Relativistic causality and conservation of energy in classical electromagnetic theory

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Causality puts certain constraints on the change of the electromagnetic field due to the change in motion of charged particles. Naive calculations of the electromagnetic energy and the work performed by the electromagnetic fields which take these constraints into account might lead to paradoxes involving the apparent nonconservation of energy. A few paradoxes of this type for the simple motion of two charges are presented and resolved in a quantitative way providing deeper insight into various relativistic effects in classical electromagnetic theory.
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I. INTRODUCTION

Starting from Einstein's work on special relativity,[1] it has become clear that classical electromagnetic theory is consistent with relativity, and no true paradoxes can be found. However, several apparent paradoxes have been extensively discussed and these discussions have enriched our understanding of electromagnetic theory. Some of these controversial topics include hidden momentum,[2] Feynman's disk,[3] the Trouton–Noble experiment,[4] and the 4/3 factor for the self-energy of an electron.[5] Here we discuss some examples of the simple motion of two charged particles which, when analyzed naively, lead to paradoxical conclusions. Our analysis is relevant to recent discussions of covariance in electromagnetic theory.[6–10]

We present our examples in the form of five paradoxes, the resolution of which is based on four different effects. We present the paradoxes without giving hints to the effects that resolve them, and we ask readers who spot the effects immediately to bear with us because they still may find the quantitative resolution of the paradoxes interesting. The first two paradoxes are based on the same effect. We present them in Secs. II and III and resolve them in Secs. IV and V. In Sec. VI we again analyze the setup of Paradox II and present Paradox III, which we resolve in Sec. VII. In Secs. VIII–XI we present and resolve the last two paradoxes and we summarize our results in Sec. XII.

II. PARADOX I: GAINING ENERGY FROM RETARDATION OF ELECTRIC FIELD

Two particles of charge q are initially separated by a distance l. We consider two ways of bringing the particles, initially and finally at rest, to a smaller distance $l-x$ (see Fig. 1).

(i) We move one particle the distance x toward the other particle. The work required for this move is

$$W^{\mathrm{i}}=U_{\mathrm{new}}-U_{\mathrm{old}}=\frac{q^2}{l-x}-\frac{q^2}{l}. \tag{1}$$

(ii) We move both particles toward each other by the distance $x/2$. We move them simultaneously and fast enough such that the motion of each particle ends before the signal about this motion can reach the location of the other particle. In this case, the external work done should be the sum of the amounts of work performed by external forces exerted on the two particles calculated as if the other particle has not moved:

$$W^{\mathrm{ii}}=W_1+W_2=2\left(\frac{q^2}{l-x/2}-\frac{q^2}{l}\right). \tag{2}$$

After the procedure is ended, we obtain the same situation in both cases, but we applied less work when we moved both particles: $W^{\mathrm{ii}}<W^{\mathrm{i}}$.

We can obtain energy equal to the work W^{i} back from the system when we reverse process (i), moving one of the charges to the original separation l. We can repeat the cycle consisting of process (ii) and the reversed process (i) gaining each time the energy:

$$W^{\mathrm{i}}-W^{\mathrm{ii}}\approx\frac{q^2x^2}{2l^3}. \tag{3}$$

Of course, there must be an error in the above argument. We have not taken all relevant effects into account. However, before explaining this paradox, we present and resolve Paradox II, which is simpler to analyze and the resolution of which follows from the same effect.

III. PARADOX II: CONSERVATION OF ENERGY FOR TWO STOPPING PARTICLES

Consider two particles of charge q and mass m located on the x axis and separated by the distance l; they are moving in the x direction with a constant velocity v. At time $t_1=0$, we stop the first particle and at time $t_2=t$ we stop the second particle (see Fig. 2). The time t is sufficiently small so that signals about the change of the velocity of the first particle cannot reach the second while it is still moving. We also require that τ, the time of deceleration, is very small. These requirements impose the following constraint:

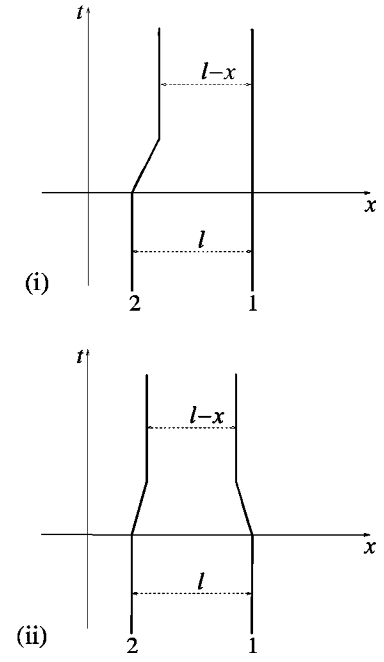

Fig. 1. Space–time diagram of the motion in the two processes: (i) one particle moves, (ii) two particles move.

$$\tau \ll t < \frac{l}{c+v}. \tag{4}$$

Let us consider conservation of energy for this process. The initial energy should be equal to the final energy plus the work done by the forces that the particles exert on external systems:

$$E_{\text{in}} = E_{\text{fin}} + W_1 + W_2 + \widetilde{W}, \tag{5}$$

where W_1 and W_2 are the work of the forces that the particles exert during the process of stopping; $\widetilde{W}$ is the work performed by the second particle moving with velocity v during the time that the other particle is at rest. Of course, no work is performed when both particles are at rest, and the net work vanishes during the time when both particles are moving with velocity v.

For a relativistic analysis, we include the rest mass energy. Thus, the final energy of the system is

$$E_{\text{fin}} = 2mc^2 + \frac{q^2}{l-x}, \tag{6}$$

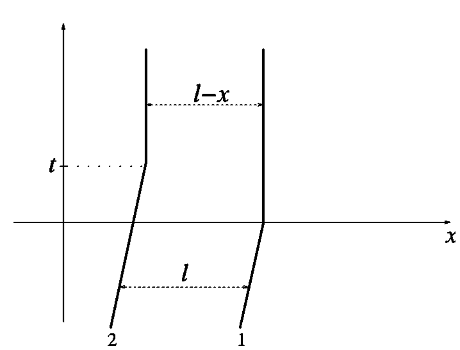

Fig. 2. Space–time diagram of the motion of the two particles.

where x is the change in the separation between the charges: $x = vt$.

When a particle moves with constant velocity, the total force exerted on it is zero. Therefore, the force it exerts on an external system is equal to the electromagnetic force the other particle exerts on it. Because the distance between the particles is l in the laboratory frame, in the Lorentz frame in which the charges are at rest, the distance between them is γl (where $\gamma \equiv 1/\sqrt{1-v^2/c^2}$). In the reference frame in which the charges are at rest, the force is given by Coulomb's law, and the Lorentz transformation between the force in the x direction in the rest frame and the force in the laboratory frame is $F_x = F'_x$. Hence, the forces the particles exert on the external systems are

$$F_{1x} = -F_{2x} = \frac{q^2}{(\gamma l)^2}. \tag{7}$$

Thus, the work $\widetilde{W}$ is

$$\widetilde{W} = -\frac{q^2 x}{(\gamma l)^2}. \tag{8}$$

Therefore, Eq. (5) for the conservation of energy becomes:

$$E_{\text{in}} = 2mc^2 + \frac{q^2}{l-x} + W_1 + W_2 - \frac{q^2 x}{(\gamma l)^2}. \tag{9}$$

The initial energy E_{in} obviously does not depend on x. Due to causality, the work W_1 and W_2 do not depend on x either. Therefore, Eq. (9) represents a paradox: it must be true for all allowed values of x, but it cannot, because the two x-dependent terms do not balance each other.

IV. RESOLUTION OF PARADOX II: INTERFERENCE OF RADIATION

Because of the constraint (4), the process of stopping charged particles cannot be arbitrarily slow. Therefore, we should expect a significant contribution due to radiation which we have not taken into account. During the process of stopping, the magnitude of the acceleration of the charges is $a = v/\tau$. According to the Larmor formula, the total energy radiated by a single charge during the whole process of stopping is

$$R_1 = R_2 = \frac{2}{3}\frac{q^2 a^2}{c^3}\tau = \frac{2}{3}\frac{q^2 v^2}{c^3 \tau}. \tag{10}$$

It is easy to see that the x-dependent term in Eq. (9), which we have to balance, is much smaller than R_1 and R_2. However, everything that happens in the close vicinity of the charges cannot depend on x, and, in particular, the radiation that each charge emits does not depend on x, so how can the radiation energy balance the x dependent terms in the equation of conservation of energy? The effect is due to the interference of the radiation. The total radiated energy is

$$R_{\text{tot}} = R_1 + R_2 + R_{\text{int}}. \tag{11}$$

The interference term depends on x and restores the balance. Next we will show this effect in detail.

In our quantitative analysis we show that the leading x-dependent term of Eq. (9) is canceled by the leading x-dependent term of R_{int}. We expand the x-dependent terms of Eq. (9) and find

$$\frac{q^2}{l-x} - \frac{q^2 x}{(\gamma l)^2} = \frac{q^2}{l} + \frac{v^2 q^2 x}{c^2 l^2} + \frac{q^2 x^2}{l^3} + \cdots. \tag{12}$$

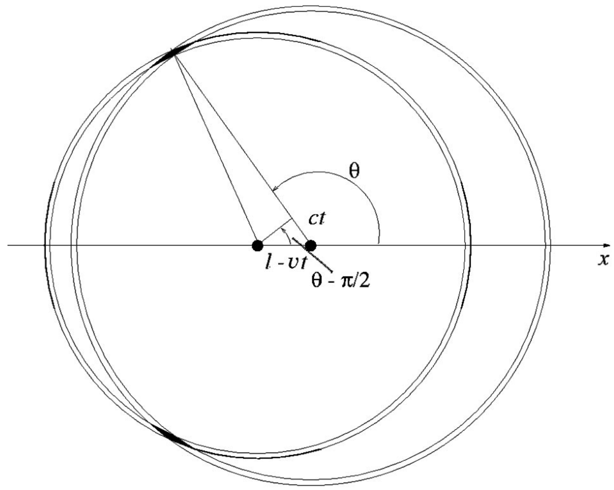

Fig. 3. Electromagnetic radiation of the two stopping particles. The area of constructive interference of the radiation field is painted in black.

Due to the constraint (4), we have

$$v\tau \ll x < \frac{lv}{c+v}. \tag{13}$$

Therefore, we will consider x to be the same order of magnitude as lv/c. In this case $q^2v^2x/c^2l^2 \ll q^2x^2/l^3$, and the leading term that has to be canceled due to the radiation is

$$\frac{q^2x^2}{l^3}. \tag{14}$$

The radiation of the stopping charge propagates inside a spherical shell of width $c\tau$, and the energy flux $\mathbf{S}$ is given by[11]

$$\mathbf{S} = \frac{q^2a^2\sin^2\theta}{4\pi c^3r^2}\hat{\mathbf{r}}, \tag{15}$$

where r is the radius of the shell and θ specifies the direction relative to the x axis. Because we have two accelerated charges, the radiation fields due to the two charges interfere in the region of the overlap (see Fig. 3). A complete overlap takes place at an angle θ defined by

$$\sin\left(\theta - \frac{\pi}{2}\right) = \frac{ct}{l - vt} = \frac{cx}{v(l-x)}. \tag{16}$$

Because the width of the shells is $c\tau$, the overlap vanishes once the deviation exceeds

$$\delta\theta = \frac{c\tau}{(l-x)\sin\theta}, \tag{17}$$

which is obtained by equating $c\tau$ to the differential of the difference between the paths of the two fields:

$$c\tau = \delta\left[(l-x)\sin\left(\theta - \frac{\pi}{2}\right)\right] = [(l-x)\sin\theta]\delta\theta. \tag{18}$$

Because the amplitudes of the fields in the region of the overlap are approximately equal, the total energy radiated in the direction of the overlap is twice as much, due to the interference, than if the two charges were radiating separately. At the interval $[(\theta-\delta\theta),(\theta+\delta\theta)]$, the overlap increases and then decreases linearly. Therefore, the interference term of the radiation energy is

$$R_{\text{int}} = 2\frac{q^2a^2\sin^2\theta}{4\pi c^3r^2}2\pi r\sin(\theta)\, r\tau\int_{-\delta\theta}^{\delta\theta}\frac{\delta\theta - |\phi|}{\delta\theta}d\phi$$
$$= \frac{q^2v^2\sin^2\theta}{c^2(l-x)}. \tag{19}$$

If we use $\sin^2\theta = 1 - \sin^2(\theta - \pi/2)$ and Eq. (16) and expand to lowest order in the parameter $x/l \approx v/c$, we obtain:

$$R_{\text{int}} = \frac{q^2v^2}{c^2(l-x)} - \frac{q^2x^2}{(l-x)^3} \approx \frac{q^2v^2}{c^2l} - \frac{q^2x^2}{l^3}. \tag{20}$$

Thus, we see that up to order v^2/c^2, the x dependent term (14) is canceled. This reasoning resolves Paradox II.

Another paradox of energy nonconservation for a system of two charged particles when radiation is neglected has been considered in Ref. 12, but the resolution of their paradox by taking into account radiation was shown only qualitatively.[13]

V. RESOLUTION OF PARADOX I

The radiation energy (10) is much larger than the term (3) which we have to compensate. However, we would like to have a quantitative resolution of this paradox that shows how the missing term (3) arises from the calculation of the radiation energy. To obtain a quantitative result we specify how we perform the processes described in Sec. II.

In case (i), we accelerate particle 1 for a small time τ until it reaches velocity v. Then it moves the distance x with constant velocity. Finally, it stops in the same manner as it was accelerated. In case (ii), both particles reach speed v and stop at time t after going the distance $x/2$. The time t is short enough such that during its motion, each particle cannot receive a signal about the motion of the other particle. Thus,

$$\tau \ll t = \frac{x}{2v} < \frac{l}{c+v}. \tag{21}$$

In case (i) the same amount of radiation energy is created due to the acceleration and due to the stopping of the particle and, therefore, it is twice the amount given by the Larmor formula (10):

$$R^{\text{i}} = \frac{4}{3}\frac{q^2v^2}{c^3\tau}. \tag{22}$$

In case (ii) there are four events in which the velocity of a particle is changed by amount v and, therefore, there are four spherical shells of radiation field of width τc (see Fig. 4). The radiation energy is four times the Larmor energy (10) with a correction due to interference. The interference is due to the radiation emitted during the acceleration of the two particles R_{aa}, the deceleration of the two particles R_{dd}, the acceleration of the first and deceleration of the second R_{ad}, and the acceleration of the second and deceleration of the first R_{da}. These four terms can be calculated in the same way as we have calculated the interference of the radiation energy of two stopping charged particles in Sec. IV.

Because the accelerations of the particles are performed simultaneously, the direction of the interference R_{aa} is $\theta = \pi/2$. This is the direction of maximal power of radiation energy, see Eq. (15). The range of the angles for which there

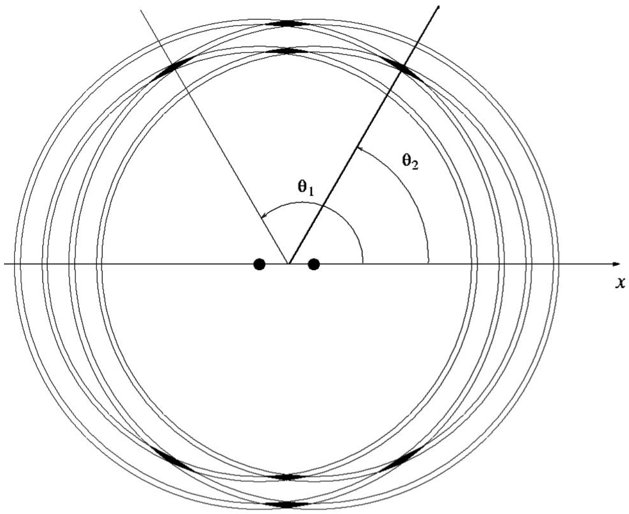

Fig. 4. Electromagnetic radiation of the two charged particles which are simultaneously accelerated toward each other and after time t stopped, case (ii). The shadowed area signifies destructive interference and the area painted in black signifies constructive interference.

is interference is given by Eq. (17) and, thus, similarly to the derivation of Eq. (20), we find that the interference term due to simultaneous acceleration is

$$R_{\mathrm{aa}}=-\frac{q^2v^2}{c^2l}, \tag{23}$$

where the minus sign (destructive interference) occurs because particles accelerate in opposite directions. The second term of Eq. (20) does not appear because simultaneity corresponds to $x=0$ in the notation of Sec. IV. The interference term due to simultaneous deceleration is the same, $R_{\mathrm{dd}}=R_{\mathrm{aa}}$.

The interference between acceleration of the first and deceleration of the second particle takes place in the direction $\hat{\theta}_1$ defined by

$$\sin\left(\theta_1-\frac{\pi}{2}\right)=\frac{ct}{l-vt}, \tag{24}$$

and the interference between acceleration of the second particle and deceleration of the first particle takes place in the direction $\hat{\theta}_2$ defined by

$$\sin\left(\frac{\pi}{2}-\theta_2\right)=\frac{ct}{l+vt}. \tag{25}$$

Now we can use Eq. (19) again, taking into account that the particles stop after going the distance $vt=x/2$. We make an appropriate approximation and obtain:

$$R_{\mathrm{ad}}=R_{\mathrm{da}}=\frac{q^2v^2}{c^2l}-\frac{q^2x^2}{4l^3}. \tag{26}$$

If we sum all the contributions, we find that the radiation energy for process (ii) is

$$R^{\mathrm{ii}}=\frac{8}{3}\frac{q^2v^2}{c^3\tau}-\frac{2q^2v^2}{c^2l}+2\left(\frac{q^2v^2}{c^2l}-\frac{q^2x^2}{4l^3}\right)$$
$$=\frac{8}{3}\frac{q^2v^2}{c^3\tau}-\frac{q^2x^2}{2l^3}. \tag{27}$$

Now we are able to analyze Paradox I taking into account the radiation energy. In case (i) the work performed by the external forces should include the radiation energy (22). Thus, instead of Eq. (1), we obtain:

$$W^{\mathrm{i}}=U_{\mathrm{new}}-U_{\mathrm{old}}+R^{\mathrm{i}}=\frac{q^2}{l-x}-\frac{q^2}{l}+\frac{4}{3}\frac{q^2v^2}{c^3\tau}. \tag{28}$$

In case (ii) we have to calculate the work taking into account the causality argument: each particle "does not know" that the other particle moved. Therefore, the work against the field and the radiated energy should be calculated as if the other particle had not moved. The work is twice the amount of work in case (i) with the change of $x\rightarrow x/2$. Thus, instead of Eq. (2), we obtain:

$$W^{\mathrm{ii}}=W_1+W_2=2\left(\frac{q^2}{l-\frac{x}{2}}-\frac{q^2}{l}+\frac{4}{3}\frac{q^2v^2}{c^3\tau}\right). \tag{29}$$

Clearly we cannot gain energy by constructing a machine with a cycle of process (ii) and reversed process (i). The work required for the reversed process (i) is

$$W^{\tilde{\mathrm{i}}}=\frac{q^2}{l}-\frac{q^2}{l-x}+\frac{4}{3}\frac{q^2v^2}{c^3\tau}. \tag{30}$$

Thus, the work during the whole cycle is

$$W_{\mathrm{tot}}=W^{\tilde{\mathrm{i}}}+W^{\mathrm{ii}}=\frac{q^2}{l}-\frac{q^2}{l-x}+\frac{4}{3}\frac{q^2v^2}{c^3\tau}$$
$$+\left(\frac{q^2}{l-\frac{x}{2}}-\frac{q^2}{l}+\frac{4}{3}\frac{q^2v^2}{c^3\tau}\right)$$
$$\approx-\frac{q^2x^2}{2l^3}+\frac{4q^2v^2}{c^3\tau}. \tag{31}$$

This work is greater than zero, because the radiation term is much larger than the gain in the potential energy, as can be seen explicitly using Eq. (21). However, even if we collect the radiation energy, we still cannot gain energy. Indeed, the total radiation energy is

$$R_{\mathrm{tot}}=R^{\mathrm{i}}+R^{\mathrm{ii}}=\frac{4}{3}\frac{q^2v^2}{c^3\tau}+\frac{8}{3}\frac{q^2v^2}{c^3\tau}-\frac{q^2x^2}{2l^3}$$
$$=\frac{4q^2v^2}{c^3\tau}-\frac{q^2x^2}{2l^3}. \tag{32}$$

Our calculations have shown (up to order v^2/c^2) that during the complete cycle $W_{\mathrm{tot}}=R_{\mathrm{tot}}$. This reasoning completes the analysis of Paradox I.

VI. PARADOX III: ANOTHER LOOK AT THE CONSERVATION OF ENERGY FOR TWO STOPPING PARTICLES

Let us return to Paradox II in which two moving particles stop at different times. We have resolved one apparent contradiction concerning the conservation of energy, but we have not checked if there are other difficulties. We next present another problem that appears in plausible but naive calculations.

The equation of conservation of energy (9) should be corrected by adding the radiation energy R:

$$E_{\rm in}=2mc^2+\frac{q^2}{l-x}+W_1+W_2-\frac{q^2x}{(\gamma l)^2}+R. \tag{33}$$

Our approach to finding the initial energy is to find the total energy of the charges in their rest frame E_0 and multiply it by the factor γ:

$$E=\gamma E_0. \tag{34}$$

In the rest frame of the moving particles the distance between the particles is γl. Therefore, the total initial energy is

$$E_{\rm in}=\gamma\left(2mc^2+\frac{q^2}{\gamma l}\right). \tag{35}$$

We now consider the situation in which particle 2 stops just before information about the stopping of particle 1 can reach it. This situation corresponds to

$$x=vt=\frac{vl}{c+v}. \tag{36}$$

For this choice of x there is no interference between radiation fields from different particles. Indeed, the overlap of the radiation fields takes place only in the direction $\theta=\pi$, and in this direction the amplitude of the radiation field vanishes (see Eq. (15)). Therefore,

$$R=R_1+R_2. \tag{37}$$

We substitute Eqs. (35)–(37) into Eq. (33) and obtain the following equation of conservation of energy for two particles:

$$2\gamma mc^2+\frac{q^2}{l}=2mc^2+\frac{q^2}{l}\left(1+\frac{v}{c}\right)+W_1+W_2 -\frac{q^2}{l}\left(\frac{v}{c}-\frac{v^2}{c^2}\right)+R_1+R_2. \tag{38}$$

We can test the consistency of Eq. (38) with the equations of conservation of energy for each particle:

$$\gamma mc^2=mc^2+W_1+R_1, \tag{39a}$$

$$\gamma mc^2=mc^2+W_2+R_2. \tag{39b}$$

If we subtract Eq. (39) from Eq. (38), we find that the energy at the end of the process is larger than the energy at the beginning by a factor

$$\frac{q^2v^2}{lc^2}. \tag{40}$$

We have obtained another paradox.

VII. RESOLUTION OF PARADOX III: LORENTZ TRANSFORMATION FOR ELECTROMAGNETIC ENERGY

Paradox III arises from the error that we made in the calculation of the initial energy. Equation (34) is, of course, correct when the system is an elementary particle. It also is true for a composite *isolated* system, but the two charges moving with constant equal velocities are not an isolated system. We have to consider the whole system, the charges together with the external system that keeps the distance between the charges unchanged. Note the similarity with the Poincaré observation that there must be nonelectromagnetic stresses in a model of a charged particle, and that these stresses solve the problem of covariance of the energy and momentum for such a model.[14]

To obtain the correct transformation of the electromagnetic energy from the rest frame to the moving frame, we consider two charges connected by a rigid rod. The energy of the whole system, charges and rod, transforms according to Eq. (34). Therefore, the anomalous term in the transformation of the electromagnetic energy equals the negative of the anomalous part in the mechanical energy of the rod. The latter is easier to calculate and we will do it now.

To calculate the expression for the transformation of the energy of the rod, we express the energy as a volume integral of the energy density $\int u\,dv$ and use the Lorentz transformation for the energy density, the component of the energy-stress tensor T_{00}:

$$u=\gamma^2\left(u'+\frac{v}{c^2}S'_x-\frac{v^2}{c^2}\sigma'_{xx}\right), \tag{41}$$

where $\mathbf{S}$ is the energy flux and σ is the stress tensor. The transformation of the energy due to the first term leads to the usual expression (34); the energy density is multiplied by γ^2, but due to the Lorentz contraction the volume is multiplied by the factor γ^{-1}. The second term does not contribute because the energy flux in the rest frame vanishes. Therefore, only the last term contributes to the anomalous term. The tension in the rod prevents the charges, separated by the distance γl, from moving; therefore, it equals $q^2/\gamma^2 l^2$. Thus, in the rest frame of the rod, the stress tensor component is

$$\sigma'_{xx}=\frac{q^2}{s\gamma^2 l^2}, \tag{42}$$

where s is the cross section of the rod. Therefore, the contribution to the energy in the laboratory frame due to the tension of the rod is

$$-\frac{\gamma^2v^2}{c^2}\int\sigma'_{xx}dv=-\frac{\gamma^2v^2}{c^2}\sigma'_{xx}ls=-\frac{v^2q^2}{c^2l}. \tag{43}$$

The correction to the electromagnetic energy is the negative of Eq. (43), and, therefore, the total initial energy is (instead of Eq. (35)):

$$E_{\rm in}=\gamma 2mc^2+\frac{q^2}{l}\left(1+\frac{v^2}{c^2}\right). \tag{44}$$

The correction cancels the unbalanced term (40) and resolves Paradox III.

VIII. PARADOX IV: ACCELERATING PARTICLES FROM REST

Let us now consider the simultaneous acceleration of two charged particles from rest to velocity v. In the frame of reference moving with velocity v, this process is the deceleration of the particles from velocity v to rest that we analyzed above. However, the transformation from one frame to another might be a difficult task, and, as in many other examples,[15–17] an analysis in a different Lorentz frame allows one to see new physical phenomena; in our case it provides yet another possibility of making an error leading to a paradox.

The equation of conservation of energy for the two particles that takes into account the radiation energy (compare with Eq. (5)) is

$$E_{\rm in}=E_{\rm fin}+W_1+W_2+\tilde{W}+R_1+R_2+R_{\rm int}. \tag{45}$$

The initial energy of the particles is

$$E_{\rm in}=2mc^2+\frac{q^2}{l}. \tag{46}$$

Because the final state of the particles (motion with velocity v and separation l) is identical to the initial state of the particles in the setup discussed in Sec. VII, the final energy $E_{\rm fin}$ is equal to the right-hand side of Eq. (44).

Because the charges start to move together, it seems that the net work is done only during the acceleration interval, that is,

$$\tilde{W}=0. \tag{47}$$

The radiated energy during the acceleration should be the same as in the process of stopping (see Eq. (20)). For simultaneous acceleration $x=0$, and hence the interference term is

$$R_{\rm int}=\frac{q^2v^2}{c^2l}. \tag{48}$$

The equations of conservation of energy for each particle are

$$mc^2=\gamma mc^2+W_1+R_1, \tag{49a}$$

$$mc^2=\gamma mc^2+W_2+R_2. \tag{49b}$$

Then, by substituting for all the terms in Eq. (45) and subtracting the single-particle equations (49), we find that the energy at the end of the process is larger than the initial energy by a factor

$$\frac{2q^2v^2}{lc^2}. \tag{50}$$

We have reached another paradox.

IX. RESOLUTION OF PARADOX IV: RETARDED FIELDS

The error we made in Sec. VIII is more transparent. It appears in the sentence stating that the only net work of the charges is done during the acceleration interval. It is true that in the case of particles moving with constant velocity, the net work of moving charges vanishes. However, at the beginning of the motion, the fields in the vicinity of the charges are different from the field of the uniformly moving charges: each particle feels the *static* field of the other particle (that is, as if it has not moved) until the signal from the motion of the other particle can arrive.

Let us calculate the contribution to the work due to the forces between the particles. Particle 2 moves in the static field of particle 1 during the time $t=l/(c+v)$ after which it feels the field of moving particle 1, which is q/γ^2l^2. Similarly, particle 1 moves in the static field of particle 2 during the time $t'=(l/c)-v$, after which it feels the field of moving particle 2. After time t', there is no contribution to the net work due to the forces between the two particles. Until time t', particle 1 covers the distance $x'=vt'$ in the static field of particle 2. Therefore, the contribution to the work from particle 1 is

$$\frac{q^2}{l}-\frac{q^2}{l+x'}=\frac{vq^2}{cl}. \tag{51}$$

The work performed by particle 2 until time t' has two parts. Until time t, it is

$$\frac{q^2}{l}-\frac{q^2}{l-x}=-\frac{vq^2}{cl}. \tag{52}$$

Between time t and t', it feels a constant field so the contribution to the work is

$$-\frac{q^2}{\gamma^2l^2}v(t'-t)=-\frac{2v^2q^2}{c^2l}. \tag{53}$$

If we sum all the contributions, Eqs. (51)–(53), we find that the net work performed by the particles during the motion with constant velocity is

$$\tilde{W}=-\frac{2v^2q^2}{c^2l}. \tag{54}$$

This term cancels the unbalanced term, Eq. (50), and restores the balance in the equation of conservation of energy.

X. PARADOX V: ACCELERATING PARTICLES MOVING IN PARALLEL

Let us consider the acceleration of two charged particles lined up along the y axis (transverse configuration) instead of along the x axis (longitudinal configuration). The particles accelerate simultaneously from rest to the velocity v in the x direction. The expression for the initial energy is again given by Eq. (46). However, the final energy is different:

$$E_{\rm fin}=\gamma\left(2mc^2+\frac{q^2}{l}\right). \tag{55}$$

Indeed, in the rest frame of the moving particles, the distance between them is l. In this case the electromagnetic energy is transformed in the usual way as in Eq. (34) because the energy of the composite system of charges and the rod connecting the charges is transformed according to Eq. (34), and the energy of the rod with the tension in the y direction is transformed according to Eq. (34). The anomalous behavior of the rod in the previous case followed from the presence of the $\sigma_{\rm xx}$ component of the stress tensor, which vanishes in the present transverse configuration.

The interference term of the radiation energy also is modified. In the longitudinal configuration, the interference (the overlap of the radiation fields) takes place in the directions that have an angle $\theta=\pi/2$ relative to the direction of the acceleration. In the transverse configuration, the interference is in the directions perpendicular to the y axes, and the angle θ attains all values. Therefore, the intensity of the radiation fields is not always maximal, and instead, it is proportional to $\sin^2\theta$ (see Eq. (15)). Averaging over θ reduces the interference term relative to that of the longitudinal configuration, Eq. (48), by a factor of 2:

$$R_{\rm int}=\frac{v^2q^2}{2c^2l}. \quad (56)$$

During the uniform motion the charges do not exert forces in the direction of motion, but in the transition period, when the charges move in the static field, there is a small component of the force in the direction of motion. The particles move in the static field during the time t which satisfies

$$ct=\sqrt{l^2+t^2v^2}. \quad (57)$$

The solution of Eq. (57) is $ct=\gamma l$. Therefore, the total work that the two particles perform is

$$\tilde{W}=2\left(\frac{q^2}{l}-\frac{q^2}{\gamma l}\right). \quad (58)$$

Of course, the single-particle equations of conservation of energy remain the same. Thus, if we include all the contributions to the conservation of energy equation (45) and subtract Eq. (49), we find:

$$\frac{q^2}{l}=\frac{\gamma q^2}{l}+\frac{2q^2}{l}\left(1-\frac{1}{\gamma}\right)+\frac{q^2v^2}{2c^2l}. \quad (59)$$

We again find a contradiction: The energy at the end of the process is larger than the initial energy. If we calculate up to second-order in v^2/c^2, we find three contributions. The increase in the potential energy contributes $v^2q^2/2c^2l$, the work of the static fields during the transition period contributes v^2q^2/c^2l, and the interference of radiation contributes $v^2q^2/2c^2l$. All terms together contribute

$$\frac{2v^2q^2}{c^2l}. \quad (60)$$

We have reached another paradox.

XI. RESOLUTION OF PARADOX V: THE WORK OF THE RADIATION FIELDS

The effect we missed in Sec. X is probably the most subtle one. We have not taken into account the work of the radiation field. The electric field at the point $\mathbf{r}$ (relative to the charge) due to the radiation of the charge q moving with acceleration $\boldsymbol{a}$, is[11]

$$\mathbf{E}=\frac{q}{c^2r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\boldsymbol{a}). \quad (61)$$

For our configuration, the field is

$$\mathbf{E}=-\frac{qa}{c^2l}\hat{x}. \quad (62)$$

This field exerts a force during the time τ during which the particle moves a distance τv. If we take into account that $a=v/\tau$, we find that the force of the radiation field changes the energy of each particle by

$$W_{\rm rad}=-\frac{q^2v^2}{c^2l}. \quad (63)$$

Both particles lose energy in this way, and, therefore, we lose $2v^2q^2/c^2l$. This loss of energy cancels the unbalanced term, Eq. (60). Note that in the longitudinal configuration, the work of the radiation fields vanishes because the radiation fields at the locations of the particles vanish.

XII. CONCLUSIONS

We have analyzed some relativistic features of classical electromagnetic theory and demonstrated quantitatively the relevance of several effects due to the conservation of energy. These effects are (i) the interference of radiation fields, (ii) the anomalous transformation of energy, (iii) the retardation of the electric field, and (iv) the work performed by the radiation field. Paradoxes I and II were caused by neglecting the interference of radiation fields (i). Paradox III was based on the anomalous transformation of energy (ii). Paradox IV was based on field retardation (iii). Finally, Paradox V was based on a surprisingly significant effect of the work performed by the radiation fields.

We provided quantitative resolution of the paradoxes up to second order in v/c. Is it a simple task to demonstrate conservation of energy to a higher order in v/c? It is not difficult to expand the algebraic expressions we have to a higher order, but such an expansion is not enough. We have used more approximations, in particular, the expressions for the radiation of the charged particles are correct only in the approximation of small acceleration and small velocities. Indeed, Eq. (10) cannot be universally correct, because it says that by reducing τ, the time of stopping the charged particle, we can obtain an unlimited amount of radiation energy: clearly we should not get more energy than the initial kinetic energy of the particle. Higher order calculations of the equation of conservation of energy are an elaborate task that goes beyond the scope of this paper.

We believe that presenting the effects in the form of paradoxes helps to achieve a deeper understanding of the subject. Obtaining quantitative resolutions of paradoxical situations bolsters our confidence in applying the equation of conservation of energy for indirect calculations of various effects.

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BEGINNINGS OF RADAR

In the 1930's, microwave research was heavily cloaked in secrecy and was simultaneously being developed under wraps in military and industrial laboratories in America, England, France, and Germany. The basic principle, that, radio waves had optical properties and could ''reflect'' solid objects, had been demonstrated in 1888 by the German scientist Heinrich Hertz. A working device for the detection of ships, based on his experiments, was tested in the early 1900's. But little was done to exploit the discovery, even though as far back as 1922, Guglielmo Marconi had urged the development of short radio waves for the detection of obstacles in the fog or darkness. It was not until the 1930's, when airplanes came of age as a military weapon—a threat made terrifyingly real by the damage inflicted by German and Italian bombers on Spain between 1936 and 1938—that the technology of radar finally began to be developed in earnest. Most of the countries exploring radar concentrated their early efforts on ''the beat method,'' or the Doppler method, which used ordinary continuous radio waves and required at least two widely separated and bulky stations, one for transmitting and one for receiving. Airplanes that penetrated between the transmitter and receiver were detected by the Doppler beat between the direct signal (from the transmitter to the receiver) and the signal scattered by the target (which traveled a longer route from the transmitter to the plane and then to the receiver). Unfortunately, the equipment was fairly limited in its effectiveness. The sharpness of the system's vision—its ability to distinguish separately the echoes from two targets close together and at the same distance from the radar—depended on the sharpness of the radar beam. For a given antenna, the beam width was proportional to the wavelength and would become sharper as the wavelength decreased. Loomis realized that if sharp radar beams were ever to be produced by an antenna not too large to carry in an airplane, they would have to develop a generator of much shorter wavelengths than was then known. It was speculative, to be sure, but the unexplored microwave spectrum promised not only to allow radar sets to become much smaller and more portable, but also to prove better at locating low-flying aircraft and to be able to distinguish targets with far greater accuracy.

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