

COMMENTS

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Validity of the Aharonov-Bergmann-Lebowitz rule

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It is argued that the proof of Cohen [Phys. Rev. A **51**, 4373 (1995)] which shows that an application of the Aharonov-Bergmann-Lebowitz (ABL) rule leads to contradiction with predictions of quantum theory is erroneous. A generalization of the ABL rule for the case of an incomplete final measurement (which is needed for the analysis of Cohen’s proof) is presented. [S1050-2947(98)03203-X]

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Cohen [1] examines a few surprising results that have been obtained for a preselected and post-selected quantum system by applying the Aharonov, Bergmann, and Lebowitz (ABL) rule [2]. Following Sharp and Shanks [3] he proves that the ABL rule is not valid in general by showing that in a particular situation the counterfactual interpretation of the ABL rule leads to a prediction contradicting quantum theory. He claimed, however, that the ABL rule is valid for a special class of situations which correspond to “consistent histories” [4]. This limitation, if true, reduces significantly the importance of the ABL rule. In this comment I will argue that there is a crucial error in Cohen’s proof of the inconsistency of the ABL rule with quantum theory.

The proof of Cohen is a variation of the proof given earlier by Sharp and Shanks [3]. I have shown in detail in Ref. [5] the flaw in these arguments, and here I will only present the key point and discuss details which are specific for Cohen’s proof.

Cohen considers a modified Mach-Zehnder apparatus with a possible measurement performed by a “which way” detector D_3 (that lets detected particles pass through), see Fig. 1. Then he applies the ABL rule in a counterfactual sense, and arrives at a contradiction with quantum theory. This leads him to reject the ABL rule for counterfactual situations. I argue that the contradiction obtained by Cohen follows from a logical error in his equation and not from an inapplicability of the ABL formula in this case.

The argument of Cohen goes as follows. If detector D_3 is not placed, then the probabilities for getting the click in D_1 and in D_2 are equal, $\text{Prob}(D_1) = \text{Prob}(D_2) = \frac{1}{2}$. This is so because the beam splitters are half transparent and the Mach-Zehnder BS₂-BS₃ is tuned in such a way that all particles moving towards BS₂ are detected by D_2 . If D_3 is in place, then the ABL formula yields (Cohen claims) a probability $\frac{1}{4}$ for a click in D_3 given a click in D_1 , $\text{Prob}(D_3|D_1) = \frac{1}{4}$, and

a probability $\frac{1}{2}$ for a click in D_3 given a click in D_2 , $\text{Prob}(D_3|D_2) = \frac{1}{2}$. Combining these three statements Cohen obtains an unconditional probability for the click in D_3 :

$$\begin{aligned} \text{Prob}(D_3) &= \text{Prob}(D_3|D_1)\text{Prob}(D_1) + \text{Prob}(D_3|D_2)\text{Prob}(D_2) \\ &= \frac{1}{4} \frac{1}{2} + \frac{1}{2} \frac{1}{2} = \frac{3}{8}. \end{aligned} \tag{1}$$

This result is in contradiction with quantum theory, which yields $\text{Prob}(D_3) = \frac{1}{4}$.

One difficulty with this derivation is that the versions of the ABL formula which have been published so far are not applicable to Cohen’s experiment. The original ABL formula is applicable to a situation in which there is a complete measurement at t_1 , a complete measurement at t_2 , and a complete measurement at time t , $t_1 < t < t_2$. “Complete” means that the outcome specifies the quantum state completely. In

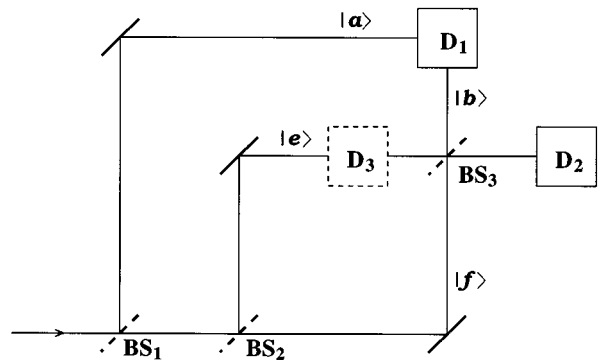


FIG. 1. Cohen’s experiment. Mach-Zehnder type interferometer with “which way” detector D_3 in place.

Ref. [6] a generalization of the ABL formula which is applicable for an arbitrary measurement at time t is given, but the measurements at t_1 and t_2 have to be complete [7]. The ABL formula of Ref. [6] yields the probability for the result $C=c_n$ at time t given that at time t_1 the system was prepared in the state $|\Psi_1\rangle$ and that at time t_2 the state $|\Psi_2\rangle$ was found:

$$\text{Prob}(C=c_n) = \frac{|\langle\Psi_2|U(t,t_2)\mathbf{P}_{C=c_n}U(t_1,t)|\Psi_1\rangle|^2}{\sum_i |\langle\Psi_2|U(t,t_2)\mathbf{P}_{C=c_i}U(t_1,t)|\Psi_1\rangle|^2}, \tag{2}$$

where U signifies unitary evolution between the measurements. In Cohen’s example, however, the measurement at time t_2 is not complete: the click in D_1 does not distinguish between $|a\rangle$, the state of the particle arriving from the left, and $|b\rangle$, the state of the particle arriving vertically from beam splitter BS_3 . I will analyze the proper generalization of the ABL formula for this case below and I will reach a different result for the conditional probability $\text{Prob}(D_3|D_1)$ which, nevertheless, will not change Cohen’s argument. However, I believe that for trying to show putative inconsistency of the ABL formula it is better to consider situations in which the present version of this formula [shown in Eq. (2)] is applicable. Therefore I will first modify Cohen’s experiment in such a way that Eq. (2) is applicable while Cohen’s argument still goes through.

In the simplest variation of the experiment which makes the final measurement complete, D_1 is modified in such a way that it distinguishes the particles in states $|a\rangle$ and $|b\rangle$. This, however, is not suitable for our purpose since it will

not lead to Cohen’s type of contradiction. Therefore we will consider, instead, a detector D_1 which distinguishes between the states $|+\rangle \equiv 1/\sqrt{2}(|a\rangle+|b\rangle)$ and $|-\rangle \equiv 1/\sqrt{2}(|a\rangle-|b\rangle)$. For such an experiment the ABL formula (2) is applicable directly. Let us take a simple model according to which the unitary evolution of a particle wave passing through a beam splitter is described by

$$\begin{aligned} |\text{horizontal}\rangle &\rightarrow \frac{1}{\sqrt{2}}(|\text{horizontal}\rangle+|\text{vertical}\rangle), \\ |\text{vertical}\rangle &\rightarrow \frac{1}{\sqrt{2}}(|\text{horizontal}\rangle-|\text{vertical}\rangle). \end{aligned} \tag{3}$$

The mirrors just change $|\text{vertical}\rangle$ to $|\text{horizontal}\rangle$ and vice versa. Then for the time t at which the particle can be detected by D_3 , we obtain

$$\begin{aligned} U(t_1,t)|\Psi_1\rangle &= \frac{1}{2}(\sqrt{2}|a\rangle+|e\rangle+|f\rangle), \\ \langle+|U(t,t_2) &= \frac{1}{2}(\sqrt{2}\langle a|+\langle f|-\langle e|), \\ \langle-|U(t,t_2) &= \frac{1}{2}(\sqrt{2}\langle a|+\langle e|-\langle f|), \end{aligned} \tag{4}$$

where $|e\rangle$ and $|f\rangle$ are states of the particle in the upper and the lower arms of the small interferometer. Detector D_3 measures the projection operator $|e\rangle\langle e|$. Since at time t the particle can only be in a superposition of states $|a\rangle$, $|e\rangle$, and $|f\rangle$, “no particle at D_3 ” corresponds to the operator $|a\rangle\langle a|+|f\rangle\langle f|$. Now we apply Eq. (2) and obtain

$$\begin{aligned} \text{Prob}(D_3|D_1,+) &= \frac{|\langle\sqrt{2}\langle a|+\langle f|-\langle e| |e\rangle\langle e| (\sqrt{2}|a\rangle+|e\rangle+|f\rangle)\rangle|^2}{|\langle\sqrt{2}\langle a|+\langle f|-\langle e| |e\rangle\langle e| (\sqrt{2}|a\rangle+|e\rangle+|f\rangle)\rangle|^2+|\langle\sqrt{2}\langle a|+\langle f|-\langle e| (|a\rangle\langle a|+|f\rangle\langle f|) (\sqrt{2}|a\rangle+|e\rangle+|f\rangle)\rangle|^2} \\ &= \frac{1^2}{1^2+|2+1|^2} = \frac{1}{10}. \end{aligned} \tag{5}$$

Similarly,

$$\text{Prob}(D_3|D_1,-) = \frac{1^2}{1^2+|2-1|^2} = \frac{1}{2}. \tag{6}$$

If detector D_3 is not placed, then the probabilities for possible outcomes of the final measurement are

$$\begin{aligned} \text{Prob}(D_1,+) &= \frac{1}{4}, \\ \text{Prob}(D_1,-) &= \frac{1}{4}, \\ \text{Prob}(D_2) &= \frac{1}{2}. \end{aligned} \tag{7}$$

Following Cohen’s proof we combine the above results and obtain the unconditional probability for a click in D_3 :

$$\begin{aligned} \text{Prob}(D_3) &= \text{Prob}(D_3|D_1,+) \text{Prob}(D_1,+) \\ &\quad + \text{Prob}(D_3|D_1,-) \text{Prob}(D_1,-) \\ &\quad + \text{Prob}(D_3|D_2) \text{Prob}(D_2) \\ &= \frac{1}{10} \frac{1}{4} + \frac{1}{2} \frac{1}{4} + \frac{1}{2} \frac{1}{2} = \frac{2}{5}. \end{aligned} \tag{8}$$

Again, this differs from the prediction of quantum theory: $\text{Prob}(D_3) = \frac{1}{4}$.

Equation (8) is indeed wrong, but not because the probabilities given by the ABL formula are incorrect. The probabilities $\text{Prob}(D_1, +)$, $\text{Prob}(D_1, -)$, and $\text{Prob}(D_2)$ are obviously wrong. Indeed, these probabilities were calculated on the assumption that detector D_3 was not placed. But Eq. (8) yields the probability for a click in D_3 . Therefore it must be in place, and the assumption on which the probabilities $\text{Prob}(D_1, +)$, $\text{Prob}(D_1, -)$, and $\text{Prob}(D_2)$ were calculated is not fulfilled.

It is easy to correct the calculation of $\text{Prob}(D_3)$. If D_3 is in place we obtain $\text{Prob}(D_1, +) = \frac{5}{8}$, $\text{Prob}(D_1, -) = \frac{1}{8}$, and $\text{Prob}(D_2) = \frac{1}{4}$. Therefore the correct calculation is

$$\begin{aligned} \text{Prob}(D_3) &= \text{Prob}(D_3|D_1, +)\text{Prob}(D_1, +) \\ &\quad + \text{Prob}(D_3|D_1, -)\text{Prob}(D_1, -) \\ &\quad + \text{Prob}(D_3|D_2)\text{Prob}(D_2) \\ &= \frac{1}{10} \frac{5}{8} + \frac{1}{2} \frac{1}{8} + \frac{1}{2} \frac{1}{4} = \frac{1}{4}. \end{aligned} \quad (9)$$

Not surprisingly, it is the same number which can be immediately obtained using quantum rules without considering the results of the final measurement.

Let us now analyze an unmodified Cohen's experiment. To this end we have to first generalize the ABL formula (2) for the case of an incomplete final measurement. The proof of Eq. (2) is given on p. 2317 of Ref. [6] and it can be repeated line by line for the case of incomplete final measurement, say with the result $B=b$. Essentially, the only change required is the replacement of " $|\Psi_f\rangle = |\Psi_2\rangle$ " by " $B=b$ " and the final result will be

$$\text{Prob}(C = c_n) = \frac{\|\mathbf{P}_{B=b}|U(t, t_2)\mathbf{P}_{C=c_n}U(t_1, t)|\Psi_1\rangle\|^2}{\sum_i \|\mathbf{P}_{B=b}|U(t, t_2)\mathbf{P}_{C=c_i}U(t_1, t)|\Psi_1\rangle\|^2}. \quad (10)$$

Now we can analyze Cohen's original example. The generalized ABL formula (10) yields $\text{Prob}(D_3|D_1) = \frac{1}{6}$ (instead of $\frac{1}{4}$ in Cohen's paper), and $\text{Prob}(D_3|D_2) = \frac{1}{2}$. If we calculate $\text{Prob}(D_3)$ using the probabilities for the clicks in D_1 and D_2 calculated on the assumption that D_3 is absent, Cohen's contradiction still holds:

$$\begin{aligned} \text{Prob}(D_3) &= \text{Prob}(D_3|D_1)\text{Prob}(D_1) + \text{Prob}(D_3|D_2)\text{Prob}(D_2) \\ &= \frac{1}{6} \frac{1}{2} + \frac{1}{2} \frac{1}{2} = \frac{1}{3} \left(\neq \frac{1}{4} \right). \end{aligned} \quad (11)$$

But the correct calculation of probabilities of the detection by D_1 and D_2 (which assumes that detector D_3 is present) yields $\text{Prob}(D_1) = \frac{3}{4}$ and $\text{Prob}(D_2) = \frac{1}{4}$. Thus we obtain again the correct result:

$$\begin{aligned} \text{Prob}(D_3) &= \text{Prob}(D_3|D_1)\text{Prob}(D_1) + \text{Prob}(D_3|D_2)\text{Prob}(D_2) \\ &= \frac{1}{6} \frac{3}{4} + \frac{1}{2} \frac{1}{4} = \frac{1}{4}. \end{aligned} \quad (12)$$

Since Cohen's proof of inconsistency of the ABL rule is erroneous, his conclusions about the limitations on the applicability of the ABL rule are unfounded. The surprising consequences of the ABL rule in the examples which Cohen considered are still valid.

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 [7] In Ref. [6] there is also another generalization of the ABL formula for systems described by "generalized states" which are defined there. However, this generalization is not relevant for the present problem.