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Applications of a simple quantum mechanical formula

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(Received 26 June 1995; accepted 14 December 1995)

New applications of the formula $A|\psi\rangle=\langle A\rangle|\psi\rangle+\Delta A|\psi_\perp\rangle$ are discussed. Simple derivations of the Heisenberg uncertainty principle and of related inequalities are presented. It is also shown that a state corresponding to a maximal uncertainty of any given observable cannot be unique.

I. INTRODUCTION

The topic of this paper concerns a simple formula, rarely mentioned in the literature, which can serve as a helpful tool in quantum mechanics. It has been shown[1] that for any Hermitian operator A and any quantum state $|\psi\rangle$, the following formula is valid:

$$A|\psi\rangle=\langle A\rangle|\psi|+\Delta A|\psi_\perp\rangle, \tag{1}$$

where $|\psi\rangle$, $|\psi_\perp\rangle$ are normalized vectors, $\langle\psi_\perp|\psi\rangle=0$, $\langle A\rangle\equiv\langle\psi|A|\psi\rangle$, and $\Delta A\equiv\sqrt{\langle A^2\rangle-\langle A\rangle^2}$. The proof is as follows: It is always possible to make a decomposition $A|\psi\rangle=\alpha|\psi\rangle+\beta|\psi_\perp\rangle$ with $\beta\geqslant 0$. Then $\langle\psi|A|\psi\rangle=\langle\psi|(\alpha|\psi\rangle+\beta|\psi_\perp\rangle)$ yields $\alpha=\langle A\rangle$, and $\langle\psi|A^\dagger A|\psi\rangle=(\alpha^*\langle\psi|+\beta^*\langle\psi_\perp|)\times(\alpha|\psi\rangle+\beta|\psi_\perp\rangle)$ yields $\beta=\Delta A$.

In Ref. 1 the formula has been applied to a composite system consisting of a large number of parts in a product state. It has been proven that such a product state is essentially an eigenstate of an operator defined as an "average" of variables corresponding to these parts. The formula has also been used in a simple derivation of the minimal time for the evolution of a quantum system to an orthogonal state.[2] Our aim here is to show new applications of this formula. In Sec. II an immediate result related to maximal uncertainty states is obtained. In Sec. III the formula is used to derive, in a simple way, the Heisenberg uncertainty principle and other related inequalities.

II. A MAXIMAL UNCERTAINTY STATE IS NOT UNIQUE

Let us start by rewriting our basic formula in the form

$$A|\psi\rangle=\langle A\rangle_{\psi}|\psi\rangle+\Delta A_{\psi}|\psi_{\perp}\rangle. \tag{2}$$

Then, the scalar product of $|\psi_{\perp}\rangle$ and $A|\psi\rangle$ is

$$\langle\psi_{\perp}|A|\psi\rangle=\Delta A_{\psi}. \tag{3}$$

For $A|\psi_{\perp}\rangle$ the formula gives

$$A|\psi_{\perp}\rangle=\langle A\rangle_{\psi_{\perp}}|\psi_{\perp}\rangle+\Delta A_{\psi_{\perp}}|\psi_{\perp\perp}\rangle, \tag{4}$$

where $\langle\psi_{\perp\perp}|\psi_{\perp}\rangle=0$. Substituting Eq. (4) in Eq. (3) yields

$$\Delta A_{\psi}=\Delta A_{\psi_{\perp}}\langle\psi|\psi_{\perp\perp}\rangle. \tag{5}$$

Since $|\langle\psi|\psi_{\perp\perp}\rangle|\leq 1$, Eq. (5) leads to

$$\Delta A_{\psi_{\perp}}\geq\Delta A_{\psi}. \tag{6}$$

Thus, we have proved the following theorem:

For any Hermitian operator A and any given state $|\psi\rangle$ there exists a state $|\psi_{\perp}\rangle$ orthogonal to $|\psi\rangle$, such that

$$\Delta A_{\psi_{\perp}}\geq\Delta A_{\psi}.$$

This implies that a state corresponding to a maximal uncertainty of any given observable cannot be unique.

III. THE HEISENBERG UNCERTAINTY PRINCIPLE

We present here a simple method, based on the formula [Eq. (1)], to obtain the Heisenberg uncertainty principle. Consider two Hermitian operators, A and B, in arbitrary Hilbert space. Then, the following equations hold:

$$A|\psi\rangle=\langle A\rangle|\psi\rangle+\Delta A|\psi_{\perp A}\rangle, \tag{7}$$

$$B|\psi\rangle=\langle B\rangle|\psi\rangle+\Delta B|\psi_{\perp B}\rangle, \tag{8}$$

where $\langle\psi_{\perp A}|\psi\rangle=0$ and $\langle\psi_{\perp B}|\psi\rangle=0$. Note that the quantities $\langle A\rangle$, $\langle B\rangle$, ΔA, and ΔB are all real numbers. Multiplying the Hermitian conjugate of Eq. (8) by Eq. (7), and using the fact that $B=B^{\dagger}$, we obtain

$$\langle\psi|BA|\psi\rangle=(\langle B\rangle\langle\psi|+\Delta B\langle\psi_{\perp B}|)(\langle A\rangle|\psi\rangle+\Delta A|\psi_{\perp A}\rangle)$$
$$=\langle B\rangle\langle A\rangle+\Delta B\Delta A\langle\psi_{\perp B}|\psi_{\perp A}\rangle. \tag{9}$$

Similarly,

$$\langle\psi|AB|\psi\rangle=\langle A\rangle\langle B\rangle+\Delta A\Delta B\langle\psi_{\perp A}|\psi_{\perp B}\rangle. \tag{10}$$

Subtracting Eq. (9) from Eq. (10) yields

$$\langle[A,B]\rangle=2i\Delta A\Delta B\ \mathrm{Im}\langle\psi_{\perp A}|\psi_{\perp B}\rangle. \tag{11}$$

Since the vectors are normalized, we can use $|\mathrm{Im}\langle\psi_{\perp A}|\psi_{\perp B}\rangle|\leq 1$ to end with

$$\Delta A\Delta B\geq\tfrac{1}{2}|\langle[A,B]\rangle|, \tag{12}$$

which is standard form of the uncertainty principle.

Another interesting inequality can be obtained by calculating the anti-commutator of A and B. We add Eq. (9) to Eq. (10) and get

$$\langle\{A,B\}\rangle=2\langle A\rangle\langle B\rangle+2\Delta A\Delta B\ \mathrm{Re}\langle\psi_{\perp A}|\psi_{\perp B}\rangle, \tag{13}$$

where $\{A,B\}=AB+BA$. Rearranging Eq. (13) and taking the absolute values of both sides, we find

$$\Delta A\Delta B|\mathrm{Re}\langle\psi_{\perp A}|\psi_{\perp B}\rangle|=|\tfrac{1}{2}\langle\{A,B\}\rangle-\langle A\rangle\langle B\rangle|. \tag{14}$$

Since $|\mathrm{Re}\langle\psi_{\perp A}|\psi_{\perp B}\rangle|\leq 1$, it follows that

$$\Delta A\Delta B\geq|\tfrac{1}{2}\langle\{A,B\}\rangle-\langle A\rangle\langle B\rangle|. \tag{15}$$

This inequality is a by-product of a conventional derivation of the uncertainty principle which is based on the Cauchy–Schwarz inequality.[3] The physical significance of Eq. (15) is that it provides an estimate for the correlations developed in time between A and B. For example, it manifests the correlation between x and p for the case of a free particle evolving in time.[4]

We can also obtain a more accurate estimate for $\Delta A\Delta B$. Adding Eq. (11) to Eq. (13), we find

$$\langle[A,B]\rangle+\langle\{A,B\}\rangle=2i\Delta A\Delta B\ \mathrm{Im}\langle\psi_{\perp A}|\psi_{\perp B}\rangle$$
$$+2\Delta A\Delta B\ \mathrm{Re}\langle\psi_{\perp A}|\psi_{\perp B}\rangle+2\langle A\rangle\langle B\rangle, \tag{16}$$

and consequently,

$$\Delta A\Delta B\langle\psi_{\perp A}|\psi_{\perp B}\rangle=\tfrac{1}{2}\langle[A,B]\rangle+\tfrac{1}{2}\langle\{A,B\}\rangle-\langle A\rangle\langle B\rangle. \tag{17}$$

Taking the norm of both sides we find

$$\Delta A\Delta B\geq|\tfrac{1}{2}\langle[A,B]\rangle+\tfrac{1}{2}\langle\{A,B\}\rangle-\langle A\rangle\langle B\rangle|. \tag{18}$$

Since $[A,B]=iC$ and $\{A,B\}=D$, where C and D are Hermitian operators, and since the expectation value of an Hermitian operator is a real number, it follows that

$$\Delta A\Delta B\geq[(\tfrac{1}{2}\langle\{A,B\}\rangle-\langle A\rangle\langle B\rangle)^{2}+\tfrac{1}{4}|\langle[A,B]\rangle|^{2}]^{1/2}. \tag{19}$$

This result combines the two previously found bounds, namely Eqs. (12) and (15).

[1] Y. Aharonov and L. Vaidman, "Properties of a Quantum System During the Time Interval Between Two Measurements," Phys. Rev. A **41**, 11–20 (1990).

[2] L. Vaidman, "Minimal Time for Evolution to an Orthogonal State," Am. J. Phys. **60**, 182–183 (1992).

[3] See, for instance, S. Wieder, *The Foundations of Quantum Theory* (Academic, New York, 1973), pp. 64–65.

[4] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951), pp. 203–207.