Emergence of Weak Values

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Abstract. Various quantum measurement procedures are analyzed and it is shown that under certain conditions they yield consistently *weak values* which might be very different from the eigenvalues, the allowed outcomes according to the standard quantum formalism. The weak value outcomes result from peculiar quantum interference of the pointer variable of the measuring device.

1 Introduction

In the standard formalism of quantum theory the outcome of a (good) measurement must be an eigenvalue of the operator corresponding to the measured variable. In this paper I will discuss a modified measuring procedures which will yield instead of an eigenvalue a *weak value*, recently introduced by Aharonov, Albert and Vaidman (1988). The weak value of an observable A is defined for a *two-state vector* $\langle\Psi_2|\,|\Psi_1\rangle$ as

$$A_w \equiv \frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle} \quad . \tag{1}$$

The expectation value of A for a system in a state $|\Psi\rangle$ is a particular case of a weak value when $|\Psi_1\rangle = |\Psi_2\rangle = |\Psi\rangle$.

The standard ideal measurements requires infinitely strong coupling. The weak values emerge only if the measuring coupling is bounded and in most (but not in all) cases the coupling must be weak and this is the reason for the name "weak value".

The important surprising feature of the weak value is that it might be far away from the range of the eigenvalues, for example, the weak value of kinetic energy might be negative, see Aharonov et al. (1993). The weak value is, in general, a complex number. The (almost) standard measurement procedure with a weakened coupling yields the real part of the weak value. The imaginary part can be measured too but we will not discuss it here.

The expectation value, $\langle\Psi|A|\Psi\rangle$ emerges in a *weak* measurement of a quantum system pre-selected in a state $|\Psi\rangle$ as well as in a *protective* measurement (Aharonov and Vaidman 1993, Aharonov, Anandan and Vaidman 1993) when the state $|\Psi\rangle$ is protected. The weak value (1) emerges in a weak measurement performed on a quantum system pre-selected in the state $|\Psi_1\rangle$ and post-selected in the state $|\Psi_2\rangle$ as well as in a protective measurement when the two-state vector $\langle\Psi_2|\,|\Psi_1\rangle$ is protected. Protective measurements consist of protection coupling and measuring coupling. The protection coupling usually protects several quantum states or several quantum two-state vectors. If the system is protected by such a coupling but not selected in one of the protected states (two-state vectors) then the outcome of the measurement is the weak value corresponding to one of the protected states (two-state vectors) chosen at random. I shall discuss all these cases below.

2 Measurement Procedure

According to standard definition, a quantum measurement of a physical variable A is described by the Hamiltonian:

$$H = g(t)PA \quad , \tag{2}$$

where P is a canonical momentum conjugate to the pointer variable Q of the measuring device. The function $g(t)$ is nonzero only for a very short time interval corresponding to the measurement, and is normalized so that $\int g(t)dt = 1$. During the time of this impulsive measurement, the Hamiltonian (2) dominates the evolution of the measured system and the measuring device. Since $[A, H] = 0$, the variable A does not change during the measuring interaction. The initial state of the pointer variable is usually modeled by a Gaussian centered at zero:

$$\Phi_{in}(Q) = e^{-Q^2/2\Delta^2}. \tag{3}$$

Here and below we omit the normalization factor. The pointer is in the "zero" position before the measurement, i.e. its initial probability distribution is

$$prob(Q) = e^{-Q^2/\Delta^2}. \tag{4}$$

If the initial state of the system is an eigenstate $|\Psi_1\rangle = |a_i\rangle$, then after the interaction (2), the state of the system and the measuring device is:

$$|a_i\rangle e^{-(Q-a_i)^2/2\Delta^2}. \tag{5}$$

The probability distribution of the pointer variable, $e^{-(Q-a_i)^2/\Delta^2}$ remained unchanged in its shape, but it is shifted by the eigenvalue a_i. In an ideal measurement, the initial probability distribution of the pointer is well localized around zero, and thus the final distribution is well localized around the eigenvalue. Thus, the reading of the pointer variable in the end of the measurement almost always yields a value of the shift (the eigenvalue of the variable).

If the initial state of the system is a superposition $|\Psi_1\rangle = \Sigma\alpha_i|a_i\rangle$, then after the interaction (2) the state of the system and the measuring device is:

$$\Sigma\alpha_i|a_i\rangle e^{-(Q-a_i)^2/2\Delta^2}. \tag{6}$$

The probability distribution of the pointer variable corresponding to the state (6) is

$$prob(Q) = \Sigma|\alpha_i|^2 e^{-(Q-a_i)^2/\Delta^2}. \tag{7}$$

In case of ideal measurement this is a weighted sum of the initial probability distribution localized around various eigenvalues. Therefore, the reading of the pointer variable in the end of the measurement almost always yields the value close to one of the eigenvalues.

In the case of the ideal measurement the measuring interaction leads to a very large uncertain change of the system due to a large uncertainty of the variable P. Indeed, in the standard measurement we require that the pointer shows zero before the measurement, i.e., Δ is very small for the initial state of the measuring device (3). This requires large uncertainty in P, and therefore the Hamiltonian (2) causes a large uncertain change.

The weak measurement is also described by the interaction Hamiltonian (2) but it kept small by taking the initial state of the measuring device such that $\langle P \rangle = 0$ and the uncertainty in P is small. We consider $\Delta \gg a_i$ for all eigenvalues a_i. Then, we can perform the Taylor expansion of the sum (7) around $Q = 0$ up to the first order and rewrite the probability distribution of the pointer in the following way:

$$prob(Q) = \Sigma|\alpha_i|^2 e^{-(Q-a_i)^2/\Delta^2} =$$
$$\Sigma|\alpha_i|^2(1 - (Q - a_i)^2/\Delta^2) = e^{-(Q-\Sigma|\alpha_i|^2 a_i)^2/\Delta^2}. \quad (8)$$

But this is exactly the initial distribution shifted by the value $\Sigma|\alpha_i|^2 a_i$. This is the the expectation value which is also the weak value in this pre-selection case: $A_w = \Sigma|\alpha_i|^2 a_i = \langle\Psi|A|\Psi\rangle$. This weak value can be found from statistical analysis of the readings of the measuring devices of such measurements performed on an ensemble of identical quantum systems. But it is different conceptually from the standard definition of expectation value which is a mathematical concept defined from the statistical analysis of the *ideal* measurements of the variable A all of which yield one of the eigenvalues a_i.

3 Protective Measurements

In general, the weak (expectation) value cannot be measured on a single system. However, it can be done if the quantum state is *protected* (Aharonov and Vaidman 1993). The appropriate measurement interaction is again described the Hamiltonian (2), but instead of impulsive interaction the adiabatic limit of slow and weak interaction is considered: $g(t) = 1/T$ for most of the interaction time T and $g(t)$ goes to zero gradually before and after the period T.

In this case the interaction Hamiltonian (2) does not dominate the time evolution during the measurement, moreover, it can be considered as a perturbation. The free Hamiltonian H_0 dominates the evolution. In order to protect a quantum state this Hamiltonian must have the state to be a nondegenerate energy eigenstate. For $g(t)$ smooth enough we then obtain an adiabatic process in which the system cannot make a transition from one energy eigenstate to another, and, in the limit $T \to \infty$, the interaction Hamiltonian changes the energy eigenstate by an infinitesimal amount. If the initial state of the system is an eigenstate $|E_i\rangle$ of H_0 then for any given value of P, the energy of the eigenstate shifts by an infinitesimal amount given by the first order perturbation theory:

$$\delta E = \langle E_i|H_{int}|E_i\rangle = \langle E_i|A|E_i\rangle P/T. \quad (9)$$

The corresponding time evolution $e^{-iP\langle E_i|A|E_i\rangle}$ shifts the pointer by the expectation value of A in the state $|E_i\rangle$. Thus, the probability distribution of the pointer variable remains unchanged in its shape, and is shifted by the expectation value $\langle A\rangle_i = \langle E_i|A|E_i\rangle$.

If the initial state of the system is a superposition of several nondegenerate energy eigenstates $|\Psi_1\rangle = \Sigma\alpha_i|E_i\rangle$, then a particular outcome $\langle A\rangle_i \equiv \langle E_i|A|E_i\rangle$ appears at random, with the probability $|\alpha_i|^2$. (Subsequent adiabatic measurements of the same observable A invariably yield the expectation value in the same eigenstate $|E_i\rangle$.)

4 Pre- and Post-Selected Systems

Aharonov, Bergmann and Lebowitz (1964) considered measurements performed on a quantum system between two other measurements, results of which were given. They proposed describing the quantum system between two measurements by using two states: the usual one, evolving towards the future from the time of the first measurement, and a second state evolving backwards in time, from the time of the second measurement. If a system has been prepared at time t_1 in a state $|\Psi_1\rangle$ and is found at time t_2 in a state $|\Psi_2\rangle$, then at time t, $t_1 < t < t_2$, the system is described by $\langle\Psi_2|e^{i\int_{t_2}^{t} H dt}$ and $e^{-i\int_{t_1}^{t} H dt}|\Psi_1\rangle$. For simplicity, we shall consider the free Hamiltonian to be zero; then, the system at time t is described by the two states $\langle\Psi_2|$ and $|\Psi_1\rangle$. In order to obtain such a system, we prepare an ensemble of systems in the state $|\Psi_1\rangle$, perform a measurement of the desired variable using separate measuring devices for each system in the ensemble, and perform the post-selection measurement. If the outcome of the post-selection was not the desired result, we discard the system and the corresponding measuring device. We look only at measuring devices corresponding to the systems post-selected in the state $\langle\Psi_2|$.

Let us show briefly how weak values emerge from a measuring procedure performed on a pre- and post-selected system with a sufficiently weak coupling. We consider a sequence of measurements: a pre-selection of $|\Psi_1\rangle$, a (weak) measurement interaction of the form of Eq. (2), and a post-selection measurement finding the state $|\Psi_2\rangle$. The state of the measuring device (which was initially in a Gaussian state) after this sequence is given (up to normalization) by

$$\Phi(Q) = \langle\Psi_2|e^{-iPA}|\Psi_1\rangle e^{-Q^2/2\Delta^2} \quad . \tag{10}$$

In the P-representation we can rewrite it as

$$\tilde{\Phi}(P) = \langle\Psi_2|\Psi_1\rangle \, e^{-iA_w P} \, e^{-\Delta^2 P^2/2} + \langle\Psi_2|\Psi_1\rangle \sum_{n=2}^{\infty} \frac{(iP)^n}{n!} [(A^n)_w - (A_w)^n] e^{-\Delta^2 P^2/2} \; . \tag{11}$$

If Δ is sufficiently large, we can neglect the second term of (11) when we Fourier transform back to the Q-representation. Large Δ corresponds to weak measurement in the sense that the interaction Hamiltonian (2) is small. Thus, in the limit of weak measurement, the final state of the measuring device (in the Q-representation) is

$$\Phi(Q) = e^{-(Q-A_w)^2/2\Delta^2} \quad . \tag{12}$$

This state represents a measuring device pointing to the weak value, A_w. Since Δ has to be large, the weak coupling between a single system and the measuring device will not, in most cases, lead to a distinguishable shift of the pointer variable, but collecting the results of measurements on an ensemble of pre- and post-selected systems will yield the weak values of a measured variable to any desired precision. Although we have showed the emergence of weak values in weak measurements for a specific von Neumann model of measurements, the result is completely general: any coupling of a pre- and post-selected system to a variable A, provided the coupling is sufficiently weak, results in effective coupling to A_w.

5 Protection of a Two-State Vector

At first sight, it seems that protection of a two-state vector is impossible. Indeed, if we add a potential that makes one state a nondegenerate eigenstate, then the other state, if it is different, cannot be an eigenstate too. (The states of the two-state vector cannot be orthogonal.) But, nevertheless, protection of the two-state vector is possible (Aharonov and Vaidman, 1995).

The procedure for protection of a two-state vector of a given system is accomplished by coupling the system to another pre- and post-selected system. The protection procedure takes advantage of the fact that weak values might acquire complex values. Thus, the effective Hamiltonian of the protection might not be hermitian. Non-hermitian Hamiltonians act in different ways on quantum states evolving forward and backwards in time. This allows simultaneous protection of two different states (evolving in opposite time directions).

Let us consider an example of a two-state vector of a spin-1/2 particle, $\langle\uparrow_y||\uparrow_x\rangle$. The protection procedure uses an external pre- and post-selected system S of a large spin N that is coupled to our spin via the interaction:

$$H_{prot} = -\lambda \mathbf{S} \cdot \sigma. \tag{13}$$

The external system is pre-selected in the state $|S_x{=}N\rangle$ and post-selected in the state $\langle S_y{=}N|$, that is, it is described by the two-state vector $\langle S_y{=}N||S_x{=}N\rangle$. The coupling constant λ is chosen in such a way that the interaction with our spin-1/2 particle cannot change significantly the two-state vector of the protective system S, and the spin-1/2 particle "feels" the effective Hamiltonian in which S is replaced by its weak value,

$$\mathbf{S}_w = \frac{\langle S_y = N|(S_x, S_y, S_z)|S_x = N\rangle}{\langle S_y = N|S_x = N\rangle} = (N, N, iN). \tag{14}$$

Thus, the effective protective Hamiltonian is:

$$H_{eff} = -\lambda N(\sigma_x + \sigma_y + i\sigma_z). \tag{15}$$

The state $|\uparrow_x\rangle$ is an eigenstates of this (non-hermitian) Hamiltonian (with eigenvalue $-\lambda N$). For backward evolving states the effective Hamiltonian is the hermitian conjugate of (15) and it has different (nondegenerate) eigenstate with this eigenvalue; the eigenstate is $\langle\uparrow_y|$.

In order to prove that the Hamiltonian (13) indeed provides the protection, we have to show that the two-state vector $\langle\uparrow_y||\uparrow_x\rangle$ will remain essentially unchanged during the measurement. See details of the proof in Aharonov and Vaidman, (1995, 1996) and Aharonov et al. (1996).

At least formally we can generalize this method to make a protective measurement of an arbitrary two-state vector $\langle\Psi_2||\Psi_1\rangle$ of an arbitrary system. However, this scheme usually leads to unphysical interaction and is good only as a gedanken experiment in the framework of non-relativistic quantum theory where we assume that any hermitian Hamiltonian is possible.

6 Weak Values and Protective Measurements

The protective Hamiltonian (13) has more interesting features than just protecting the two state vector $\langle\uparrow_y||\uparrow_x\rangle$. There is another two-state vector which is protected:

the two state $\langle\downarrow_x||\downarrow_y\rangle$ with corresponding eigenvalue λN.

In general, a nondegenerate non-hermitian Hamiltonian yields protection for a set of pairs consisting from "bras" and "kets". The Hamiltonian can be written in the following form

$$H = \Sigma_i \omega_i \frac{|\Phi_i\rangle\langle\Psi_i|}{\langle\Psi_i|\Phi_i\rangle}, \tag{16}$$

where $\langle\Psi_i|$ are the "eigen-bras" of H, and $|\Phi_i\rangle$ are the "eigen-kets" of H. The $\langle\Psi_i|$ form a complete but, in general, non-orthogonal set, and so do the $|\Phi_i\rangle$. They obey mutual orthogonality condition:

$$\langle\Psi_i|\Phi_i\rangle = \delta_{ij}\langle\Psi_i|\Phi_i\rangle. \tag{17}$$

If the initial state is a superposition of the eigenstates $|\Psi\rangle = \Sigma_i \alpha_i |\Psi_i\rangle$ then its time evolution is given by

$$|\Psi(t)\rangle = \mathcal{N}(t)\Sigma_i \alpha_i e^{-i\omega_i T}|\Psi_i\rangle \tag{18}$$

An adiabatic measurement coupling of a variable A performed on such system leads to the state of the system and the measuring device given by

$$\Sigma_i \alpha_i e^{-i\omega_i T}|\Psi_i\rangle \Phi(Q - \frac{\langle\Phi_i|A|\Psi_i\rangle}{\langle\Phi_i|\Psi_i\rangle}). \tag{19}$$

The state of the measuring device is then amplified to a macroscopically distinguishable situation and, according to standard interpretation, a collapse takes place to the reading of one of the *weak values* of A with the relative probabilities given by $|\alpha_i e^{-i\omega_i T}|^2$.

In summary, the main properties of such adiabatic measurements are (Aharonov et al. 1996):

a) The only possible outcomes of the measurement are the weak values A_w^i corresponding to one of the pairs of states $\langle\psi_i||\phi_i\rangle$ associated with the non hermitian Hamiltonian.

b) A particular outcome A_w^i appears at random, with a probability which depends only on the initial state of the measured system and is independent of the details of the measurement.

c) The measurement leads to an effective collapse to the two-state vector $\langle\psi_i||\phi_i\rangle$ corresponding to the observed weak value A_w^i. Subsequent adiabatic measurements of the same observable A invariably yield the same weak value.

d) Simultaneous measurements of different observables yield the weak values corresponding to the same two-state vector $\langle\psi_i||\phi_i\rangle$.

An effective non-hermitian Hamiltonian can be obtained in a real laboratory in a natural way when we consider a decaying system and we post-select the cases in which it has not decayed during the period of time T which is larger than its characteristic decay time. Kaon decay is such an example. $|K_L^0\rangle$ and $|K_S^0\rangle$ are the eigen-kets of the effective Hamiltonian and they have corresponding eigen-bras $\langle K_L'^0|$ and $\langle K'^0_S|$ evolving backward in time. Due to the $CP - violation$ the states $|K_L^0\rangle$ and $|K_S^0\rangle$ are not orthogonal. However, the mixing is small: $|\langle K_S^0|K_L^0\rangle| \ll 1$, and therefore the corresponding backward evolving states are almost identical to the forward evolving states: $|\langle K'^0_S|K_S^0\rangle| = |\langle K'^0_L|K_L^0\rangle| = \frac{1}{\sqrt{1-|\langle K_S^0|K_L^0\rangle|^2}}$. Thus, it is difficult to expect a large effect in this system and for a realistic experimental proposal one should look, probably, for another system.

7 Conclusions

We have shown that weak values emerge in procedures which are very close to the standard quantum measurements. The procedures are: (i) weak measurement performed on ensemble of pre-selected quantum systems, (ii) adiabatic measurement on a single system with a non-degenerate energy spectrum, (iii) weak measurement on pre- and post-selected ensemble, (iv) adiabatic measurement on a single system described by a non-hermitian Hamiltonian. In cases (i-ii) the weak values are just expectation values but in cases (iii-iv) the weak values might lie outside the range of eigenvalues. These results can be explained as a peculiar interference effect of the pointer variable of the measuring device (for computer simulation of these interference effects see Vaidman, 1995 and Unruh, 1995) but they are most naturally explained in the framework of the two-state vector formalism.

In fact, the measurements discussed above are not just gedanken experiments. Experiments of type (i) are frequently performed in laboratories: in many cases the individual measurement can not reach the required precision and the measured quantity is found from a measurement on an ensemble of identically prepared systems (but not all such cases correspond to weak measurements). Some types of elastic scattering experiments might fall under category (ii). There were several experiments of the type (iii). The best example, probably, is photon polarization measurement (Ritchie, 1991). I do not know about any performed experiment of type (iv). The most promising is a subclass of such experiments which consist of adiabatic measurements performed on a decaying system which has not decayed yet. We do not know for what decaying system the weak values can emerge in adiabatic measurements in today's laboratory. We leave it as a challenge to find such realistic proposals.

This research was supported in part by grant 614/95 of the Basic Research Foundation (administered by the Israel Academy of Sciences and Humanities).

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