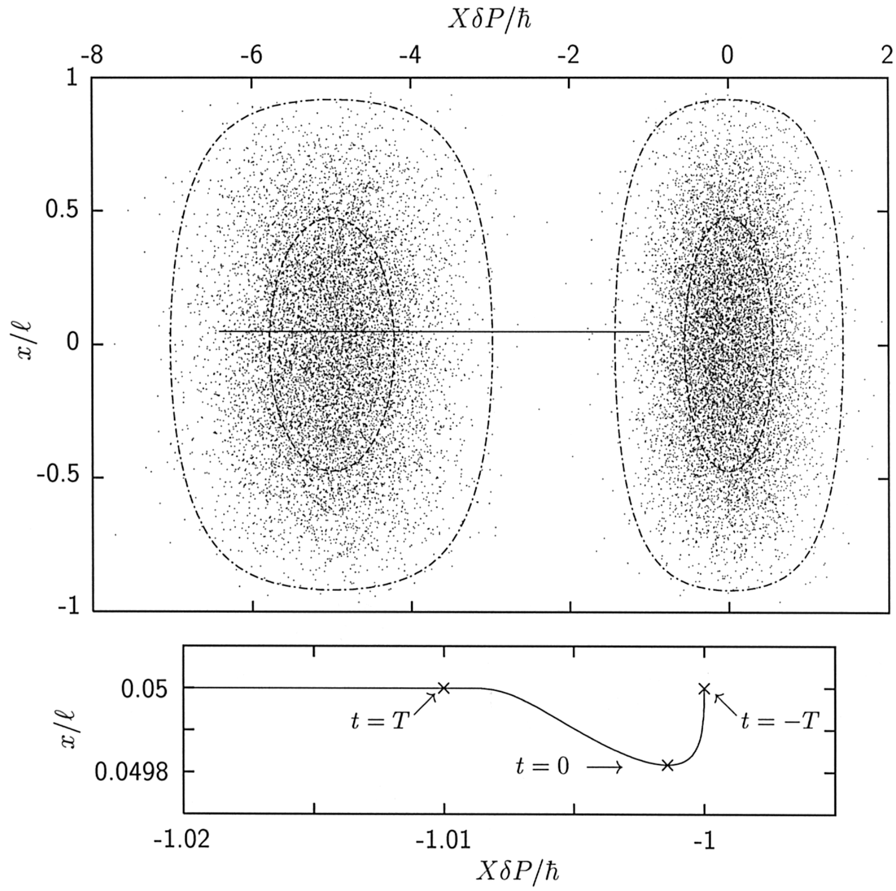

Fig. 5. Probability distributions before and after the protective measurement, and a typical Bohm trajectory. The meter coordinate X (abscissa) is given in units of the coherence length $\hbar/\delta P$, and the particle coordinate x in units of the box size ℓ. The plots are for the $\beta = 1$ function $f(t/T)$ of Fig. 1 and the following parameter values: $t_0 = -T$, $(\delta P)^2 T/(\hbar M) = 0.001$ [see (6)]; $\ell \delta P/\hbar = 0.01$ [see (26)]; $m/M = 0.01$, $m\ell/(T\delta P) = 0.1$ [see (27)]; as well as $\epsilon = 0.1$, so that $\Delta P_{\rm net}/\delta P = -10$, $\Delta X_{\rm net}(t)\delta P/\hbar = -0.01\,t/T$ [see(17)]. The right and left clouds of the top plot show the probability distributions $|\Psi_{\rm ini}(t = -T, x, X)|^2$ and $|\Psi_{\rm fin}(t = 500T, x, X)|^2$, respectively, in accordance with Eqs. (5) and (23). The inner iso-$|\Psi|^2$ lines enclose 50% of the probability, the outer ones 99%. The clouds are very well separated, so that the meter's position has changed by a clearly recognizable amount. The solid line is the Bohm trajectory that passes through $x = 0.05\ell$ and $X = -\hbar/\delta P$ at $t = -T$. In the bottom plot we take a much closer look at the relevant part of this Bohm trajectory. The positions at the instants $t = -T$ (interaction begins), $t = 0$ (interaction is strongest), and $t = T$ (interaction ends) are marked by crosses. Obviously, the trajectory does not come anywhere near the interaction region at $x = 0$.

trajectories are just mathematical constructs with no relation to the actual motion of the particle.

Acknowledgements

Y.A. thanks for the support by the Basic Research Foundation of the Israel Academy of Sciences and Humanities and by the National Science Foundation. B.G.E. is grateful for the hospitality extended to him at Texas A&M University. M.O.S. acknowledges the support of the Office of Naval Research, the National Science Foundation, the Welch Foundation, and the US Air Force.

Appendix A. Derivation of the Bohm velocity fields (30) and (31)

We proceed from noting that the t dependence in $|\Psi(t,x,X)|^2$ of (18) originates in $\gamma(t)$ as well as $\Delta X(t)$ and $\delta X(t)$. The latter two are related to the motion of the meter, the former to that of the particle. The time derivative of $|\Psi|^2$ splits naturally into two contributions,

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial\gamma}{\partial t}\frac{\partial}{\partial\gamma}|\Psi|^2 + \frac{\partial\Delta X}{\partial t}\frac{\partial}{\partial\Delta X}|\Psi|^2 + \frac{\partial\delta X}{\partial t}\frac{\partial}{\partial\delta X}|\Psi|^2, \tag{A.1}$$

(where the first term on the right-hand side is related to the motion of the particle and the last two terms to the the motion of the meter) and the comparison with the continuity Eq. (29) therefore establishes

$$\frac{\partial}{\partial x}\left(v|\Psi|^2\right) = -\frac{\partial\gamma}{\partial t}\frac{\partial}{\partial\gamma}|\Psi|^2 \tag{A.2}$$

for the particle, and

$$\frac{\partial}{\partial X}\left(V|\Psi|^2\right) = -\frac{\partial\Delta X}{\partial t}\frac{\partial}{\partial\Delta X}|\Psi|^2 - \frac{\partial\delta X}{\partial t}\frac{\partial}{\partial\delta X}|\Psi|^2 \tag{A.3}$$

for the meter.

The gaussian meter factor in $|\Psi|^2$ does not depend on γ and x, so that we can simplify (A.2) by the replacement

$$|\Psi|^2 \to \ell|\psi_\gamma|^2 = \cos^2\left(\tfrac{1}{2}\pi x/\ell\right) + \left(\gamma-\tfrac{1}{2}\pi\right) \times\Bigg[\left(1-|x|/\ell\right)\sin\left(\pi|x|/\ell\right) - \frac{2}{\pi}\cos^2\left(\tfrac{1}{2}\pi x/\ell\right)\Bigg]. \tag{A.4}$$

This takes already into account that $\gamma-\frac{1}{2}\pi$ is of first order in ϵ, see Eq. (12), and higher-order terms are disregarded consistently. Accordingly, we get

$$v(t,x,X)\cos^2\left(\tfrac{1}{2}\pi x/\ell\right) = \frac{\partial\gamma}{\partial t}\ell\int_{-1}^{x/\ell}\mathrm{d}\xi\left[\frac{2}{\pi}\cos^2\left(\tfrac{1}{2}\pi\xi\right) - (1-|\xi|)\sin(\pi|\xi|)\right], \tag{A.5}$$

where the boundary condition of vanishing probability current at $x=-\ell$ is incorporated. After evaluating the integral and collecting all the factors, Eq. (30) obtains.

Turning to (A.3) now, we note that for a gaussian of the form (25), with ΔX_{net} replaced by ΔX, the identities

$$\frac{\partial}{\partial\Delta X}G = -\frac{\partial}{\partial X}G, \tag{A.6a}$$

$$\frac{\partial}{\partial\delta X}G = -\frac{\partial}{\partial X}\frac{X-\Delta X}{\delta X}G, \tag{A.6b}$$

$$\frac{X-\Delta X}{\delta X}G = -\frac{\partial}{\partial X}\delta X\, G, \tag{A.6c}$$

hold, where G is a stand-in for the gaussian. With the aid of (43a) and (43b), we get

$$\frac{\partial}{\partial X}\left(V|\Psi|^2\right) = \frac{\partial}{\partial X}\left\{\frac{\partial\Delta X}{\partial t}|\Psi|^2 + \frac{\partial\delta X}{\partial t}\frac{X-\Delta X}{\delta X}|\Psi|^2\right\} - \frac{\partial\delta X}{\partial t}\frac{\partial\gamma}{\partial X}\frac{\partial|\psi_\gamma|^2}{\partial\gamma}\frac{X-\Delta X}{\delta X}G \tag{A.7}$$

for $|\Psi|^2=|\psi_\gamma|^2 G$. The curly-bracket term supplies the first two terms of (31). The additional last term comes from a partial integration that affects the X dependence of γ. It gives the second line of (31), as soon as (A.6c), (A.4), and (12) are employed. The requirement of vanishing probability current for $X \to -\infty$ specifies the boundary condition for the X integration.

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