15 An Impossible Necklace

Lev Vaidman

A game which a quantum team can win against any classical team is proposed. The game is based on a Zeno-type proof of Bell's inequality. The unusual feature of this game is that a person who does not know about quantum mechanics might be led to belive in the existence of impossible necklaces.

When I approached John Bell in 1987 asking to be his Post-Doctoral student working on foundations of quantum mechanics, his reply was that his main work is particle physics and, therefore, he cannot take anybody who wants to work on foundations. Nevertheless, John Bell played an important role in my research. I find his works to be exceptionally clear and of tremendous importance. I met Bell in the relaxed atmosphere of Erice in 1989. Our discussions there and following e-mail correspondence played a very important role in forming my views. The most important influence on me was, however, made by his works showing the miraculous features of quantum theory. In this brief note I want to present an elaboration of a Bell-type proof [1] which exhibits one of such miracles.

Conceptually, the most simple, surprising, and convincing among the Bell-type experiments is *Mermin's* version [2] of the *Greenberger–Horne–Zeilinger* (GHZ) setup [3]. I find that it can be best explained as a game [4]. A team of three players is allowed to make any preparations before the players are taken to three remote locations. Then, at a certain time, each player is asked one of two possible questions: "What is X?" or "What is Y?" to which they must quickly give one of the answers: "1" or "-1". According to the rules of the game, either all players are asked the X question, or only one player is asked the X question and the other two are asked the Y question. The team wins if the product of their three answers is -1 in the case of three X questions and is 1 in the case of one X and two Y questions. It is a simple exercise to prove that if the answer of each player is determined by a local hidden variable theory, then the best strategy of the team will lead to 75% probability to win. However, a quantum team equipped with ideal devices can win with certainty. Each player performs a spin measurement of a spin-1/2 particle: σ_x measurement for the X question and σ_y measurement for the Y question and gives the answer 1 for spin "up" and -1 for spin "down". Quantum theory ensures that if the players have particles prepared in the GHZ state, the team always wins. Actually constructing such devices and seeing that, indeed, the quantum team wins the game with probability significantly larger than 75% will be a very convincing proof of Bell-type inequalities.

Apart from the GHZ game just described there have been several other proposals: an interesting variation of the GHZ game by *Steane* and *van Dam* [5], a game based on the original Bell proof by *Tsirelson* [6], the "quantum cakes" game based on a non-maximally entangled state by *Kwiat* and *Hardy* [7] (see also related experiment [8]). Note also the proposal of *Cabello* [10] for a two-party Bell-inequality proof which can be transformed into a game, too. Let me present here one more game. My game is called an "impossible necklace" and it is based on the Zeno-type Bell inequalities proof [9].

A team of two players wants to persuade a third party, "the interrogator," that they found a secret of making an "impossible necklace". The impossible two-colored neclace has an even number of beads N and all adjacent beads are of different colors except beads 1 and N which are of the same color. The team does not want to reveal the "secret coloring", but the players are ready to reveal the colors of any two adjacent beads of the necklace. They claim to have identical necklaces of this kind, one necklace for each player. The interrogator arranges to ask one player the color of any single bead and ask the other player, at a space-like separated region, the color of one of the adjacent beads. If the team succeeds in giving the correct answers in many repeated experiments (with new necklaces each time), a naive interrogator might be persuaded that the team knows how to make such necklaces. Indeed, if it is a "classical team", and the players decide in advance what answer they will give for every question, then the probability to fail is at least $1/N$. (There are N different pairs and there is no way to arrange that all have correct coloring.) Therefore, the probability to pass the test, say $5N$ times is

$$prob_{\text{classical}} = \left(1 - \frac{1}{N}\right)^{5N} \sim \mathrm{e}^{-5} \sim 0.01. \tag{15.1}$$

The quantum team can do much better. The players do not make any necklaces. Each player takes with him a spin $-\frac{1}{2}$ particle from the EPR (Einstein–Podolski–Rosen) pair. When a player is asked the color of a bead i, he measures the spin component in the direction $\hat{\theta}_i$ in the x–z plane which makes an angle $\theta_i = \pi i/N$ with the z-axis. He says "green" if the result is "up", and "red" if the result is "down". His partners do the same. For all pairs, the measurements are in the directions which differ by the angle π/N except for the pair $\{1, N\}$ in which case the angle is $\pi(N-1)/N$. Therefore, the probability to fail the test is $\sin^2(\pi/2N)$. The probability to pass $5N$ tests is

$$prob_{\text{quantum}} = \left(1 - \sin^2\frac{\pi}{2N}\right)^{5N} \sim \left(1 - \frac{\pi^2}{4N^2}\right)^{5N} \sim \mathrm{e}^{-\frac{5\pi^2}{4N}}. \tag{15.2}$$

For $N = 100$ the quantum team has probability of almost 90% to succeed, compared with 1% of a classical team.

Technological problems will not allow an experiment with a large number N in a near future. Putting aside the attempt to "fool" the interrogator that the team has impossible necklaces, the game can be defined as the competition of two-player teams to pass the interrogator tests a maximal number of times. For any number $N \geq 4$, the quantum team has an advantage over a classical team, so this game is a realistic proposal for demonstrating Bell-type inequalities. A succesful experiment of this type will show that quantum technology is capable of performing communication tasks which are impossible when classical devices are used.

Acknowledgements

It is a pleasure to thank Hans Christian von Bayer for helpful discussions. This research was supported in part by grant 62/01 of the Israel Science Foundation and the EPSRC grant GR/N33058.

References

1. J.S. Bell, Physics **1**, 195 (1964)
2. D.N. Mermin, Am. Phys. **58**, 731 (1990)
3. D.M. Greenberger, M.A. Horne, A. Zeilinger, in *Bell Theorem, Quantum Theorey and Conceptions of the Universe*, M. Kafatos, (ed.) (Kluwer Academic, Dordrecht 1989) p.69
4. L. Vaidman, Found. Phys. **29**, 615 (1999)
5. A. Steane, W. van Dam, Phys. Today **53**, 35 (2000)
6. B. Tsirelson, Lecture Notes, Tel-Aviv University (1996)
7. P.G. Kwiat, L. Hardy, Am. J. Phys. **68**, 33 (2000)
8. G. Brida et al., Phys. Lett. A 268 (2000)
9. E.J. Squires, L. Hardy, H.R. Brown, Stud. Hist. Philos. Sci. **25**, 425 (1994)
10. A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001)