CHAPTER 8

The assignment game

8.1 The formal model

This chapter presents a model in which there may be many sellers and many buyers, or many firms and workers. Formally, there are two finite disjoint sets of players P and Q, containing m and n players, respectively. Members of P will sometimes be called P-agents and members of Q called Q-agents, and the letters i and j will be reserved for P- and Q-agents, respectively. Associated with each possible partnership (i,j) in $P \times Q$ is a nonnegative real number α_{ij}. A game in coalitional function form with side payments is determined by (P, Q, α), with the numbers α_{ij} being equal to the worth of the coalitions $\{i,j\}$ consisting of one P-agent and one Q-agent. The worth of large coalitions is determined entirely by the worth of the pairwise combinations that the coalition members can form. That is, the coalitional function v is given by

$v(S) = \alpha_{ij}$ if $S = \{i,j\}$ for i in P and j in Q;

$v(S) = 0$ if S contains only P-agents or only Q-agents; and

$v(S) = \max(v(i_1, j_1) + v(i_2, j_2) + \cdots + v(i_k, j_k))$ for arbitrary coalitions S, with the maximum to be taken over all sets $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ of k distinct pairs in $S_P \times S_Q$, where S_P and S_Q denote the sets of P- and Q-agents in S (i.e., the intersection of the coalition S with P and with Q) respectively. Of course the number k of pairs in this maximization problem cannot exceed the minimum of $|S_P|$ and $|S_Q|$.

So the rules of the game are that any pair of agents (i,j) in $P \times Q$ can together obtain α_{ij}, and any larger coalition is valuable only insofar as it can organize itself into such pairs. The members of any coalition may divide among themselves their collective worth in any way they like. An imputation of this game is thus a nonnegative vector (u, v) in $\boldsymbol{R}^m \times \boldsymbol{R}^n$ such that $\sum_{i \in P} u_i + \sum_{j \in Q} v_j = v(P \cup Q)$. The easiest way to interpret this is to

take the quantities α_{ij} to be amounts of money, and to assume that agents' preferences are concerned only with their monetary payoffs.

We might think of this kind of game as arising from the multiseller generalization of the model of Chapter 7, where P is a set of potential buyers of some objects offered for sale by the set Q of potential sellers, and each seller owns and each buyer wants exactly one indivisible object. If each seller has a reservation price of zero, then the α_{ij}'s represent each buyer i's reservation price for the object offered by seller j. In this case if buyer i buys from seller j at a price p, and if no other monetary transfers are made or received by i and j, then the resulting utilities to the two agents are $u_i = \alpha_{ij} - p$ and $v_j = p$. More generally, if each seller j has a reservation price c_j, and each buyer i has a reservation price r_{ij} for object j, we may take α_{ij} to be the potential gains from trade between i and j; that is $\alpha_{ij} = \max\{0, r_{ij} - c_j\}$. In this case if buyer i buys object j from seller j at a price p, and if no other monetary transfers are made, the utilities are $u_i = r_{ij} - p$ and $v_j = p - c_j$. (It will be convenient to normalize each seller's utility function in this way, with the utility of keeping his own object being zero rather than c_j as in the previous chapter, so that these utilities u_i and v_j sum to α_{ij}. There is no loss of generality in doing so.) Note that transfers between agents are not restricted to those between buyers and sellers; for example, buyers may make transfers among themselves as in the bidder rings of Section 7.2.1.

Of course, in a similar way we can think of the P- and Q-agents as being firms and workers, and so on. As in the marriage model, we look here at the simple case of one-to-one matching, with firms constrained to hire at most one worker. In such a case, the α_{ij}'s represent some measure of the joint productivity of the firm and worker, and transfers between a matched firm and worker represent salary. Transfers can also take place between workers (as when workers form a labor union in which the dues of employed members help pay unemployment benefits to unemployed members) or between firms.

Note that since money is freely transferable and since each agent's preferences are assumed to be essentially monetary in nature, we are assuming that no agent has strict preferences. That is, for every pair of objects and any buyer, there is a pair of prices that makes the buyer indifferent between purchasing either of the objects.

The evaluation of the maximization problem to determine $v(S)$ for a given matrix α is called an *optimal assignment problem* or simply an *assignment problem*, so games of this form are called *assignment games*. We will be particularly interested in the value of the coalition $P \cup Q$, since $v(P \cup Q)$ equals the maximum total payoff available to the players in this game, and hence determines the Pareto set and the set of imputations.

Consider the following linear programming (LP) problem P_1:

$$\text{Maximize } \sum_{i,j} \alpha_{ij} \cdot x_{ij}$$

$$\text{subject to (a) } \sum_i x_{ij} \le 1$$

$$\text{(b) } \sum_j x_{ij} \le 1$$

$$\text{(c) } x_{ij} \ge 0.$$

Note that constraints (a), (b), and (c) are almost the same as constraints (1), (2), and (4) in Theorem 3.2.1. (The difference is that the inequalities in (a) and (b) allow agents to be unmatched.) So we may interpret x_{ij} as, for example, the probability that a partnership (i, j) will form. Then the linear inequalities of type (a), one for each j in q, say that the probability that j will be matched to some i cannot exceed 1. The inequalities of form (b), one for each i in P, say the same about the probability that i will be matched.

It can be shown as in Section 3.2.4 (see, e.g., Dantzig 1963, 318) that there exists a solution of this LP problem that involves only values of zero and one. (The extreme points of systems of linear inequalities of the form (a), (b), and (c) have integer values of x_{ij}; i.e., each x_{ij} equals zero or one.) Thus the fractions artificially introduced in the LP formulation disappear in the solution and the (continuous) LP problem is equivalent to the (discrete) assignment problem for the coalition of all players, that is, the determination of $v(P \cup Q)$. Then $v(P \cup Q) = \sum \alpha_{ij} \cdot x_{ij}$, where x is an optimal solution of the LP problem.

Definition 8.1. *A **feasible assignment** for* (P, Q, α) *is a matrix* $x = (x_{ij})$ *(of zeros and ones) that satisfies* (a), (b), *and* (c) *above.*

Then using the interpretation of x given above we can say that $x_{ij} = 1$ if i and j form a partnership and $x_{ij} = 0$ otherwise. If $\sum_j x_{ij} = 0$, then i is *unassigned*, and if $\sum_i x_{ij} = 0$, then j is likewise unassigned. A feasible assignment x corresponds exactly to a matching μ as defined in Definition 2.1, with $\mu(i) = j$ if and only if $x_{ij} = 1$. And it is equivalent to say that an agent i or j is unassigned at x or is unmatched (single) at μ.

Any solution of the preceding LP problem is called an *optimal assignment*.

Definition 8.2. *A feasible assignment x is **optimal** for* (P, Q, α) *if, for all feasible assignments* x', $\sum_{i,j} \alpha_{ij} \cdot x_{ij} \ge \sum_{i,j} \alpha_{ij} \cdot x'_{ij}$.

An assignment problem always has a solution, since there are only a finite number of assignments. For example, consider the assignment problem given by

$$\alpha = \begin{pmatrix} 10 & 12 & 7 \\ 6 & 8 & 2 \\ 5 & 5 & 9 \end{pmatrix}.$$

There are two optimal assignments given by

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with value $\alpha_{11} + \alpha_{22} + \alpha_{33} = \alpha_{12} + \alpha_{21} + \alpha_{33} = 27$.

Definition 8.3. *The pair of vectors* (u, v), *with* u *in* R^m *and* v *in* R^n, *is called a **feasible payoff** for* (P, Q, α) *if there is a feasible assignment* x *such that*

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_j = \sum_{\substack{i \in P \\ j \in Q}} \alpha_{ij} \cdot x_{ij}.$$

In this case we say (u, v) and x are *compatible* with each other, and we call $((u, v); x)$ a *feasible outcome*. Note again that a feasible payoff vector may involve monetary transfers between agents who are not assigned to one another.

As in the models of earlier chapters, the key notion is that of stability.

Definition 8.4. *A feasible outcome* $((u, v); x)$ *is **stable** (or the payoff* (u, v) *with an assignment* x *is stable) if*

(i) $u_i \ge 0, \quad v_j \ge 0$

(ii) $u_i + v_j \ge \alpha_{ij}$ for all (i, j) in $P \times Q$.

Condition (i) (individual rationality) reflects that a player always has the option of remaining unmatched (recall that $v(i) = v(j) = 0$ for all individual agents i and j). Condition (ii) requires that the outcome is not blocked by any pair: If (ii) is not satisfied for some agents i and j, then it would pay them to break up their present partnership(s) (either with one another or with other agents) and form a new partnership together, because this could give them each a higher payoff.

From the definition of feasibility and stability it follows that

Lemma 8.5. *Let $((u,v),x)$ be a stable outcome for (P,Q,α). Then*

(i) $u_i+v_j=\alpha_{ij}$ *for all pairs (i,j) such that $x_{ij}=1$*

(ii) $u_i=0$ *for all unassigned i, and $v_j=0$ for all unassigned j at x.*

Proof: Let R (respectively S) be the set of all unassigned i (respectively j) at x. Then by feasibility of $((u,v)x)$:

$$\sum_P u_i+\sum_Q v_j=\sum_{P\times Q}(u_i+v_j)x_{ij}+\sum_{i\in R}u_i+\sum_{i\in S}v_j=\sum_{P\times Q}\alpha_{ij}\cdot x_{ij}.$$

Now apply the definition of stability.

The lemma implies that at a stable outcome, the only monetary transfers that occur are between P- and Q-agents who are matched to each other. (Note that this is an implication of stability, not an assumption of the model.)

8.2 The core of the assignment game

Consider the LP problem P_1^* that is the dual of P_1, that is, the LP problem of finding a pair of vectors (u,v) in $R^m\times R^n$, that minimizes the sum

$$\sum_{i\in P}u_i+\sum_{i\in Q}v_j$$

subject, for all i in P and j in Q, to

(a*) $u_i\geq 0,\ v_j\geq 0$

(b*) $u_i+v_j\geq\alpha_{ij}$.

Because we know that P_1 has a solution, we know also that P_1^* must have an optimal solution. A fundamental duality theorem (see Dantzig, 1963, 129) asserts that the objective functions of these dual LP's must attain the same value. That is, if x is an optimal assignment and (u,v) is a solution of P_1^*, we have that

$$\sum_{i\in P}u_i+\sum_{i\in Q}v_j=\sum_{P\times Q}\alpha_{ij}\cdot x_{ij}=v(P\cup Q). \tag{8.1}$$

This means that $((u,v),x)$ is a feasible outcome. Moreover, $((u,v),x)$ is a stable outcome for (P,Q,α) since (a*) ensures individual rationality and $u_i+v_j\geq\alpha_{ij}$ for all (i,j) in $P\times Q$ by (b*).

On the other hand, condition (b*) says that

$$u_i+v_j\geq v(i,j)\quad\text{for all } i \text{ in } P,\ j \text{ in } Q.$$

It follows, by the definition of $v(S)$, that for any coalition $S=S_P\cup S_Q$, where S_P is contained in P and S_Q in Q,

$$\sum_{i\in S_P}u_i+\sum_{j\in S_Q}v_j\geq v(S). \tag{8.2}$$

But (8.1) and (8.2) are exactly how the core of the game is determined (recall Proposition 7.1): (8.1) ensures the feasibility of (u,v) and (8.2) ensures its nonimprovability by any coalition. Conversely, any payoff vector in the core, that is satisfying (8.1) and (8.2), satisfies the conditions for a solution to P_1^*.

Hence we have shown that

Theorem 8.6 (Shapley and Shubik). *Let (P,Q,α) be an assignment game. Then*

(a) *the set of stable outcomes and the core of (P,Q,α) are the same.*

(b) *the core of (P,Q,α) is the (nonempty) set of solutions of the dual LP of the corresponding assignment problem.*

The following two corollaries make clear why, in contrast to the discrete models considered earlier, we can concentrate here on the payoffs to the agents rather than on the underlying assignment (matching).

Corollary 8.7. *If x is an optimal assignment, then it is compatible with any stable payoff (u,v).*

Proof: Immediate from the fact that if (u,v) is a stable payoff, then it satisfies (8.1) for any optimal assignment.

Corollary 8.8. *If $((u,v),x)$ is a stable outcome, then x is an optimal assignment.*

Proof: Immediate from the fact that

$$\sum_j u_i+\sum_j v_j=v(P\cup Q)=\sum_{i,j}\alpha_{ij}\cdot x_{ij}.$$

As in the marriage model, if i prefers a stable payoff (u,v) to another stable payoff (u',v'), his or her mate(s) will prefer (u',v') (recall Corollary 2.21).

Proposition 8.9. *Let $((u,v),x)$ and $((u',v'),x')$ be stable outcomes for (P,Q,α). Then if $x'_{ij}=1$, $u'_i>u_i$ implies $v'_j<v_j$.*

Proof: Suppose $v'_j\geq v_j$. Then $\alpha_{ij}=u'_i+v'_j>u_i+v_j\geq\alpha_{ij}$, which is a contradiction.

Just as Proposition 8.9 shows how the interests of P- and Q-agents are opposed in the core, the following theorem shows that among themselves, the P-agents and Q-agents have common interest in the core. Specifically, as in the marriage market, the core is a lattice; that is, the greatest lower (or least upper) bound to any two points in the core is also in the core (recall Theorems 2.16 and 3.8).

Define the partial order $(u',v') >_P (u,v)$ if $u_i' > u_i$ for all i in P and $u_i' > u_i$ for at least one i in P. It follows from Proposition 8.9 that for stable outcomes, if $(u',v') >_P (u,v)$ then $v_j' \leq v_j$ for all j in Q. Then we have

Theorem 8.10 (Shapley and Shubik). *The core of the assignment game endowed with the partial order $\geq_P$ forms a complete lattice (dual to the lattice with ordering $\geq_Q$).*

Proof: Let (u,v) and (u',v') be any two payoff vectors in the core. Let x be some optimal assignment. Let

$$\underline{u}_i = \min\{u_i, u_i'\} \qquad \underline{v}_j = \min\{v_j, v_j'\}$$
$$\bar{u}_i = \max\{u_i, u_i'\} \qquad \bar{v}_j = \max\{v_j, v_j'\}.$$

We will show that $((\underline{u},\bar{v}),x)$ and $((\bar{u},\underline{v}),x)$ are also in the core. For any i and j we have either

$$\underline{u}_i + \bar{v}_j = u_i' + \bar{v}_j \geq u_i' + v_j' \geq \alpha_{ij} \quad \text{or}$$
$$\underline{u}_i + \bar{v}_j = u_i + \bar{v}_j \geq u_i + v_j \geq \alpha_{ij}.$$

By Corollary 8.7, (u,v) and (u',v') are compatible with x. Clearly $\underline{u}_i \geq 0$ and $\bar{v}_j \geq 0$. It remains to show that $\sum_i \underline{u}_i + \sum_j \bar{v}_j = v(P \cup Q)$. But it is immediate, from Proposition 8.9 and Lemma 8.5, that if $x_{ij} = 1$ then

$$\underline{u}_i + \bar{v}_j = u_i' + v_j' = \alpha_{ij} \quad \text{or}$$
$$\underline{u}_i + \bar{v}_j = u_i + v_j = \alpha_{ij}.$$

Hence

$$\sum_i \underline{u}_i + \sum_j \bar{v}_j = \sum_{i,j} \alpha_{ij} \cdot x_{ij} = v(P \cup Q).$$

Analogously, $(\bar{u},\underline{v})$ is stable. Hence we have shown that the core is a lattice. Since it is a convex polytope it is also a compact set, from which it follows that it is a complete lattice.

As in the marriage market, this implies the existence of P- and Q-optimal stable outcomes. That is, there is a vertex in the core at which every player from one side gets the maximum payoff and every agent from the other side gets the minimum payoff. There is another vertex with symmetric properties. This is an immediate consequence of Theorem 8.10 and Proposition 8.9.

Theorem 8.11 (Shapley and Shubik). *There is a P-optimal stable payoff $(\bar{u},\underline{v})$, with the property that for any stable payoff (u,v), $\bar{u} \geq u$ and $\underline{v} \leq v$; there is a Q-optimal stable payoff $(\underline{u},\bar{v})$ with symmetrical properties.*

8.3 A multiobject auction mechanism

In this section we will interpret P as a set of *bidders* and Q as a set of *objects*. Each object j has a reservation price of c_j. The value of object j to bidder i is $\alpha_{ij} \geq 0$. A feasible price vector p is a function from Q to $\mathbf{R}^+$ such that $p_j = p(j)$ is greater than or equal to c_j. As a notational convention we will also assume in this section that Q contains an artificial "null object," O, whose value α_{iO} is zero to all bidders and whose price is always zero. Then if a bidder is unmatched we will say that he or she is assigned to O. (More than one bidder may be assigned to O.) The *demand set* of a bidder i at prices p is defined by

$$D_i\{p\} = \{j \in Q;\ \alpha_{ij} - p_j = \max_{k \in Q}\{\alpha_{ik} - p_k\}\}.$$

The price vector p is called *quasi-competitive* if there is a matching μ from P to Q such that if $\mu(i) = j$ then j is in $D_i(p)$, and if i is unmatched under μ then O is in $D_i(p)$. Thus at quasi-competitive prices p each buyer can be assigned to an object in his or her demand set. The matching μ is said to be *compatible* with the price p. The pair (p,μ) is a *competitive equilibrium* if p is quasi-competitive, μ is compatible with p, and $p_j = c_j$ for all $j \notin \mu(P)$. Thus at a competitive equilibrium, not only does every buyer get an object in his or her demand set, but no unsold object has a price higher than its reservation price. If (p,μ) is a competitive equilibrium, p will be called a *competitive* or an *equilibrium* price vector.

It is easy to verify that if (p,μ) is a competitive equilibrium, then the corresponding payoffs (u,v) are stable (where $u_i = \alpha_{ij} - p_j$ and $v_j = p_j - c_j$ for $j = \mu(i)$). The existence of a P-optimal stable payoff is equivalent to the statement that there is a unique vector of equilibrium prices that is optimal for the P-agents, in the sense that it is at least as small in every component as any other equilibrium price vector. This price is called the *minimum equilibrium price*. We will describe an algorithm for computing this price, which is an auction mechanism that generalizes the Vickrey second-price auction described in Chapter 7. (Note that the Vickrey auction of a single object also produces the minimum equilibrium price.) As we will see in Section 8.4, one important property of the single-object auction that generalizes to the multiobject case is that submitting true valuations is a dominant strategy for the bidders.

To describe the mechanism, we will make use of the following well-known result from graph theory. Let B and C be two finite disjoint sets (e.g., of buyers and objects, respectively). For each i in B, let D_i be a subset of C (e.g., D_i is i's demand set at some set of prices). A *simple assignment* is an assignment of objects to buyers such that each buyer i is assigned exactly one object j such that j is in D_i, and each object is assigned to at most one buyer. (So a simple assignment assigns an object to every buyer but may not assign every object to a buyer.) Then it is apparent that if a simple assignment exists, each buyer in every subset B' of B must be matched to a different object, so there must be at least as many objects in $D(B') \equiv \bigcup_{i \in B'} D_i$ as there are buyers in B'. Hall's theorem says that this necessary condition is also sufficient.

Theorem 8.12 (Hall's theorem). *A simple assignment exists if and only if, for every subset B' of B, the number of objects in $D(B')$ is at least as great as the number of buyers in B'.*

The auction mechanism for the multobject case that we will now present produces the minimum price equilibrium in a finite number of steps.

We will take all prices and valuations to be integers. At the first step of the auction the auctioneer announces an initial price vector, $p(1)$, equal to the vector c of reservation prices. Each bidder "bids" by announcing which object or objects (including the null object O) are in his or her demand set at price $p(1)$.

Step $(t+1)$: After the bids are announced, if it is possible to match each bidder to an object in his or her demand set at price $p(t)$ the algorithm stops. If no such matching exists, Hall's theorem implies that there is some *overdemanded* set, that is, a set of objects such that the number of bidders demanding only objects in this set is greater than the number of objects in the set. The auctioneer chooses a *minimal* overdemanded set (i.e., an overdemanded set S such that no strict subset of S is an overdemanded set) and raises the price of each object in the set by one unit. All other prices remain at the level $p(t)$. This defines $p(t+1)$. (Note that the nonexistence of the matching implies the minimal overdemanded set does not contain the null object O, since we allow any number of agents to be matched to O if O is in their demand sets.)

It is clear that the algorithm stops at some step t, because as soon as the price of an object becomes higher than any bidder's valuation for it, no bidder can demand it. It follows that the final price obtained by this algorithm is a quasi-competitive price vector. Indeed it is the minimum equilibrium price vector, although this fact is not so obvious.

Theorem 8.13 (Demange, Gale, Sotomayor). *Let p be the price vector obtained from the auction mechanism. Then p is the minimum quasi-competitive price.*

Proof: Suppose instead that there exists a quasi-competitive price q such that $p \not\leq q$. Now at step $t=1$ of the auction we have $p(1)=c$ so $p(1) \leq q$. Let t be the last step of the auction at which $p(t) \leq q$ and let $S_1 = \{j;\ p_j(t+1) > q_j\}$. Let S be the minimal overdemanded set whose prices are raised at stage $t+1$, thus $S = \{j;\ p_j(t+j) > p_j(t)\}$, so S_1 is contained in S. Furthermore $q_j = p_j(t)$ for all j in S_1 (since we are working with all integers). We will show that $S - S_1$ is nonempty and overdemanded, hence S is not a minimal overdemanded set, contrary to the rules of the auction.

Define $T = \{i;\ D_i(p(t))$ is contained in $S\}$. That S is overdemanded means exactly that

$$|T| > |S|. \tag{1}$$

Define $T_1 = \{i \in T;$ the set of objects in S_1 demanded by i at price $p(t)$ is nonempty$\}$.

We claim that $D_i(q)$ is contained in S_1 for all i in T_1. Indeed, choose j in S_1 and in $D_i(p(t))$. If $k \notin S$, then i prefers j to k at price $p(t)$ because i is in T, but $p_k(t) \leq q_k$ and $p_j(t) = q_j$. So i prefers j to k at price q. On the other hand, if k is in $S - S_1$, then i likes j at least as well as k at price $p(t)$, but $p_k(t) < p_k(t+1) \leq q_k$ (and, again, $p_j(t) = q_j$) so i prefers j to k at price q, as claimed. Now since q is quasi-competitive there are no overdemanded sets at price q so

$$|T_1| \leq |S_1|. \tag{2}$$

Now from (1) and (2), $|T - T_1| > |S - S_1|$ so $T - T_1 \neq \emptyset$ and $T - T_1 = \{i \in T;\ D_i(p(t)) \in S - S_1\}$. So $S - S_1 \neq \emptyset$ and $S - S_1$ is overdemanded, giving the desired contradiction.

Theorem 8.14 (Demange, Gale, Sotomayor). *If p is the minimum quasi-competitive price, then there is a matching μ^* such that (p, μ^*) is an equilibrium (so p is a competitive price vector).*

Proof: Let μ be a matching corresponding to p. Call an object j *overpriced* if it is unmatched by μ but $p_j > c_j$. If (p, μ) is not an equilibrium, there is at least one overpriced object. We will give a procedure for altering μ so as to eliminate overpriced objects. For this purpose we construct a directed graph whose vertices are $P \cup Q$. There are two types of arcs. If $\mu(i)=j$ there is an arc from i to j. If j is in $D_i(p)$ there is an arc from j to i. Now let k be an overpriced object. Then k is in $D_i(p)$ for some i, for if not we could decrease p_k and still have quasi-competitive prices, which contradicts the minimality of p. Let $\bar{P} \cup \bar{Q}$ be all vertices that can be reached by a directed path starting from k.

Case 1: $\bar{P}$ contains an unmatched bidder, i. Let $(k, i_1, j_2, i_2, j_3, i_3, \ldots, j_t, i)$ be a path from k to i. Then we may change μ by matching i_1 to k, i_2 to $j_2, \ldots, i$ to j_t. The matching is still competitive and k is no longer overpriced so the number of overpriced objects has been reduced.

Case 2: All i in $\bar{P}$ are matched. Then we claim that there must be some j in $\bar{Q}$ such that $p_j = c_j$, for suppose not. By definition of $\bar{P} \cup \bar{Q}$ we know that if $i \notin \bar{P}$ then i does not demand any object in $\bar{Q}$. Therefore we can decrease the price of each object in $\bar{Q}$ by some positive δ and still have quasi-competitiveness, contradicting the minimality of p. So choose j in $\bar{Q}$ such that $p_j = c_j$ and let $(k, i_1, j_2, i_2, \ldots, j_t, i_t, j)$. Again change μ by matching i_1 to k, i_2 to $j_2, \ldots$, leaving j unmatched. Again the number of overpriced objects has been reduced.

8.4 Incentives

Denote by $(\bar{u}, \underline{v})$ the P-optimal stable payoff for the market $M=(P, Q, \alpha)$. In this section it will continue to be convenient to think of P-agents as buyers, and Q-agents as sellers. (But we will no longer speak of unmatched buyers as demanding an artificial null object O, nor will we continue to take all prices to be integers.) For simplicity we will take the reservation prices c to all be zero, so $\underline{v}$ is the minimum equilibrium price vector. Let v be the coalitional function of the game, that is, for every S contained in P and R contained in Q, $v(S,R)=\max \sum_{S \times R} \alpha_{ij} x_{ij}$, for all assignments x. The demand set of buyer i at prices $\underline{v}$ is defined by $D_i(\underline{v}) = \{j \in Q \text{ such that } \alpha_{ij} - \underline{v}_{ij} \geq 0,\ \alpha_{ij} - \underline{v}_{ij} = \max_{k \in Q}\{\alpha_{ik} - \underline{v}_{ik}\}\}$. (Note that now that we have dispensed with the null object, the demand set of a buyer may be empty.)

The following lemma shows a critical way in which the Vickrey second-price auction is generalized by the mechanism that sets prices equal to $\underline{v}$ (i.e., that gives buyers their optimal stable outcome). Both mechanisms give buyers their marginal contribution to coalitional values.

Lemma 8.15 (Demange; Leonard). *For all i in P,*

$$\bar{u}_i = v(P, Q) - v(P-\{i\}, Q).$$

Proof: Let x be an optimal assignment for $M=(P, Q, \alpha)$. Construct a graph whose vertices are $P \cup Q$. There are two kinds of arcs. If $x_{ij}=1$ there is an arc from i to j. If j is in $D_i(\underline{v})$ and $x_{ij}=0$ there is an arc from j to i. Let j be an object whose price is greater than zero. Then there is an oriented path starting from j and ending at an unmatched buyer or at an object of price zero. To see this, suppose there is no such path, and denote by S and T the sets of objects and buyers, respectively, that can be reached from j. Then $\underline{v}_k > 0$ for all k in S. Furthermore, if $i \notin T$, then there is no object in S that is demanded by i at price $\underline{v}$. (If k is demanded by i then there is an arc from k to i if $x_{ik}=0$, or an arc from i to k if $x_{ik}=1$. In both cases, if i is not in T, k cannot be in S.) Then we can decrease $\underline{v}_k$ for all k in S, and still have an equilibrium, which contradicts the minimality of $\underline{v}$.

So, let i' be any buyer. If i' is assigned to some object j_1, we may consider a path c beginning at j_1 and ending at an unmatched buyer i_s or at an object k of price zero. (Note that k might be j_1.) That is, $c=(j_1, i_1, j_2, i_2, \ldots, j_s, i_s)$ or $c=(j_1, i_1, j_2, i_2, \ldots, j_s, i_s, k)$. Consider now the assignment x' in $M'=(P-\{i'\}, Q, \alpha)$ that assigns j_1 to i_1, j_2 to $i_2, \ldots, j_s$ to i_s, and that leaves k unmatched if k is in the path, and that otherwise agrees with x on every buyer in $P-\{i'\}$ who is not in the path. We claim the outcome $((u^*, \underline{v}); x')$ is stable for M', where $u_i^* = \bar{u}_i$ for all $i \neq i'$. This is immediate from the fact that $x'_{i_t j_t}=1$, j_t is demanded by i_t at price $\underline{v}_{j_t}$ for all $t=1, \ldots, s$, and $((\bar{u}, \underline{v}), x)$ is stable for (P, Q, α). Then x' is an optimal assignment for M', so

$$\sum_{\substack{i \neq i' \\ j \in Q}} \alpha_{ij} x'_{ij} = v(P-\{i'\}, Q). \tag{a}$$

On the other hand,

$$\sum_{\substack{i \neq i' \\ j \in Q}} \alpha_{ij} x'_{ij} = \sum_i u_i^* + \sum_j \underline{v}_j = \sum_{i \neq i'} \bar{u}_i + \sum_j \underline{v} = v(P, Q) - \bar{u}_{i'}. \tag{b}$$

From (a) and (b) we obtain $\bar{u}_{i'} = v(P, Q) - v(P-\{i'\}, Q)$, which completes the proof.

Let x' be any optimal assignment for $(P-\{i\}, Q-\{j\}, \alpha)$, where i is assigned to j under the optimal assignment x for (P, Q, α). Then

$$\sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} \cdot x'_{\ell k} + \alpha_{ij} \leq \sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} \cdot x_{\ell k} + \alpha_{ij},$$

from optimality of x. Then

$$\sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} x'_{\ell k} \leq \sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} \cdot x_{\ell k}. \tag{1}$$

On the other hand,

$$\sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} \cdot x_{\ell k} \leq \sum_{\substack{\ell \neq i \\ k \neq j}} \alpha_{\ell k} \cdot x'_{\ell k}, \tag{2}$$

from optimality of x'.

By (1) and (2) we get that

$$v(P, Q) = \alpha_{ij} + v(P - \{i\}, Q - \{j\}), \quad \text{if } x_{ij} = 1. \tag{*}$$

Note that Lemma 8.15 and (*) together imply that if buyer i gets object j in the auction,

$$\bar{u}_i = \alpha_{ij} - [v(P - \{i\}, Q) - v(P - \{i\}, Q - \{j\})]. \tag{**}$$

That is, buyer i buys object j at the price

$$p_j = [v(P - \{i\}, Q) - v(P - \{i\}, Q - \{j\})].$$

The critical observation for the proof of the next theorem is that this price does not depend on any valuations α_{ik} of buyer i. So as in the Vickrey second-price auction for a single object, the price a buyer pays is not determined by the reserve prices he or she states. This permits us to prove the following.

Theorem 8.16 (Demange; Leonard). *In the multiobject auction mechanism, truth telling is a dominant strategy for each buyer.*

Proof: If buyer i tells the truth and gets object j at the end of the auction, his or her profit will be $\bar{u}_i = \alpha_{ij} - [v(P - \{i\}, Q) - v(P - \{i\}, Q - \{j\})]$, by (**). Suppose the buyer misrepresents his or her valuations. If he or she is assigned the same object j at the end of the auction under the new valuations, the buyer's true payoff will be the same, since he or she will pay the same price p_j [given by (**)] for object j. If assigned to some other object k, the buyer will pay $[v(P - \{i\}, Q) - v(P - \{i\}, Q - \{k\})]$ and his or her true profit will be $\bar{u}'_i = \alpha_{ik} - [v(P - \{i\}, Q) - v(P - \{i\}, Q - \{k\})]$. But,

$$\alpha_{ik} + v(P - \{i\}, Q - \{k\}) = \alpha_{ik} + \max_{x'} \sum_{\substack{t \neq i \\ \ell \neq k}} \alpha_{t\ell} \cdot x'_{t\ell} \leq v(P, Q)$$
$$= \alpha_{ij} + v(P - \{i\}, Q - \{j\}),$$

by (*). So $\bar{u}_i \geq \bar{u}'_i$, and buyer i has not profited from misstating his or her valuations. If buyer i is unmatched, then he or she also does not profit, since $\bar{u}'_i = 0 \leq \bar{u}_i$.

If buyer i is unmatched under the true valuations, then $v(P - \{i\}, Q) = v(P, Q)$, so if he or she is matched to k under the misstated valuations, $\bar{u}'_i = [\alpha_{ik} + v(P - \{i\}, Q - \{k\})] - v(P, Q) \leq v(P, Q) - v(P, Q) = 0 = \bar{u}_i$. Thus in every case, $\bar{u}_i \geq \bar{u}'_i$ and truth telling is a dominant strategy for i.

8.5 The effect of new entrants

In this section we return to another question we have previously considered for the marriage market, namely, What is the effect on the set of stable outcomes of changing the market by introducing a new agent? Aside from being able to prove results parallel to those we have seen for the marriage model, we will see that the special assumptions of the assignment model allow us to draw some even stronger conclusions.

Suppose some P-agent i^* enters the market $M = (P, Q, \alpha)$. The new market is then $M^{i^*} = (P \cup \{i^*\}, Q, \alpha')$, where $\alpha'_{ij} = \alpha_{ij}$ for all i in P and j in Q. The first result, whose proof we will defer until the more general model of the next chapter (Theorem 9.12), is parallel to Theorem 2.25 for the marriage market. It compares the optimal stable outcomes of the two markets.

Proposition 8.17. (a) *Let $(\bar{u}, \underline{v})$ and $(\bar{u}', \underline{v}')$ be the P-optimal stable payoffs for M and M^{i^*}, respectively. Then $\bar{u}'_i \leq \bar{u}_i$ for all i in P and $\underline{v}'_j \geq \underline{v}_j$ for all j in Q.*

(b) *Let $(\underline{u}, \bar{v})$ and $(\underline{u}', \bar{v}')$ be the Q-optimal stable payoffs for M and M^{i^*}, respectively. Then $\underline{u}'_i \leq \underline{u}_i$ for all i in P and $\bar{v}'_j \geq \bar{v}_j$ for all j in Q.*

The next result (analogous to Theorem 2.26 for the marriage market) shows that there will be some P- and Q-agents for whom we can unambiguously compare *all* stable outcomes of the two markets.

Theorem 8.18: Strong dominance (Mo). *If i^* is matched under some optimal assignment for M^{i^*}, then there is a nonempty set A of agents in $P \cup Q$ such that every Q-agent in A is better off and every P-agent in A is worse off at any stable outcome of the new market than at any stable outcome for the old market. That is, for all (u', v') and (u, v) stable for M^{i^*} and M, respectively, we have*

(a) *if a P-agent i is in A, then $u_i \geq u'_i$*
(b) *if a Q-agent j is in A, then $v_j \leq v'_j$.*

Before proving this theorem we need to recall Lemma 8.15, which implies that if $(\bar{u}', \underline{v}', x')$ is the P-optimal stable outcome for M^{i^*}, then $\bar{u}'_{i^*} = v(P \cup \{i^*\}, Q) - v(P, Q)$.

Recall that the central idea of the proof of Lemma 8.15 involved showing that if i^* is assigned by x' to some agent j_1, then there is an oriented path $c=(j_1, i_1, j_2, i_2, \ldots, j_s, i_s, (j_{s+1}))$, starting from j_1, with the following properties.

P1: c ends at i_s if i_s is unassigned by x' or c ends at j_{s+1} if i_s is assigned to j_{s+1} by x' and $\underline{v}'_{s+1}=0$.
P2: i_m is assigned by x' to j_{m+1} for all $m=1,\ldots,s-1$.
P3: $\bar{u}'_m+\underline{v}'_m=\alpha_{mm}$ for all $m=1,\ldots,s$ (since j_m is in the demand set of i_m at prices $\underline{v}'$).

Furthermore if x is the assignment (in M) defined by

(i) $x_{mm}=1$ for all $m=1,\ldots,s$,
(ii) $x_{ij}=1$ if i and j are not in the path and $x'_{ij}=1$,
(iii) if j_{s+1} is in the path he or she is unassigned by x,

then x is an optimal assignment for M and the outcome $(\bar{u}, \underline{v}', x)$ is stable for M, where $\bar{u}_i=\bar{u}'_i$ for all i in P. Mo calls the path c a "turnover chain," with the last element being the "crowd-out" i_s or the "draw-in" j_{s+1}. The idea is that if x and x' are the assignments before and after i^* enters the market, then the agents in the chain c are those whose assignments change. If, for example, the P-agent i_s is unassigned by x', he or she has been crowded out of the market by the entry of the new P-agent i^*.

The existence of the path c will be needed to prove Theorem 8.18. The following lemma takes advantage of the special assumptions of the assignment game, namely, that all payoffs are essentially monetary in nature, to compare the benefits and losses that agents in a turnover chain experience when a new player enters the game.

For each i in P and j in Q, define the "benefit functions" B_i and B_j as follows. For all pairs of payoff vectors (u,v) and (u',v'), with (u,v) stable for M and (u',v') stable for M^{i^*},

$$B_i((u,v),(u',v'))=u'_i-u_i, \quad \text{and}$$
$$B_j((u,v),(u',v'))=v'_j-v_j.$$

Lemma 8.19: Benefit lemma (Mo). *Let x' be an optimal assignment for M^{i^*}. If i^* is matched to some j_1 under x' and $(j_1, i_1, j_2, i_2, \ldots, j_s, i_s, (j_{s+1}))$ is some oriented path satisfying properties* P1, P2, *and* P3, *then*

$$B_{j_1}\geq B_{j_2}\geq \cdots \geq B_{j_s}\geq B_{j_{s+1}}; \quad \textit{and}$$
$$B_{i_s}\geq B_{i_{s-1}}\geq \cdots \geq B_{i_1}.$$

The lemma compares the "benefits" that accrue to agents in a turnover chain resulting from the entry of the P-agent i^*. Looking ahead for a moment to when we have completed the proof of Theorem 8.18, we know that these benefits will be nonnegative for all the Q-agents and nonpositive for all the P-agents in the chain. So the lemma says that the greatest benefit will come to agent j_1, who will be matched to i^*, with decreasing benefits to j_2 and so on for Q-agents more distant in the chain from i^*. And the greatest harm (i.e., the most negative benefit) will come to agent i_1, who was matched to j_1 before i^* entered the market, with less harm done to P-agents further down the chain from i^*. Note that these comparisons are meaningful here because we are speaking of monetary gains and losses. (In the marriage model, no similar comparison is possible, since it would involve comparisons of, e.g., how a change from my second to my third choice mate compares with your change from your seventh to your ninth choice.)

Proof of Lemma 8.19: Let (u',v',x') be stable for M^{i^*} and let (u,v,x) be stable for M, where x is defined from x' by rules (i)–(iii).

Since $x_{11}=1$, it follows from the stability of (u,v,x) that

$$\alpha_{11}-v_1\geq \alpha_{12}-v_2. \tag{1}$$

Since $x'_{12}=1$, the stability of (u',v',x') implies

$$\alpha_{12}-v'_2\geq \alpha_{11}-v'_1. \tag{2}$$

Adding (1) and (2) gives us that $v'_1-v_1\geq v'_2-v_2$.

In the same manner, from the fact that $x_{22}=1$ and $x'_{23}=1$, we obtain

$$v'_2-v_2\geq v'_3-v_3.$$

Repeating this procedure we get that

$$v'_1-v_1\geq v'_2-v_2\geq \cdots \geq v'_s-v_s\geq v'_{s+1}-v_{s+1}.$$

In an analogous way we obtain

$$u'_1-u_1\leq u'_2-u_2\leq \cdots \leq u'_s-u_s.$$

Since (u,v) and (u',v') are arbitrary, we have concluded the proof.

Proof of Theorem 8.18: Consider any path starting from some partner of i^* under some optimal assignment for M^{i^*} and satisfying properties P1, P2, and P3. Let A be the union of all agents belonging to all these such paths. Since i^* is matched under some optimal assignment for M^{i^*}, $A\neq\emptyset$. It is enough to prove the theorem for any such path in A.

Suppose x' is an optimal assignment for M^{i^*} under which i^* is matched to j_1. Let $c=(j_1, i_1, \ldots, j_s, i_s, (j_{s+1}))$ be some oriented path starting from j_1 satisfying properties P1, P2, and P3. Let x be the optimal assignment for M derived from x' by rules (i)–(iii). Let (u', v', x') and (u, v, x) be stable outcomes for M^{i^*} and M, respectively.

Case 1: The path c ends at i_s. So i_s is unmatched under x'. Then $u_s' = 0$ and so $u_s \geq u_s'$. Since (u, v) and (u', v') are arbitrary, $B_{i_s} \leq 0$. From Lemma 8.19 it follows that $B_{i_m} \leq 0$ for all $m = 1, 2, \ldots, s-1$. In particular, $u_m' \leq u_m$ for all $m = 1, 2, \ldots, s-1$. Now, since $x_{mm} = 1$ for all $m = 1, \ldots, s$, we have $u_m + v_m = \alpha_{mm}$ and $u_m' + v_m' \geq \alpha_{mm}$ by stability, from which it follows that $(v_m' - v_m) + (u_m' - u_m) \geq 0$. We already know that $u_m' - u_m \leq 0$. Therefore $v_m' - v_m \geq 0$ for all $m = 1, \ldots, s$, which concludes the proof for this case.

Case 2: The path c ends at j_{s+1}. So $\underline{v}_{s+1}' = 0$. Then j_{s+1} is not assigned under x. Hence $v_{s+1} = 0$ and $v_{s+1}' = v_{s+1}$, which implies $B_{j_{s+1}} = 0$. Hence from Lemma 8.19, $B_{j_m} \geq 0$ and in particular, $v_m' - v_m \geq 0$ for all $m = 1, \ldots, s$. As before, since $x_{s,s+1}' = 1$ we obtain that $(v_{s+1} - v_{s+1}') + (u_s - u_s') \geq 0$.

Hence we have that $u_s - u_s' \geq 0$. This implies that $B_{i_s} \leq 0$, which in turn implies that $B_{i_m} \leq 0$ for all $m = 1, \ldots, s$. Then $u_m' - u_m \leq 0$ for all $m = 1, \ldots, s$ and the proof is complete.

The final result of this section can be thought of as describing how much the entry of an agent i^* can move the core of the game. There will be some agents whose worst core payoff in one of the two games (with and without i^*) is exactly equal to their best core payoff in the other.

Corollary 8.20 (Mo). *Let $(\bar{u}', \underline{v}')$ be the P-optimal stable payoff for M^{i^*}. Let $(\underline{u}, \bar{v})$ be the Q-optimal stable payoff for M. If i^* is matched under some optimal assignment for M^{i^*}, there exists a nonempty set A of agents in $P \cup Q$ such that*

(a) *if a P-agent i is in A, then $\bar{u}_i' = \underline{u}_i$*
(b) *if a Q-agent j is in A, then $\underline{v}_j' = \bar{v}_j$.*

Proof: Construct A in the same way as in Theorem 8.18. We know that $(u^*, \underline{v}')$ is a stable payoff for M, where $u_i^* = \bar{u}_i'$, for all i in P. Then, from the Q-optimality of $(\underline{u}, \bar{v})$ it follows that $\underline{v}_j' \leq \bar{v}_j$ for all $j \in Q$ and $\bar{u}_i' \geq \underline{u}_i$ for all $i \in P$. Now use Theorem 8.18 (strong dominance) to get

$$\underline{u}_i \geq \bar{u}_i' \geq \underline{u}_i \quad \text{for all } i \text{ in } A,$$

$$\bar{v}_j \leq \underline{v}_j' \leq \bar{v}_j \quad \text{for all } j \text{ in } A,$$

from which it follows that $\underline{u}_i = \bar{u}_i'$ and $\bar{v}_j = \underline{v}_j'$ for all i and j in A.

8.6 Guide to the literature

The assignment game is a model formulated and studied by Shapley and Shubik (1972). All the initial results presented here are from that paper, although the proofs are not the same.

Section 8.3 follows the paper of Demange, Gale, and Sotomayor (1986). The auction mechanism is a version of the Hungarian algorithm for the assignment problem (see, e.g., Dantzig 1963). Hall's theorem is due to P. Hall (1935). Two simple proofs of Hall's theorem are given by Gale (1960) in a book that deals with some other linear assignment models. Demange, Gale, and Sotomayor (1986) also consider another auction mechanism that is a version of the deferred acceptance algorithm proposed by Crawford and Knoer (1981), which in turn is a special case of the algorithm of Kelso and Crawford that we considered in Section 6.2. They make precise the observation of Crawford and Knoer that the outcome of this algorithm for the discrete case can be made to approximate arbitrarily closely the buyer-optimal core outcome for the continuous assignment problem. They showed that the final price obtained in this algorithm has upper and lower bounds that can be made arbitrarily close to the minimum equilibrium price. Mo (1988b) considers a generalization of the Hungarian algorithm in this context by defining an overdemanded set that contains all minimally overdemanded sets, which he calls the largest pure overdemanded set. Mo, Tsai, and Lin (1988) observe that Demange, Gale, and Sotomayor (1986) incorrectly assert that an algorithm in Gale (1960) computes a minimal overdemanded set, but they show that a variant of this algorithm computes the largest pure overdemanded set.

Section 8.4 follows the independent work of Leonard (1983) and Demange (1982). The proof of Theorem 8.15 presented here follows that of Demange, whereas our proof of Theorem 8.16 follows Leonard's paper.

Section 8.5 on new entrants follows the work of Mo (1988a), although the proofs are somewhat different. Proposition 8.17 will be proved for a generalization of the assignment model in the next chapter. A particular case was also proved (for a different generalization of the assignment model) by Kelso and Crawford (1982). As mentioned in connection with Theorem 2.25, earlier related results in the context of linear programming are found in Shapley (1962). Although most of the results in the literature concern the effect of new entrants on the core of the game, Mo and Gong (1989) show that the same qualitative effects (i.e., agents on the same side of the market are substitutes and agents on opposite sides are complements) are found using the Shapley value. (The Shapley value selects a unique imputation for each game with side payments: See Shapley 1953b, and the collection of papers on the subject in Roth 1988b.)

Becker (1981), who uses the assignment model to study marriage and household economics, makes use of the fact that stable outcomes all correspond to optimal assignments (and that the optimal assignment is typically unique) to study which men are matched to which women, for different assumptions about how the assignment matrix is derived.

Rochford (1984) characterized certain points in the interior of the core of an assignment game as fixed points of a "rebargaining" process, in which matched pairs are thought of as bargaining over their transfer payments. In Roth and Sotomayor (1988) it was observed that Tarski's celebrated fixed point theorem (for order-preserving functions from a complete lattice to itself) implies that these interior fixed points in the core share the lattice property of the core, and have P- and Q-optimal elements. A similar rebargaining process was explored for a generalization of the assignment game by Moldovanu (1988). A different formulation of the bargaining process led Crawford and Rochford (1986) to consider outcomes outside of the core. A similarly motivated reformulation by Bennett (1988), however, again led to points in the core. A strategic model of bargaining and matching that yields core points as equilibria is studied in Kamecke (1989).

Geometric properties of the core of assignment games have also received attention. Some measures of the degree to which the core is "elongated," reflecting the polarization of interests between the two sides of the market, have been considered by Quint (1987a). Balinski and Gale (1987) showed that the number of vertices of the core polytope of the assignment game is at most $\binom{2m}{m}$ where $m = \min\{|P|, |Q|\}$. They gave a characterization of the games that realize these numbers when $|P| = |Q|$ and also studied games where $|P| \neq |Q|$.

A presentation of the assignment game as part of a general introduction to game theory is given by Shubik (1984). A number of generalizations and related models have been explored, for example, by Curiel (1988), Curiel and Tijs (1985), Kaneko (1976, 1982), Kaneko and Wooders (1982), Kaneko and Yamamoto (1986), Kamecke (1987), Quint (1987b, 1988a), and Thompson (1980) (who uses a nonstandard definition of the core, however). Sotomayor (1986b) uses a standard definition of the core for Thompson's model, which allows multiple partners, and observes that this model differs in many respects from the assignment game.

Sotomayor (1988) considers two generalizations of the assignment game that allow many-to-many matching. In a model that keeps track of the individual transactions between each firm and its workers, the results parallel those of Chapter 5. That is, the relationship between the results for her model and the results presented in this chapter and the next for the one-to-one case are very similar to the relationship we have observed between the college admissions model and the marriage model. However when only the aggregate payoffs to each agent are modeled, the core no longer corresponds to the set of pairwise stable outcomes.

Quint (1988b) considers some conditions under which games with more than two "sides" may have nonempty cores.

Samet and Zemel (1984) consider the relationship between linear programs and their duals in connection with the core of side payment games whose coalitional function value for each coalition is given by a linear program. See also Owen (1975) who studies games of this kind.