The Impossibility of the Counterfactual Computation for all Possible Outcomes.

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Recent proposal for counterfactual computation [Hosten et al., Nature, 439, 949 (2006)] is analyzed. It is argued that the method does not provide counterfactual computation for all possible outcomes. The explanation involves a novel paradoxical feature of pre- and post-selected quantum particles: the particle can reach a certain location without being on the path that leads to this location.

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The foundations of quantum theory which is almost 100 years old is still a subject of a heated debate. Where is a quantum particle passing through a Mach-Zehnder interferometer? Is it present in two places simultaneously? If not, how to explain the interference at the output? Recently, Hosten et al. [1] claimed that quantum mechanics allows to build another counterintuitive device: a computer which yields the outcome of a computation without running. It is an improvement of the device: a computer which yields the outcome of a computation (CFC) proposal [2]. Two identical optical cavities are connected by a common wall - an almost 100% reflection mirror. During the period of the oscillation in a cavity, a photon starting in the left \( |L\rangle \) cavity evolves into a superposition of being in the left and in the right \( |R\rangle \) cavities according to the following law:

\[
|L\rangle \rightarrow \cos \alpha |L\rangle + \sin \alpha |R\rangle, \\
|R\rangle \rightarrow -\sin \alpha |L\rangle + \cos \alpha |R\rangle, \quad \alpha \ll 1. 
\]  

(1)

The computer device is placed inside the right cavity, see Fig. 1. A single photon localized wavepacket starts in cavity \( L \) and it is left to evolve \( N \) periods of the oscillation of a single cavity. It is arranged that \( N\alpha = \frac{\pi}{2} \).

Assume first that the outcome is 0. After one period, the state of the photon is \( \cos \alpha |L\rangle + \sin \alpha |R\rangle \). It is easy to see that after \( n \) periods, the state is \( \cos n\alpha |L\rangle + \sin n\alpha |R\rangle \). When the number of periods equal \( N = \frac{\pi}{2\alpha} \), the final state of the photon is \( |R\rangle \).

Now assume that the outcome is 1. After one period, the state of the photon is \( \cos \alpha |L\rangle + \sin \alpha |ABS\rangle \). The state \( |ABS\rangle \) signifies the photon absorbed by the computer device on the first round. (The process might include macroscopic amplification, in which case the situation is usually described as a collapse to the state \( |L\rangle \) with probability \( \cos^2 \alpha \), or collapse to the absorption of the photon with probability \( \sin^2 \alpha \).) After \( N \) periods the state is

\[
\cos^N \alpha |L\rangle + \sum_{n=1}^{N} \cos^{n-1} \alpha \sin \alpha |ABS \rangle n. 
\]  

(2)

The final step of the procedure is a measurement which tests the presence of the photon in cavity \( L \). If we find the photon there, we know that the outcome is 1, since, if the outcome is 0, after \( N \) periods it has to be in cavity \( R \). And we know that it was computed counterfactually, since computation of the outcome 1 ends by the

\[
\text{Fig. 1: Jozsa counterfactual computation method. If the outcome is 1, the photon remains in the left cavity, and if the outcome is 0, the photon moves to the right cavity.}
\]
absorption of the photon by the computing device, but the photon was detected by another detector.

If we do not find the photon in cavity \( L \), we are pretty sure (although we cannot be certain) that the outcome is 0, but there is no any reason to claim that in this case there have been a counterfactual computation, since at every period part of the photon wave went through the computer device.

The probability to absorb the photon by the computer device in case the outcome is 1 is \( 1 - \cos^2 N \frac{\pi}{2N} \approx \frac{2N}{\pi} \), so we have constructed a CFC procedure for a single outcome with efficiency which can be made arbitrary close to 100%.

In order to perform a counterfactual computation for both outcomes, Hosten et al. proposed to construct three identical optical cavities using two identical almost 100% reflection mirrors. The computer device is placed inside the third cavity, see Fig. 2.

Again, a single photon localized wavepacket starts in cavity \( A \) and moves toward the almost 100% reflection mirror which separates cavity \( A \) from cavity \( B \). After it bounces on the mirror transmitting a localized packet of an amplitude \( \sin \alpha \) to the cavity \( B \), the mirror between cavities \( A \) and \( B \) is switched to be 100% reflective. It remains to be 100% reflective for \( N \) periods. During this time, the part of the photon wave in cavity \( A \) remains as it is, and the part in cavity \( B \) evolves as in the previous two-cavity case. Thus, if the outcome is 0, the state of the photon becomes

\[
\cos \alpha |A\rangle + \sin \alpha |C\rangle, \tag{3}
\]

and if the outcome is 1, the state of the photon becomes

\[
\cos \alpha |A\rangle + \sin \alpha \left( \cos^N \alpha |B\rangle + \sum_{n=1}^{N} \cos^{n-1} \alpha \sin \alpha |ABS n\rangle \right). \tag{4}
\]

At this stage a measurement is performed which tests the presence of the photon in cavity \( C \). The probability to find the photon in this measurement does not vanish only if the outcome is 0, but even then it is very small: \( p_{\text{fail}} \approx \frac{\pi}{2N} \). If we find the photon, we know the outcome of the computation, but CFC fails, since the photon did pass through the computer. If the outcome is 1, we have even smaller probability of the CFC failure, i.e. absorption of the photon by the computer device. If the photon was not absorbed, then the state of the photon in case of the outcome 0 is just \( |A\rangle \) and in case of the outcome 1, the state is, up to normalization, \( \cos \alpha |A\rangle + \sin \alpha |B\rangle \).

This ends the first subroutine of the computation process.

Now, the mirror between cavities \( A \) and \( B \) is opened again for one bounce of the photon, i.e., it leads to the evolution described by (1), after which it is closed for \( N \) periods as before. The second run of the subroutine, as the first one, ends by the test of the presence of the photon in \( C \). If the outcome is 0, everything is the same as in the first subroutine: probability of the failure of the CFC is again \( p_{\text{fail}} \approx \frac{\pi}{2N} \), and if the photon is not found in \( C \), its final state is \( |A\rangle \). If the outcome is 1, we have a tiny probability for a failure and if the photon is not absorbed by the computer, the final state of the photon is, up to normalization, \( \left( \cos^2 \alpha - \sin^2 \alpha \cos \alpha |A\rangle + (\sin \alpha \cos^2 \alpha + \sin \alpha \cos^2 \alpha) |B\rangle \right) \).

The state is approximately equal to \( \cos 2\alpha |A\rangle + \sin 2\alpha |B\rangle \). After \( N \) rounds the state is approximately equal to \( \cos N\alpha |A\rangle + \sin N\alpha |B\rangle \).

Under this approximation, after \( N \) rounds of the subroutine we end up with the state \( |A\rangle \) if the outcome is 0 and the state \( |B\rangle \) if the outcome is 1. The parameter \( \alpha \) can and should be tuned a little such that this will be an exact and not an approximate statement. Exact, except for a possibility of the failure which can be made arbitrary small, since it is of the order of \( \frac{\pi}{2N} \). The whole procedure ends with the test in which cavity, \( A \) or \( B \), the photon is located, and this yields the outcome of the computation, 0 or 1.

It seems that it is a counterfactual computation. Indeed, if the outcome is 1, we know that the photon was not inside the computer (otherwise it would be absorbed), and, if the outcome is 0, we also apparently can claim that the photon was not inside the computer, because our tests of the cavity \( C \) at the end of each subroutine checked every time when the photon could enter the computer and we found that it did not.

The core of the controversy is that it is possible to make CFC when the process of computation is just the passage of the photon through the device without any change in the device and without absorption of the photon; it corresponds to the CFC of the outcome 0 in the examples above. So, following Hosten et al. [8], let us consider a simpler scheme. It is not optimally efficient and it makes decisive computation only in the case of the outcome 0.

The scheme consists of one Mach-Zehnder interferometer (MZI) nested inside another, and the computer placed in one arm of the inner interferometer, see Fig. 3. The inner interferometer is tuned by a \( \pi \) phase shifter in such a way that if the outcome is 0, i.e., the computer is trans-
parent, there is a destructive interference towards the output beam splitter of the large interferometer. If the outcome is 1, the photon can reach the output beamsplitter of the external MZI, and it is tuned in such a way that there is a destructive interference toward detector $D_1$.

A possible implementation of this system is that the external MZI has two identical beamsplitters which transmit two thirds of the beam and reflect one third. The evolution of the photon passing through such a beamsplitter is:

$$|H\rangle \rightarrow \sqrt{\frac{2}{3}}|H\rangle + \sqrt{\frac{1}{3}}|V\rangle,$$
$$|V\rangle \rightarrow -\sqrt{\frac{1}{3}}|H\rangle + \sqrt{\frac{2}{3}}|V\rangle, \quad (5)$$

where $H$ and $V$ signify horizontal and vertical modes. The inner MZI has two identical half and half beamsplitters. The evolution of the photon passing through such beamsplitter is:

$$|H\rangle \rightarrow \sqrt{\frac{1}{2}}|H\rangle + \sqrt{\frac{1}{2}}|V\rangle,$$
$$|V\rangle \rightarrow -\sqrt{\frac{1}{2}}|H\rangle + \sqrt{\frac{1}{2}}|V\rangle. \quad (6)$$

A simple calculation shows that, indeed, there is a destructive interference toward vertical output mode of the inner interferometer if the computer is transparent, and there is a destructive interference toward $D_1$ if the lower arm of the inner interferometer is blocked by the computer.

Consider now a run of our device in which a single photon enters the interferometer and it is detected by detector $D_1$. In this case we get the information that the result of the computation is 0 and it is apparently a counterfactual computation. It seems that the photon has not passed through the computer since photons passing through the inner interferometer, where the computer is transparent, cannot reach detector $D_1$.

In spite of the fact that it is a very vivid and persuasive explanation, I will argue that it is incorrect. The photon is a quantum pre- and post-selected system, and what is correct for classical systems and quantum pre-selected only systems might be wrong for a pre- and post-selected system.

Let us consider an example of a pre- and post-selected systems, usually known as the “there-box paradox” [9]. A single particle is pre-selected in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle), \quad (7)$$

and post-selected in the state

$$\langle\phi| = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle), \quad (8)$$

where the mutually orthogonal states $|A\rangle$, $|B\rangle$, and $|C\rangle$ denote the particle being in box $A$, $B$, and $C$, respectively. (We use “bra” and “ket” notation to distinguish between standard, forward evolving quantum state and backward evolving quantum state from the post-selection measurement.)

The paradox is that at the intermediate time, the particle is to be found with certainty in box $A$ if searched there and, at the same time, it is to be found with certainty in box $B$ if it is searched there instead. Indeed, if a particle is not found, e.g., in box $B$, then its state collapses to $1/\sqrt{2}(|A\rangle + |C\rangle)$, but this is impossible since this state is orthogonal to the post-selected state (8).

One thing is obvious in this example: there are no grounds to say that the particle was not present at box $B$. Nevertheless, I can "show" that it was not present there using similar arguments to those showing that the photon was not passing through the computer in Hosten et al. example.

In fact, Hosten et al. setup is an implementation of the three-box experiment. Consider three boxes on the way of the photon in the three arms of the nested MZI experiment, Fig. 4. Taking into account the evolution laws (5,6) we see that indeed, quantum state of the photon in these boxes is described by (7), and, given that the computer is transparent, detection by $D_1$ corresponds to post-selection of the state (8).

According to Hosten et al. argument, the box $B$ is empty. But this conclusion is not tenable. We know that

FIG. 3: Hosten et al. counterfactual computation method for outcome 0. The first figure shows that if the outcome is 0, then the photon entering the inner interferometer cannot reach detector $D_1$, and the second figure shows that if the outcome is 1, then no photon at all can reach detector $D_1$. 

\[\text{FIG. 3: Hosten et al. counterfactual computation method for outcome 0. The first figure shows that if the outcome is 0, then the photon entering the inner interferometer cannot reach detector $D_1$, and the second figure shows that if the outcome is 1, then no photon at all can reach detector $D_1$.} \]
FIG. 4: Both forward and backward evolving quantum states are present in all three boxes: A, B, and C; they are described by (7) and (8). However, on the way toward inner interferometer, backward evolving wave is not present and on the way out, the forward evolving wave is not present.

the photon to be found with certainty if searched in B. One can argue that a strong nondemolition measurement of the projection on B changes the physical situation. Then, we can perform a weak measurement [10] of the projection on B which does not change, at the limit, neither forward evolving state (7) nor the backward evolving state (8). This weak measurement requires a pre- and post-selected ensemble, but according to Hosten et al., for all members of the ensemble the photon is not in B, so we should not see any effect in the measurement in B. The experiment, however, will show a different result: In the CFC experiment the weak value, the outcome of the weak measurement of the projection inside the “computer” is 1.

Note that the weak values of the projections on the paths E, F, inside the external interferometer which lead to and from the inner interferometer, do vanish. The probabilities to find in strong nondemolition measurements the photon on the way toward the interferometer and on the way out of the interferometer vanish as well. The photon did not enter the interferometer, the photon never left the interferometer, but it was there! This is a new paradoxical feature of a pre- and post-selected quantum particle.

It is interesting to note that Bohmian interpretation of quantum mechanics does not exhibit such behavior and, in fact, supports the claim of Hosten et al. Bohmian particle has to “ride” on the quantum wave, but there is no quantum wave on path F, from the inner interferometer to $D_1$. Therefore, the (Bohmian) particle detected by $D_1$ did not pass through the interferometer and did not pass through the computer.

It has already been mentioned that weak measurements can lead to observable effects even in the region where the Bohmian particle is not present [11]. The most surprising observable effect of “empty waves” is, however, the so called “surrealistic trajectories” [12]. The present setup can also exhibit such behavior. The interaction with the spin at the vicinity of the computer leads to a spin flip even so the Bohmian particle does not pass near the spin.

This strange behavior of Bohmian trajectories and especially the noncovariance of Bohmian interpretation lead us not to rely on it when we ask questions such as: “Where is the particle?” We think that the Bohmian argument cannot stand against testable results of (weak) measurements. But together with Bohmian trajectories we have to dismiss a very persuasive explanation (Hosten et al. argument) based solely on the forward evolving quantum wave.

Discarding the hidden variables approach, we should be able to answer any physical question based on the complete description of the quantum system which consists (for pre- and post-selected system) of both forward and backward evolving quantum states. It cannot depend on the particular way of preparing and post-selecting these states as it happens to be if we adopt Hosten et al. approach.

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