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How to Ascertain the Values of σ_x, σ_y, and σ_z of a Spin-$\frac{1}{2}$ Particle

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A procedure is described whereby the result of the measurement of *any* of the three Cartesian components of the spin of a single spin-$\frac{1}{2}$ particle at a single time can be inferred with certainty from the result of two other measurements, one of which is carried out before, and the other after, the time in question.

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The standard quantum-mechanical prescriptions for predicting or retrodicting the results of experiments never yield dispersion-free inferences about the results of measurements of two or more noncommuting observables of a single system at a single time. However, as is well known by now, the appropriate combination of the predictive and the retrodictive prescriptions can produce dispersion-free inferences about two such measurements (either one of which may actually occur) on a single system at a single time. We shall here describe a method whereby the results of any one of *more* than two such measurements can be inferred with certainty.[1]

Suppose that a measurement of σ_x, σ_y, or σ_z (but only *one* of those) is performed on a spin-$\frac{1}{2}$ particle at a given time. By means of measurements carried out both before and after the time in question (and no matter what the results of those measurements may happen to be), we shall show how to ascertain, with probability 1, the result of the spin measurement, even though we do not know in which direction the spin was measured. That is, our method will produce inferences such as the following: If σ_x was measured the result was "up," if σ_y was measured the result was "down," and if σ_z was measured the result was "down."

This seems to us to be a very striking and surprising result. To make inferences about *two* noncommuting observables at a single time is, after all, fairly straightforward. We can, for example, ascertain both σ_x and σ_y at a given moment simply by measuring the spin in the x direction before, and in the y direction after, this time. But to make definite inferences about σ_x, σ_y, and σ_z is quite another matter, the possibility of which is by no means obvious.

We shall describe our method here in the conventional language of quantum mechanics, albeit this method was discovered by thinking in a new and more flexible language (the language of multiple-time states), which we shall present in detail elsewhere.[2]

Our method runs as follows. Before the spin measurement we take another, "external," spin-$\frac{1}{2}$ particle and we prepare the composite system of these two spin-$\frac{1}{2}$ particles in the correlated (EPR-type) state,

$$\Psi = \tfrac{1}{2}\sqrt{2}(|\uparrow\rangle_{\text{ext}}|\uparrow\rangle + |\downarrow\rangle_{\text{ext}}|\downarrow\rangle), \tag{1}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ_z. After the spin measurement, we perform on the composite system a measurement of an operator A which has nondegenerate

TABLE I. Possible outcome of spin measurement.

The value of A	The result of a possible measurement of any one of the three Cartesian components of the spin On the first particle x	 y	 z	 On the external particle x	 y	 z
a_1	↑	↑	↑	↑	↓	↑
a_2	↓	↓	↑	↓	↑	↑
a_3	↑	↓	↓	↑	↑	↓
a_4	↓	↑	↓	↓	↓	↓

eigenstates Φ_i,

$$\begin{aligned}
\Phi_1&=\tfrac{1}{2}\sqrt{2}\,|\uparrow\rangle_{\text{ext}}|\uparrow\rangle+\tfrac{1}{2}(|\uparrow\rangle_{\text{ext}}|\downarrow\rangle e^{i\pi/4}+|\downarrow\rangle_{\text{ext}}|\uparrow\rangle e^{-i\pi/4}),\\
\Phi_2&=\tfrac{1}{2}\sqrt{2}\,|\uparrow\rangle_{\text{ext}}|\uparrow\rangle-\tfrac{1}{2}(|\uparrow\rangle_{\text{ext}}|\downarrow\rangle e^{i\pi/4}+|\downarrow\rangle_{\text{ext}}|\uparrow\rangle e^{-i\pi/4}),\\
\Phi_3&=\tfrac{1}{2}\sqrt{2}\,|\downarrow\rangle_{\text{ext}}|\downarrow\rangle+\tfrac{1}{2}(|\uparrow\rangle_{\text{ext}}|\downarrow\rangle e^{-i\pi/4}+|\downarrow\rangle_{\text{ext}}|\uparrow\rangle e^{i\pi/4}),\\
\Phi_4&=\tfrac{1}{2}\sqrt{2}\,|\downarrow\rangle_{\text{ext}}|\downarrow\rangle-\tfrac{1}{2}(|\uparrow\rangle_{\text{ext}}|\downarrow\rangle e^{-i\pi/4}+|\downarrow\rangle_{\text{ext}}|\uparrow\rangle e^{i\pi/4}),
\end{aligned} \tag{2}$$

and eigenvalues given by

$$A\Phi_i=a_i\Phi_i,\quad i=1,\ldots,4,\quad a_i\neq a_j. \tag{3}$$

The probability for a given result $C=c_n$ for the measurement of a variable C at any time between the two measurements described above is[3]

$$p(C=c_n)=\frac{|\langle\Phi_i|P_{C=c_n}|\Psi\rangle|^2}{\sum_j|\langle\Phi_i|P_{C=c_j}|\Psi\rangle|^2}, \tag{4}$$

where $P_{C=c_j}$ is a projection operator on a space of eigenstates with eigenvalues c_j, and the sum in the denominator goes over all possible values of C.

Straightforward calculations based on this formula give us the results that are shown in Table I. (In our case $P_{C=c_j}$ is a projection operator on a space of certain states of the system of two spin-$\frac{1}{2}$ particles: those for which the value of spin of the observed particle in some direction is fixed.) The measuring procedure is almost symmetric for the external and the observed particles: If at the time between the initial and final measurements a measurement of σ_x, σ_y, or σ_z is performed on the external particle, then we can ascertain the result of that measurement as well. (These results are also presented in Table I.) However, we cannot ascertain the results of the spin measurements in the event that they were performed on both particles: In that case the measurements would disturb one another.

As a matter of fact, for any given value of the variable A, we can ascertain the result of the spin measurement not only in the x, y, and z directions, but for a continuum of directions that forms a cone. For example, for $A=a_1$, a measurement of the spin in the direction of any ray which is part of the cone including the x, y, and z directions will always give the result "up"; obviously, if we choose a ray in the cone which if formed by the continuation of the lines of the first cone, the result will be "down." For other values of A, the cones are similar, but they are rotated in space in an appropriate way to include the Cartesian rays in the correct directions (see Table I). The only lines common to all these cones, the directions for which we can ascertain the result of the spin measurement for any outcome of the A measurement, are the Cartesian axes.

We have described a procedure here whereby the result of the measurement of any one of three specified components of the spin of a single spin-$\frac{1}{2}$ particle at a given time can be inferred with certainty from the results of certain other experiments. We have also been able to prove that no procedure whatsoever can possibly suffice to produce such inferences for *more* than three such components, and that no such procedure can be developed for three such components along directions which are not mutually orthogonal.

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[1] Some curious properties of a quantum system within time intervals between two experiments have been discussed recently; see D. Albert, Y. Aharonov, and S. D'Amato, Phys. Rev. Lett. **54**, 5 (1985), and **56**, 2427 (1986); J. Bub and H. Brown, Phys. Rev. Lett. **56**, 2337 (1986); Y. Aharonov, A. Casher, D. Albert, and L. Vaidman, in Proceedings of the Conference

on New Techniques and Ideas in Quantum Measurement Theory, New York, 1986 (to be published); L. Vaidman and Y. Aharonov, *ibid.*

[2] The main ideas of the new language have already been presented [see Y. Aharonov and D. Albert, Phys. Rev. D **24**, 359 (1981); Y. Aharonov, D. Albert, and S. D'Amato, Phys. Rev. D **32**, 1975 (1985)], and a systematic description of this language will be soon submitted for publication to Phys. Rev. D.

[3] This is a generalization of the formula that appears in Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. B **134**, 1418 (1964).