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Weak-Measurement Elements of Reality

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A brief review of the attempts to define "elements of reality" in the framework of quantum theory is presented. It is noted that most definitions of elements of reality have in common the feature to be a definite outcome of some measurement. Elements of reality are extended to pre- and post-selected systems and to measurements which fulfill certain criteria of weakness of the coupling. Some features of the newly introduced concepts are discussed.

1. INTRODUCTION

Experiments performed in laboratories of the twentieth century tell us that the picture of reality we have from our everyday experience cannot be a true representation of nature. This leads us to develop a new language which will be more adequate for the description of our (quantum) world. The fact that our world is quantum seems indisputable, because the predictions of quantum theory have been confirmed with incredible precision in all experiments which have been performed until today.

There have been numerous attempts to describe quantum reality, but a consensus has not been reached. I do not pretend to give here *the* correct definition of elements of reality in quantum theory. I believe, however, that the concepts which I introduce are helpful tools for developing intuition to see and to understand quantum phenomena.

Einstein, Podolsky, and Rosen (EPR) were the pioneers in their attempt to define *elements of reality* in quantum theory. The difficulty of considering elements of reality in the framework of quantum theory which

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they discussed in their seminal paper[1] led Bohr to claim that the only elements of reality in quantum theory are results of (quantum) measurements. (This is, of course, an oversimplification. Max Jammer has illuminating writings on this subject clarifying this important historical issue.[2,3]) The position of Bohr is certainly consistent, but, I believe, is not very fruitful: it implies that no reality is associated with the quantum system between measurements. This approach refuses to consider various concepts which are helpful tools to see the bizarre features of our quantum world.

The EPR definition of element of reality is:

> "If, without in any way disturbing the system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

The key to understanding of the meaning of their definition is the interpretation of the phrase "without in any way disturbing the system." They assumed a strong version of relativistic causality, i.e., that no operation in a space-like separated region can disturb the system. I would say that Bell[4] showed that the existence of EPR elements of reality is inconsistent with predictions of quantum theory. However, there are many important historial aspects of that issue, as Max Jammer[3] has taught us.

The next definition of element of reality I want to bring here was inspired by EPR but it makes no assumptions about relativistic causality. This is the definition of element of reality due to Redhead[5]:

> "If we can predict with certainty, or at any rate with probability one, the result of measuring a physical quantity at time t, then, at time t, there exists an element of reality corresponding to this physical quantity and having value equal to the predicted measurement result."

Although the definitions look very similar, they are very different conceptually. The EPR element of reality is defined by the mere possibility of finding the outcome "without distrubing the system in any way," i.e., from some measurements, while according to Redhead, there is an element of reality when we *know* the outcome of the measurement without performing any other measurement. For example, if we consider an EPR pair of spin-1/2 particles (i.e., the two particles in the singlet state) then, according to EPR, the spin components of each particle in all directions are elements of reality since they can be found from the spin measurement performed on the other particle (and the assumption of no action at a distance ensures that this measurement does not disturb the first particle). Accoding to Redhead, however, there is not any (local) element of reality for a spin variable of any particle; we cannot predict the result of such measurement.

The three alternative definitions of element of reality which I presented above are all related to definite results of measurements. In the first (spirit of Bohr) these are the results of measurements which were actually performed, in the last (Redhead) these are the results of measurements which (if performed) are known with certainty, and in the second (EPR) these are the results of measurements which we can ascertain with certainty using other measurements (performed in a space-like separated region). Thus, it is natural to adopt the following principle: *For any definite result of a measurement there is a corresponding element of reality.*

2. ELEMENTS OF REALITY AS A DEFINITE SHIFT OF THE PROBABILITY DISTRIBUTION OF THE POINTER VARIABLE

In order to understand the meaning of "definite result of measurement" we have to analyze the concept of quantum measurement. The standard definition of quantum measurement is due to von Neumann[6]: The measurement of a physical variable A is described by the Hamiltonian

$$H = g(t)\, PA \tag{1}$$

where P is a canonical momentum conjugate to the pointer variable Q of the measuring device. The function $g(t)$ is nonzero only for a very short time interval corresponding to the measurement, and is normalized so that $\int g(t)\, dt = 1$. During the time of this impulsive measurement, the Hamiltonian (2) dominates the evolution of the measured system and the measuring device. Since $[A, H] = 0$, the variable A does not change during the measuring interaction. The initial state of the pointer variable is usually modeled by a Gaussian centered at zero:

$$\Phi_{in}(Q) = (\Delta^2\pi)^{-1/4} e^{-Q^2/2\Delta^2} \tag{2}$$

The point is in the "zero" position before the measurement, i.e., its initial probability distribution is

$$prob(Q) = (\Delta^2\pi)^{-1/2} e^{-Q^2/\Delta^2} \tag{3}$$

If the initial state of the system is an eigenstate $|\Psi_1\rangle = |a_i\rangle$, then after the interaction (1), the state of the system and the measuring device is

$$(\Delta^2\pi)^{-1/4}\, |a_i\rangle\, e^{-(Q-a_i)/2\Delta^2} \tag{4}$$

The probability distribution of the pointer variable, $(\Delta^2\pi)^{-1/2} e^{-(Q-a_i)^2/\Delta^2}$, remained unchanged in its shape, but it is shifted by the eigenvalue a_i. This eigenvalue is considered to be the element of reality. Thus we can translate the meaning of "definite result of quantum measurement" as *definite shift of the probability distribution of the pointer variable.* I suggest to take this property to be the definition: *If we are certain that a procedure for measuring a certain variable will lead to a definite shift of the unchanged probability distribution of the pointer, then there is an element of reality: the variable equal to this shift.*

In an ideal measurement, the initial probability distribution of the pointer is well localized around zero, and thus the final distribution is well localized around the eigenvalue. Thus, the reading of the pointer variable in the end of the measurement almost always yields the value of the shift (the eigenvalue of the variable). The generalization I suggest in the above definition is applicable also to the situations in which the initial probability distribution of the pointer variable has a significant spread. Although, in this case, the reading of the measuring device at the end of the measurement is not definite, the shift of the distribution is. In such a case, a measurement performed on a single system does not yield the value of the shift (the element of reality), but such measurements performed on a large enough ensemble of identical systems yield the shift with any desirable precision.

If the initial state of the system is a superposition $|\Psi_1\rangle = \sum \alpha_i |a_i\rangle$, then after the interaction (2) the state of the system and the measuring device is

$$(\Delta^2\pi)^{-1/4} \sum \alpha_i |a_i\rangle \, e^{-(Q-a_i)^2/2\Delta^2} \tag{5}$$

The probability distribution of the pointer variable corresponding to the state (5) is

$$prob(Q) = (\Delta^2\pi)^{-1/2} \sum |\alpha_i|^2 e^{-(Q-a_i)/\Delta^2} \tag{6}$$

In case of ideal measurement this is a weighted sum of the initial probability distribution localized around various eigenvalues. It is not a single shifted original probability distribution, and therefore, according to our definition in this case there is no element of reality for the value of the measured variable. (If one adds a "collapse" to one of the positions of the pointer, he ends up with one of the eigenvalues. So, *à la* Bohr, after the measurement is completed, there is an element of reality, but here we are looking for an element of reality which can be predicted with certainty before the measurement.) In general, when the initial probability distribution is not very well localized, the final distribution (6) is very different from the original distribution. However, as I will show in the next section, even in this case there is a well-defined limit in which the final distribution converges to the unchanged initial distribution shifted by a well-defined value. I will suggest to call this well-defined shift an element of reality of a new type.

3. WEAK-MEASUREMENT ELEMENT OF REALITY

I propose to consider the standard measuring procedure (1) in which we weaken the interaction in such a way that the state of the quantum system is not changed significantly during the interaction. Usually, the measuring interaction leads to a very large uncertain change of the system due to a large uncertainty of the variable P. Indeed, in the standard measurement we require that the pointer shows zero before the measurement, i.e., Δ is very small for the initial state of the measuring device (2). This requires a large uncertainty in P, and therefore the Hamiltonian (1) causes a large uncertain change. I propose to take the initial state of the measuring device in which $\langle P \rangle = 0$ and the uncertainty in P is small, and I will show that this is enough to ensure that the pointer probability distribution after the measuring interaction is essentially equal to the shifted initial distribution. In this case the interaction Hamiltonian (1) is small and this is why we call such procedure a *weak measurement*.[7]

The limit of weak measurement corresponds to $\Delta \gg a_i$ for all eigenvalues a_i. Then, we can perform the Taylor expansion of the sum (6) around $Q = 0$ up to the first order and rewrite the probability distribution of the pointer in the following way:

$$prob(Q) = (\Delta^2 \pi)^{-1/2} \sum |\alpha_i|^2 e^{-(Q - a_i)^2/\Delta^2}$$
$$\simeq (\Delta^2 \pi)^{-1/2} \sum |\alpha_i|^2 (1 - (Q - a_i)^2/\Delta^2) \simeq (\Delta^2 \pi)^{-1/2} e^{-(Q - \sum |\alpha_i|^2 a_i)^2/\Delta^2} \tag{7}$$

But this is exactly the initial distribution shifted by the value $\sum |\alpha_i|^2 a_i$. Thus we will say that there is here a *weak-measurement element of reality* $A_w = \sum |\alpha_i|^2 a_i$.

The mathematical expression of this weak measurement of reality is not something new. This is the expectation value: $A_w = \sum |\alpha_i|^2 a_i = \langle \Psi | A | \Psi \rangle$. The weak value is obtained from statistical analysis of the readings of the measuring devices of the measurements on an ensemble of identical quantum systems. But it is different conceptually from the standard definition of expectation value which is a mathematical concept defined from the statistical analysis of the *ideal* measurements of the variable A all of which yield one of the eigenvalues a_i.

I have showed that expectation values fall under the definition of the elements of reality as a definite shift of unchanged probability distribution of the pointer variable in the limit of weak-coupling measurement (Δ large). The advantage of this definition is that it is applicable for any quantum system in any pure or mixed state and for any quantum variable of this system. The disadvantage is that usually we need an ensemble of identical systems in order to find the expectation values of their quantum variables with good precision. It is important to mention that sometimes the *weak* measurement need not be too weak and the expectation value can be found with relatively good precision from a single such measurement. This is the case when the uncertainty of A is small. The latter, in particular, is a generic property of "average" variables of large composite systems.

4. ELEMENTS OF REALITY OF PRE- AND POST-SELECTED SYSTEMS

The concept of the weak-measurement element of reality yields novel results when considered on pre- and post-selected systems. But the ideal-measurement element of reality of the pre- and post-selected system also have novel features and I will consider them first.

For the pre- and post-selected systems it is fruitful to consider a modification of Redhead's definition when I replace "predict" by "infer"[8,9]:

> "If we can infer with certainty, or at any rate with probability one, the result of measuring a physical quantity at time t, then at time t, there exists an element of reality corresponding to this physical quantity and having a value equal to the predicted measurement result."

Essentially, Redhead's definition says that $A = a$ if and only if the system is in the appropriate eigenstate or mixture of such eigenstates. In pre- and post-selected situations we might have $A = a$ even if the system is not in an eigenstate of A; we obtain the inference both from the preparation and from the post-selection measurement.

Elements of reality in the pre- and post-selected situations might be very peculiar. One such example is a single particle inside three boxes A, B, and C, with two elements of reality: "the particle is in box 1" and "the particle is in box 2." This is the case with the pre-selection of the state of the particle $|\Psi_1\rangle = 1/\sqrt{3}\,(|A\rangle + |B\rangle + |C\rangle)$ which was found later in the state $|\Psi_2\rangle = 1/\sqrt{3}\,(|A\rangle + |B\rangle - |C\rangle)$. If in the intermdiate time it was searched for in box A, it has to be found there with probability one, and if, instead, it was searched for in box B, it has to be found there too with probability one. (Indeed, not finding the particle in box A would project the initial state $|\Psi_1\rangle$ onto $1/\sqrt{2}\,(|B\rangle + |C\rangle)$ which is orthogonal to the final state $|\Psi_2\rangle$.)

This example shows that the "product rule" does not hold for elements of reality of pre- and post-selected systems. Indeed $\Pi_A = 1$ is an element of reality and $\Pi_B = 1$ is an element of reality, but $\Pi_A \Pi_B = 1$ is not an element of reality. In fact, $\Pi_A \Pi_B = 0$ is an element of reality. The meaning of this equation is a trivial point that the probability to find the particle both in A and B is equal to zero. The "sum rule" does not hold either. Indeed, there is no element of reality $\Pi_A + \Pi_B = 2$. In fact, there is no element of reality for the value of the sum $\Pi_A + \Pi_B$. This means that when we perform a measurement which tells us that the particle is inside one of the boxes A or B, but without telling in which one, the probability to find it is neither zero nor one. (The probability is equal to 2/3, but this does not correspond to any element of reality.) Note, however, that there is an element of reality for the sum of the three projection operators: $\Pi_A + \Pi_B + \Pi_C = 1$; clearly, the measurement testing the existence of the particle in the three boxes will say yes with probability one.

The elements of reality for pre- and post-selected quantum systems have unusual and counterintuitive properties. But, may be this is not because of the illness of their definition, but due to bizarre features of quantum systems which goes against the intuition built during thousands of years, when the results of quantum experiments were not known.

5. WEAK-MEASUREMENT ELEMENTS OF REALITY OF PRE- AND POST-SELECTED SYSTEMS

The next natural step is to consider the limit of weak measurements performed on pre- and post-selected quantum systems. Again we consider the measuring Hamiltonian (1), the initial state of the measuring device (2), and the limit of large Δ. In general, this will lead to a very low precision of the measurement, so we consider an ensemble of identical systems with such measurements. The difference here from the weak measurement of Sec. 3 is that now, before reading the outcomes of the measuring devices, we post-select a certain state of the system and discard the readings of measuring devices corresponding to the systems for which the post-selection was not successful.

I will not repeat here the calculations; they can be found in Ref. 7. The above procedure is called *weak measurement* and it indeed converges to a well-defined value. At the limit of large Δ, the probability distribution of the final state of the measuring device converges to the initial distribution shifted by the real part of the *weak value* of the variable A:

$$A_w = \frac{\langle \Psi_2 | A | \Psi_1 \rangle}{\langle \Psi_2 | \Psi_1 \rangle} \tag{8}$$

Our definition of elements of reality, i.e., a definite shift of the probability distribution of the pointer variable yields for pre- and post-selected systems the weak value (8). Even the imaginary part of the weak value falls under this definition, but only for a particular case of the Gaussian as the initial measuring device state. In order to see that, we have to consider, instead of the pointer position distribution, the distribution of the conjugate momentum. It turns out[7] that the original Gaussian in P does not change its shape (again, in the limit of a large Δ) and is shifted by the value $\mathrm{Im}(A_w)$.

The weak-measurement elements of reality of pre-selected only systems, the expectation values, are a particular case of the weak values. Indeed, we can consider a future measurement which tests that we are in the initial state $|\Psi_1\rangle$. The weak measurement, by assumption, does not change the state of the system significantly, and therefore, this future measurement *must* yield the state $|\Psi_1\rangle$. But then, the definition of weak value (8) yields the expectation value.

The advantage of weak-measurement elements of reality is that they are well defined for any situation and for all variables. It also has some classical type features. The "sum rule" holds. Indeed, if $C = A + B$, then $C_w = A_w + B_w$. Therefore, if $A_w = a$ is a weak-measurement element of reality and $B_w = b$ is a weak-measurement element of reality, then $(A+B)_w = a + b$ is also a weak-measurement element of reality. The "product rule," however, does not hold. From $C = AB$ does not follow that $C_w = A_w B_w$.

The main disadvantage of weak-measurement elements of reality is again that usually we need an ensemble of identical pre- and post-selected systems in order to find the weak values of their quantum variables with good precision. However, there are certain important cases in which the *weak* measurement need not be too weak and the "weak" value can be found with relatively good precision from a single such measurement. For example, if a spin N particle is prepared in the state $S_x = N$ and later found in the state $S_y = N$, then, at the intermediate time the weak value of the spin component in the direction ξ which bisects $\hat{x}$ and $\hat{y}$ is larger than N. Indeed, $(S_\xi)_w = \sqrt{2}\, N$. An experimenter can repeatedly see this "forbidden" value in a standard measurement with precision of order $\sqrt{N}$. Note, however, that for any "unusual" weak values the probability to obtain the required result of the post-selection is extremely small. (In the last example this probability is equal to 2^{-N}.)

The concept of weak-measurement elements of reality is a generalization of the usual conept of the (strong-measurement) element of reality. Indeed, if we know with certainty that a strong measurement of A will yield $A = a$ with probability one, then we know that the weak measurement will also yield $A_w = a$.[10] Thus, all (strong) elements of reality are also weak elements of reality. The class of weak elements of reality is much wider; it is defined for all variables for any realizable pre- and post-selected situation (as well as for the pre-selection only situation). In contrast, the strong elements of reality are defined only for some variables in each situation and somtimes they do not exist at all. (There is no *local* element of reality for spin variables of an EPR pair.)

Let us analyze the example of the three boxes we have introduced above. Since we know several (ideal-measurement) elements of reality, we can immediately write down the corresponding weak-measurement elements of reality for the discussed pre- and post-selected particle:

$$(\Pi_A)_w = 1, \qquad (\Pi_B)_w = 1, \qquad (\Pi_A + \Pi_B + \Pi_C)_w = 1 \tag{9}$$

Now using the sum rule we obtain another weak-measurement element of reality:

$$(\Pi_C)_w = (\Pi_A + \Pi_B + \Pi_C)_w - (\Pi_A)_w - (\Pi_B)_w = -1 \tag{10}$$

To say that there is a "reality" of having -1 particle in a box sounds paradoxical. However, when we test this reality weakly this is what we see. We cannot see this "reality" for one particle because the uncertainty of the appropriate weak measurement has to be much larger than 1, but if we have a larger number of such pre- and post-selected particles in the three boxes, then a realistic measurement of the pressure in the boxes will yield p for boxes A and B and the negative pressure, $-p$, for the measurement in the box C. Note, however, that the probability to obtain in the post-selection measurement the state $|\Psi_2\rangle$ for a macroscopic number of particle is extremely small.

6. CONCLUSION

The name "element of reality" suggests an ontological meaning. However, historically, and in the present paper, the element of reality is an epistemological concept. It is better to have ontological elements of reality, but there are severe difficulties in constructing them. Probably, the most serious attempt in this direction is the causal interpretation[12] which introduces "real" point-like particles moving according to a simple (but nonlocal) law. However, it seems that we are foced to accept that the total quantum wave in the causal interpretation has also ontological status.[13] I feel that if we do consider the quantum wave function of the universe as an ontological reality, we need not add anything else, e.g., Bohmian particles. I am perfectly ready to accept that the reality of our physical universe *is* its wave function. But since that reality is very far from what we experience, I think it is fruitful to define epistemological reality as it has been done here.

I certainly see a deficiency of weak-measurement elements of reality defined above in the situations in which they cannot be measured on a single system. Still, I do not think that the fact that a weak value cannot be measured on a single system prevents it from being a "reality." We know that the measuring device shifts its pointer exactly accoding to the weak value, even though we cannot find it because of the large uncertainty of the pointer position. We can verify this knowledge performing measurements on an ensemble. In the cases where weak values can be found with good precision on a single system, the concept of weak-measurement elements of reality is fully justified. In fact, the word "weak" is not exactly appropriate, since the measurements in question are the usual one. (However, they are not "ideal" in the von Neumann sense, since the measuring Hamiltonian has to be bounded.)

Another example of measurements which are "good measurements" in the sense that they yield a measured quantity with good precision and which yield weak-measurement elements of reality are *protective measurements*. For pre-selected protected systems these elements of reality are again expectation values.[14] It has been shown that even the two-state vector can be protected and that the weak values can be measured on any single (appropriately protected) system with good precision.[15] Finally, it has been shown recently[16] that adiabatic measurements performed on decaying systems which were post-selected not to decay, yield one of the (nontrivial, i.e., not expectation value) weak values even without specific pre- and post-selection, manifesting again the physical meaning of the weak values.

The concept of elements of reality for pre- and post-selected systems fits very well the many-worlds interpretation[17] which I believe is the best available interpretation today.[18] It answers in a very convincing way the following difficulty. Consider a present moment of time t. We can assign a list of elements of reality and weak-measurement elements of reality based on the quantum state at that time. It includes the eigenvalues of variables for which the quantum state is an eigenstate and expectation values for all variables. In particular, let us assume that a spin N particle has an element of reality $S_x = N$, and consequently it has no element of reality regarding the value of S_y, but it has weak-measurement elements of reality $(S_y)_w = 0$. Let us assume that at a later time the spin in the y direction was measured and the (very improbable for large N) result $S_y = N$ was obtained. If now we assign elements of reality for the time t, we see that the list is different. Indeed, we have to add some elements of reality, e.g., $S_y = N$ is our case, and we have to change some weak-measurement elements of reality, e.g., $(S_y)_w = N$. How can we associate different elements of reality for the same moment of time? The answer is natural in the framework of the many-worlds interpretation. We discuss here epistemological element of reality of conscious beings. Before the measurement of S_y we considered one world corresponding to a certain quantum state and in this world we had certain elements of reality. The measurement of S_y generated $2N+1$ new worlds corresponding to different outcomes of the measurement. The conscious beings (the experimenters) in these different worlds have, not surprisingly, different sets of elements of reality.

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