Negative Kinetic Energy between Past and Future State Vectors[a]

DANIEL ROHRLICH, YAKIR AHARONOV,[b]
SANDU POPESCU, AND LEV VAIDMAN

Raymond and Beverly Sackler Faculty of Exact Sciences
School of Physics and Astronomy
Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel

INTRODUCTION

When the word "quantum" first entered the language of physics, it meant a restriction on possible values of energy, and it is still axiomatic that the only observable values of a physical quantity are the eigenvalues of a corresponding quantized operator. When we obtain values that are not eigenvalues, we interpret them as errors. Still, measurements are uncertain in practice and can even yield classically forbidden, "unphysical" values. We have uncovered remarkable regularities in the way that "unphysical" values can appear in sequences of measurements, suggesting that these values may not be unphysical at all. In quantum theory, it seems, not only are physical quantities not restricted: they can take values outside the classically allowed range. Here, we discuss this new effect in the context of barrier penetration by quantum particles.

Barrier penetration, such as tunneling out of a potential well, is a classically forbidden quantum process. Quantum particles can be found in regions where a classical particle could never go: it would have negative kinetic energy. However, the eigenvalues of kinetic energy cannot be negative. How, then, can a quantum particle "tunnel"? The apparent paradox is resolved by noting that the wave function of a tunneling particle only partly overlaps the forbidden region, whereas a particle found within the forbidden region may have taken enough energy from the measuring probe to offset any energy deficit. Nevertheless, actual measurements of kinetic energy can yield negative values. Here, we present a model experiment in which we measure the kinetic energy of a bound particle to any desired precision. We then attempt to localize the particle within the classically forbidden region. The attempt rarely succeeds, but, whenever it does, we find that the kinetic energy measurements gave an "unphysical" negative result; moreover, these results cluster around the appropriate value, that is, the difference between the total and the potential energy. This consistency, which seems to come from nowhere (a background of errors), suggests strongly that the notion of a quantum observable is richer than generally

[a]This research was supported by Grant No. 425/91-1 of the Basic Research Foundation (administered by the Israel Academy of Sciences and Humanities) and by Grant No. PHY-8807812 of the National Science Foundation.

[b]Also at Department of Physics, University of South Carolina, Columbia, South Carolina 29208.

realized. Previous papers making this suggestion have analyzed a measurement of spin[1] and a quantum time machine[2] as well as negative kinetic energy.[3,4]

NEGATIVE KINETIC ENERGY

Our example may be summarized as follows: We prepare a large ensemble of particles bound in a potential well, in an eigenstate of energy, and measure the kinetic energy of each particle to a given precision. Then, we measure the position of each particle and select only those cases where the particle is found within some region "far enough" from the well, with "far enough" depending on how precisely the kinetic energy was measured. In almost all such cases, we find that the measured kinetic energy values are *negative* and cluster around the particular negative value appropriate to particles in the classically forbidden region. Also, the spread of the clustering is the characteristic spread for kinetic energy measurements with this device.

We begin with a particle trapped in a potential well. The Hamiltonian is $H = (p^2/2m) + V(x)$, with $V(x) = -V_0$ for $|x| < a$ and $V(x) = 0$ for $|x| > a$. We prepare an ensemble of particles in the ground state, with energy $E_0 < 0$: $|\Psi_{in}\rangle = |E_0\rangle$. Following von Neumann,[5] we model a measurement of kinetic energy with an interaction Hamiltonian $H_{int} = g(t)P(p^2/2m)$, where P is a canonical momentum conjugate to the position, Q, of a pointer on the measuring device. The time-dependent coupling constant $g(t)$ is nonzero only for a short time interval and is normalized so that $\int g(t)dt = 1$. When the time interval is very short, we call the measurement impulsive. For an impulsive measurement, H_{int} dominates the Hamiltonians of the measured system and the measuring device. Then, since $\dot{Q} = (i/\hbar)\,[H_{int}, Q]$, we obtain

$$Q_{fin} - Q_{in} = \frac{p^2}{2m} \qquad (1)$$

for the operator Q.

In an ideal measurement, the position of the pointer is precisely defined and thus we read a precise value of kinetic energy. However, in practice, measurements involve uncertainty. To model a source of uncertainty, we take the initial state of the pointer to be

$$\Phi_{in}(Q) = (\epsilon^2\pi)^{-1/4}e^{-Q^2/2\epsilon^2}. \qquad (2)$$

The uncertainty in the initial position of the pointer produces errors of order ϵ; when $\epsilon \to 0$, we recover the ideal measurement. Thus, *any* measured value is possible, although large errors are exponentially suppressed. There is no mystery in such errors; they are expected, given the uncertainty associated with the measuring device. Measurements can even yield negative values. The negative values may be unphysical, but they are part of a distribution representing the measurement of a physical quantity. They should not be thrown out because they give information about the distribution and contribute to the best estimate of the peak value. Given the fact that these errors originate in the measuring device and not in the system under study, it seems that they cannot depend on any property of the system.

However, closer analysis of these errors reveals a pattern that clearly reflects properties of the system under study. The pattern emerges only after selection of a particular final state of the system.

Initially, the particle and device are in a product state $\Psi_{in}(x)\Phi_{in}(Q)$; after the interaction is complete, the state is $e^{-(i/\hbar)Pp^2/2m}\Psi_{in}(x)\Phi_{in}(Q)$, in which the particle and the device are correlated. Now, we consider kinetic energy measurements followed by a final measurement of position, with the particle found far outside the potential well. For the final state, we choose a Gaussian wave packet with its center far from the potential well,

$$\Psi_{fin}(x) = (\delta^2\pi)^{-1/4}e^{-(x-x_0)^2/2\delta^2}, \tag{3}$$

and we require $\delta > \alpha\hbar^2/m\epsilon$. The condition for the particle to be "far enough" from the potential well is

$$\alpha x_0 \gg (\alpha^2\hbar^2/2m\epsilon)^2. \tag{4}$$

Since $\alpha^2\hbar^2/2m = |E_0|$, the expression in the parentheses is the ratio of the magnitude of the effect, $|E_0|$, to the precision of the measurement, ϵ. For more precise measurements of kinetic energy ($\epsilon \to 0$), the final state is selected at increasing distances from the potential well ($x_0 \to \infty$).

The state of the measuring device after the measurement, and after the particle is found in the state $\Psi_{fin}(x)$, is obtained by projecting the correlated state of the particle and measuring device onto the final state of the particle, $\Psi_{fin}(x)$. Apart from normalization, it is $\Phi_{fin}(Q) = \langle\Psi_{fin}|e^{-(i/\hbar)Pp^2/2m}|\Psi_{in}\rangle\Phi_{in}(Q)$. For simplicity, we take $V(x)$ to be a delta-function potential ($a \to 0$). Then, $\Psi_{in}(x)$ is $\sqrt{\alpha}\,\exp(-\alpha|x|)$. As an integral over x, the final state is

$$\Phi_{fin}(Q) = \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/2\delta^2}e^{-(i/\hbar)Pp^2/2m}e^{-\alpha|x|}\Phi_{in}(Q) \tag{5}$$

up to normalization. Note that the exponential of $-iPp^2/2m\hbar$ acts to translate Q in $\Phi_{in}(Q)$. If we could ignore the part of the integral near $x = 0$, we could replace p^2 with $-\alpha^2$ in equation 5 and the final state of the measuring device would be $\Phi_{fin}(Q) = \Phi_{in}(Q + \alpha^2\hbar^2/2m)$. We cannot ignore this part of the integral, but we can suppress it by choosing x_0 in $\Psi_{fin}(x)$ to be large. If we express $\Psi_{in}(x)$ via its Fourier transform and replace the operator p with its eigenvalue, we obtain (up to a normalizing factor)

$$\Phi_{fin}(Q) = \left(\frac{\pi}{\hbar\alpha}\right) e^{\alpha x_0 - \alpha^2\delta^2/2} \int dp \left[\frac{e^{-p^2\delta^2/2\hbar^2 - ipx_0/\hbar}}{(\alpha^2\hbar^2 + p^2)}\right] \Phi_{in}(Q - p^2/2m). \tag{6}$$

This integral has poles at $p = \pm i\alpha\hbar$; we evaluate it by integration on a contour including a line of p with imaginary part $-ip_0$, for any $p_0 > \hbar\alpha$. The integral in equation 6 then reduces to two terms: a pole term,

$$\Phi_{in}(Q + \alpha^2\hbar^2/2m), \tag{7}$$

and a correction term, the integral in equation 6 with p replaced by $p - ip_0$. The pole term represents the measuring device with its pointer shifted to the negative value of $-\alpha^2\hbar^2/2m$. A short computation (see reference 4) shows that the correction term can be made arbitrarily small by taking x_0 large, as in equation 4. For x_0 large, the final state of the measuring device shows the "unphysical" result of $-\alpha^2\hbar^2/2m$ for the kinetic energy, up to a scatter ϵ characteristic of the device.

We thus obtain a correlation between position measurements and prior kinetic energy measurements: nearly all particles found far outside the potential well yielded negative values of kinetic energy. On the other hand, we could consider all particles that produced negative values of kinetic energy and could ask about their final position. We would find nearly all these particles *inside* the well. The correlation works one way only. Prior kinetic energy measurements on particles found far from the well cluster around a negative value, but position measurements on particles yielding negative values of kinetic energy cluster around zero. How do we interpret this one-way correlation?

INTERPRETATION

Our example suggests that particles in a classically forbidden region have negative kinetic energy. The conventional interpretation of quantum mechanics has no place for negative kinetic energy. However, the conventional interpretation involves an assumption about how measurements are made. The conventional interpretation considers measurements on ensembles of systems prepared in an initial state, without any conditions on the final state of the systems. Such an ensemble, defined by initial conditions only, may be termed a *preselected* ensemble. By contrast, we consider measurements made on *preselected and postselected* ensembles, defined by both initial and final conditions. The experiment of the previous section is an example of a measurement on a preselected and postselected ensemble. It is natural to introduce preselected and postselected ensembles in quantum theory: in the quantum world, unlike the classical world, complete specification of the initial state does not determine the final state.

Also, the measurements that we consider are not *ideal.* Real measurements are subject to error. At the same time, the disturbance they make is bounded. These two aspects of nonideal measurements go together. Suppose our measuring device interacts very weakly with the systems in the ensemble. We pay a price in precision. On the other hand, the measurements hardly disturb the ensemble and therefore they characterize the ensemble during the whole intermediate time. Even noncommuting operators can be measured at the same time if the measurements are imprecise. When such measurements are made on preselected and postselected ensembles, they yield surprising results. An operator yields *weak* values that need not be eigenvalues or even classically allowed.[1,6] The negative kinetic energy of the previous section is an example of a weak value. Another is a measurable value of 100 for a spin component of a spin-1/2 particle.[1]

Let us briefly review how weak values arise. The initial wave function of the measuring device is $\Phi_{in}(Q)$. After an impulsive measurement of an operator C on an initial state $|a\rangle$, and projection onto a final state $|b\rangle$, the final state of the measuring device is $\langle b|e^{-iPC/\hbar}|a\rangle\Phi_{in}(Q) = \Sigma_i\langle b|c_i\rangle\langle c_i|a\rangle\Phi_{in}(Q - c_i)$. If $\Phi_{in}(Q)$ is sharply peaked, then the various terms $\Phi_{in}(Q - c_i)$ will be practically orthogonal. However, suppose $\Phi(Q)$ has a width of ϵ. Its Fourier transform has a width in P of $\hbar/\epsilon$. Small $|P|$ corresponds to a measuring device that is coupled weakly to the measured system. If

ϵ is large, then $|P|$ is small and we can expand the exponential $e^{-iPC/\hbar}$ to first-order in P to obtain $\langle b|e^{-iPC/\hbar}|a\rangle\Phi(Q) \approx \langle b|1 - iPC/\hbar|a\rangle\Phi(Q) \approx \langle b|a\rangle e^{-iPC_w/\hbar}\Phi(Q)$. Here, $C_w \equiv \langle a|C|b\rangle/\langle a|b\rangle$ is the *weak value* of the operator C for the preselected and postselected ensemble defined by $\langle b|$ and $|a\rangle$.

The definition of a weak value provides us with a new and intuitive language for describing quantum processes. In our example, the operators of total energy E, kinetic energy K, and potential energy V do not commute. Therefore, the classical formula $E = K + V$ applies only to their expectation values, and the expectation value of K in any state is positive. However, the formula applies to weak values, $E_w = K_w + V_w$, and the weak value of K is *not* necessarily positive. We know that $E_w = E_0 = -\alpha^2\hbar^2/2m$ because the preselected state is an energy eigenstate, and that V_w vanishes because the postselected state is far from the potential well. Then, $K_w = -\alpha^2\hbar^2/2m$, the "unphysical" result obtained above in our example.

In our example, instead of the condition on the initial state of the measuring device (ϵ large), we had a condition on the final state of the particle (x_0 large and $\delta > \alpha\hbar^2/m\epsilon$). The price is that we must wait for increasingly rare events. As measurements of kinetic energy become more precise ($\epsilon \to 0$), they disturb the particle more. To get negative kinetic energies, we must postselect particles further from the potential well ($x_0 \to \infty$). As the precision of the measurement increases, negative kinetic energies become less and less frequent; in the limit of ideal measurements, the probability vanishes and thus ideal measurements of kinetic energy never yield negative values.

CONCLUSIONS

From the point of view of standard quantum theory, all that we have produced is a game of errors of measurement. Ideal measurements of kinetic energy can yield only positive values because all eigenvalues of the kinetic energy operator are positive. However, in practice, measurements are not exact and, even if their precision is very good, sometimes—rarely—they yield negative values. If particles are subsequently found far from the potential well, we have seen that the measured kinetic energy of these particles comes out negative. Consistently, large measurement "errors" did occur, producing a distribution peaked at the "unphysical" negative value E_0.

What special properties of nonideal measurements led to this result? First, these measurements involve only bounded disturbances of particle position. Second, because their precision is limited, they can supply, "by error", the necessary negative values. These two properties are intimately connected: any measurement of kinetic energy causing only bounded changes of position must occasionally yield negative values for the kinetic energy. The change of x due to the measurement is $\dot{x} = (i/\hbar)g(t)[x, Pp^2/2m]$. P and p are unchanged during the measurement, so $x_{fin} - x_{in} = Pp/m$. From here, it follows that the change of x is bounded only if the pointer is in an initial state with P bounded, that is, if the Fourier transform of $\Phi_{in}(Q)$ has compact support. Then, however, the support of $\Phi_{in}(Q)$ is unbounded,[7] which immediately implies a nonzero probability for the pointer to indicate negative values ($Q < 0$). Indeed, the "game of errors" displays a remarkable consistency, and this consistency allows negative kinetic energies to enter physics in a natural way. The concept of a *weak* value of a quantum operator gives precise meaning to the statement that the kinetic energy of a particle in a classically forbidden region is negative: namely, the weak value of the kinetic energy is negative.

REFERENCES AND NOTES

1. AHARONOV, Y., D. ALBERT & L. VAIDMAN. 1988. Phys. Rev. Lett. **60:** 1351.
2. AHARONOV, Y., J. ANANDAN, S. POPESCU & L. VAIDMAN. 1990. Phys. Rev. Lett. **64:** 2965.
3. AHARONOV, Y., S. POPESCU, D. ROHRLICH & L. VAIDMAN. 1993. *In* Proceedings of the Fourth International Symposium on Foundations of Quantum Mechanics (Tokyo). JJAP Series No. 9, p. 251.
4. AHARONOV, Y., S. POPESCU, D. ROHRLICH & L. VAIDMAN. 1993. Phys. Rev. **A48:** 4084.
5. VON NEUMANN, J. 1955. Mathematical Foundations of Quantum Theory. Princeton University Press. Princeton, New Jersey.
6. AHARONOV, Y. & L. VAIDMAN. 1990. Phys. Rev. **A41:** 11; AHARONOV, Y. & D. ROHRLICH. 1990. *In* Quantum Coherence (Proceedings of the Conference on Fundamental Aspects of Quantum Theory, Columbia, South Carolina). J. S. Anandan, Ed. World Scientific. Singapore.
7. If the Fourier transform of $\Phi_{in}(Q)$ has compact support, then $\Phi_{in}(Q)$ is analytic. The two derivations of our result, via contour integration in the second section and via Taylor expansion of the exponential in the third section, both require $\Phi_{in}(Q)$ to be analytic.