"Elements of Reality" and the Failure of the Product Rule

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Abstract

The concept of "elements of reality" is analyzed within the framework of quantum theory. It is shown that elements of reality fail to fulfill the product rule. This is the core of recent proofs of the impossibility of a Lorentz-invariant interpretation of quantum mechanics. A generalization and extension of the concept of elements of reality is presented. Lorentz-invariance is restored by giving up the product rule. The consequences of giving up the "and" rule, which must be abandoned together with the product rule, are discussed.

1. "Elements of Reality"

One of the main tasks of physics is to give a (mathematical) description of reality. At the beginning of the century physicists thought that they were close to completing this task. Newtonian mechanics and classical electrodynamics explained very well most of the observed phenomena. The theory of relativity showed that "reality" is much more bizarre than the Laplacian mechanics in Cartesian space, but physics is still capable of describing it. Experiments showed, however, relativistic classical physics does not describe reality. Experiments are in extremely good agreement with another theory: quantum mechanics.

If quantum mechanics is the correct physics theory, then it is very difficult to see how physics can describe reality. Einstein, Podolsky, and Rosen[1] (EPR) argued that quantum mechanics cannot give a complete description of reality and Bell[2] even showed that it is impossible to have a (local) complete description of reality which is compatible with quantum mechanics. Essentially, he showed that using our common "classical" concepts we cannot describe the world as it is.

One way out (close to Bohr's position) is to postulate that reality is what we find in our measurements, and that quantum mechanics is a mathematical tool for calculating probabilities for these results. However, if you believe that the moon is there even if nobody looks at it, and if you have the same belief about an electron as you have about the moon, you must search for a different definition of physical reality.

Beyond the limitations on local realistic theories demonstrated by Bell, recently, several authors[3–6] claimed that there are even stronger restrictions on realistic theories: it is also impossible to build a realistic Lorentz-invariant theory consistent with quantum mechanics. Technically their results are correct, but we disagree with their conclusions. A crucial issue is what we understand as an "element of reality". Since classical physics is incorrect, it is not surprising that common classical concepts are not appropriate for a description of physical reality. As the theory of relativity taught us to change radically our concepts of space and time, quantum mechanics leads us to alter our concept of reality. We will show that the claim that Lorentz-invariance conflicts with realism relies on an assumed classical property of elements of reality, which does not hold in our (quantum) world.[7]

2. The Product Rule

The proofs of the impossibility of realistic Lorentz-invariant quantum theory used the *product rule*: If A and B commute, $A = a$ and $B = b$, then $AB = ab$.

In fact, Fine and Teller,[8] based on Bell's paper,[2] showed already in 1977 that one cannot construct a hidden variable (i.e. realistic) theory, compatible with quantum mechanics, which obeys the product rule. It was shown even more explicitly in the works of Peres.[9–10] However, the product rule holds in standard quantum mechanics, and the recent definitions of elements of reality go only half way between quantum mechanics and local hidden variables, so it is not obvious what is the status of the product rule in this case. We claim that according to the recent definition, the product rule *should not* be used in the situations which have been considered. We present simple examples in which the product rule clearly fails. But, let us first review the product rule in usual situations of quantum mechanics, where it certainly holds.

In every textbook of quantum mechanics we can find a condition for simultaneous measurability of variables A and B; the corresponding operators must commute:

$$[A, B] = 0. \tag{1}$$

Commutativity of the operators A and B is a strong sufficient condition; for a given quantum state $|\Psi\rangle$, it is sufficient to have commutativity with respect to that state:

$$[A, B]|\psi\rangle = 0. \tag{2}$$

The commutativity condition (2) is a necessary and sufficient condition for simultaneous measurability of A and B. If the operators A and B do not commute, the measurement of one disturbs the outcome of the other. For example, consider a standard measuring procedure[11] with an interaction Hamiltonian given by

$$H = g(t)pA. \tag{3}$$

Here p is a canonical momentum of the measuring device; the conjugate position q corresponds to the position of a pointer on the device. The coupling $g(t)$ is non-zero for a short time interval, and during the measurement we obtain (in the Heisenberg picture)

$$\frac{dB}{dt} = i[H, B] = ig(t)p[A, B]. \tag{4}$$

Thus, commuting operators are measurable without mutual disturbance, while non-commuting operators disturb one another.

If A and B commute, and if we know that at a given moment a measurement of A (if performed) must yield $A = a$ while a measurement of B (if performed) must yield $B = b$, we can safely claim that the product a AB is also known and equal ab. We repeat this well-known fact because, surprisingly, it is not true when we consider a pre- and post-selected quantum system.

3. The Pre- and Post-Selected Ensemble

To define a pre- and post-selected quantum system, we consider a quantum system at time t. For simplicity we let the free Hamiltonian be zero. At time $t_1 < t$ the system is prepared in a quantum state $|\Psi_1\rangle$, and at a time $t_2 > t$ a measurement is performed and the system is found in the state $|\Psi_2\rangle$. We ask about possible measurements at time t. Suppose A is measured at time t. If either $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate of A, then clearly the outcome of the measurement is determined; it is the corresponding eigenvalue of A. Measuring the commuting operator B before, after, or even during the measurement of A does not, in principle, disturb the measurement of A. However, for a pre- and post-selected quantum system it might be that the result of measuring A is certain, even if neither $|\Psi_1\rangle$ nor $|\Psi_2\rangle$ is an eigenstate of A. In this case a measurement at any time between t_1 and t_2 of certain operators commuting with A *invariably disturbs the A-measurement.*

A simple example is the setup proposed by Bohm for analyzing the EPR argument: two separate spin-1/2 particles prepared, at time t_1, in a singlet state

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle). \tag{5}$$

At time t_2 measurements of σ_{1x} and σ_{2y} are performed and certain results are obtained. If at time t, $t_1 < t < t_2$, a measurement of σ_{1y} is performed (and if this is the only measurement performed between t_1 and t_2), then the outcome of the measurement is known with certainty: $\sigma_{1y}(t) = -\sigma_{2y}(t_2)$. If, instead, only a measurement of σ_{2x} is performed at time t, the result of the measurement is also certain: $\sigma_{2x}(t) = -\sigma_{1x}(t_2)$. The operators σ_{1y} and σ_{2x} obviously commute, but nevertheless, measuring $\sigma_{2x}(t)$ clearly disturbs the outcome of the measurement of $\sigma_{1y}(t)$: it is not certain anymore.

Measuring the product $\sigma_{1y}\sigma_{2x}$, is, in principle, different from the measurement of both σ_{1y} and σ_{2x} separately. In our example the outcome of the measurement of the product *is* certain, but it does not equal the product of the results which must come out of the measurements of σ_{1y} and σ_{2x} when every one of them is performed without the other. To measure the product $\sigma_{1y}\sigma_{2x}$ we may write it as a modular sum, $\sigma_{1y}\sigma_{2x} = (\sigma_{1y} + \sigma_{2x})\text{mod}4 - 1$. It has been shown[12] that nonlocal operators such as $(\sigma_{1y}+\sigma_{2x})\text{mod}4$ can be measured using solely local interactions.

In order to find out the results of the measurements we can use the generalization of the formula of Aharonov, Bergmann, and Lebowitz[13] (ABL) for calculating probabilities for the results of an intermediate measurement performed on a pre- and post-selected system. If the initial state is $|\Psi_1\rangle$ and the post-selected state is $|\Psi_2\rangle$, then the probability for an intermediate measurement of A to yield $A = a_n$ is given by[14]

$$\text{prob}[A = a_n] = \frac{|\langle\Psi_2|\mathbf{P}_{A=a_n}|\Psi_1\rangle|^2}{\sum_k |\langle\Psi_2|\mathbf{P}_{A=a_k}|\Psi_1\rangle|^2} . \tag{6}$$

where the sum is over all eigenvalues of A and $\mathbf{P}_{A=a_k}$ is the projection operator onto the subspace with eigenvalue a_k. The formula immediately yields probability 1 when $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate, but it also can yield 1 when neither of the states is an eigenstate, as we now show.

The state $|\Psi_1\rangle$ is given by Eq. (5). Suppose the results of the post-selection measurements are $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$. Then the state $|\Psi_2\rangle = |\uparrow_{1x}\uparrow_{2y}\rangle$. To predict the outcome of a measurement of σ_{1y} we have to use the projection operators $\mathbf{P}_{[\sigma_{1y}=1]} = |\uparrow_{1y}\rangle\langle\uparrow_{1y}|$ and $\mathbf{P}_{[\sigma_{1y}=-1]} = |\downarrow_{1y}\rangle\langle\downarrow_{1y}|$. Applying formula (6), we indeed obtain $\text{prob}[\sigma_{1y} = -1] = 1$. In the same way we obtain $\text{prob}[\sigma_{2x} = -1] = 1$. For calculation of the probabilities of the measurement of the product $\sigma_{1y}\sigma_{2x}$ we use the projection operators

$$\begin{aligned} \mathbf{P}_{[\sigma_{1y}\sigma_{2x}=1]} &= |\uparrow_{1y}\uparrow_{2x}\rangle\langle\uparrow_{1y}\uparrow_{2x}| + |\downarrow_{1y}\downarrow_{2x}\rangle\langle\downarrow_{1y}\downarrow_{2x}|, \\ \mathbf{P}_{[\sigma_{1y}\sigma_{2x}=-1]} &= |\uparrow_{1y}\downarrow_{2x}\rangle\langle\uparrow_{1y}\downarrow_{2x}| + |\downarrow_{1y}\uparrow_{2x}\rangle\langle\downarrow_{1y}\uparrow_{2x}|. \end{aligned} \tag{7}$$

Then Eq. (6) yields $\text{prob}[\sigma_{1y}\sigma_{2x} = 1] = 0$, contrary to the product rule, which requires $\sigma_{1y}\sigma_{2x} = 1$ with probability 1. It follows that the value of the product $\sigma_{1y}\sigma_{2x}$ is certain, but it equals -1.

Another striking example was discussed by Albert, Aharonov and D'Amato.[15] Consider a particle which can be located in one of three boxes. We denote the state of the particle when it is in box i by $|i\rangle$. At time t_1 the particle is prepared in the state

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle). \tag{8}$$

At time t_2 the particle is found to be in the state

$$|\Psi_2\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle). \tag{9}$$

We assume that in the time interval $[t_1, t_2]$ the Hamiltonian is zero. Then, if at time t between t_1 and t_2 we open box 1, we are certain to find the particle in box 1; and if we open box 2 instead, we are certain to find the particle in box 2. Nevertheless, if we open both of them, we might not see the particle at all. We can obtain these results by straightforward application of the formula (6). Opening box i corresponds to measuring the projection operator $|i\rangle\langle i|$. Opening two boxes is equivalent to opening the third box, and, therefore, corresponds to measuring the projection operator on the state in the third box.

We have shown that for a pre- and post-selected quantum system it might happen that the operators corresponding to two observables A and B commute, $[A, B] = 0$, but measuring B invariably disturbs the results of the measurement of A. Therefore, for a pre- and post-selected quantum system one cannot apply a "product rule" that asserts that if measurements of A and B yield $A = a$ and $B = b$ with certainty, then a measurement of AB yields ab. In fact, the value of AB might also be certain, but not equal to ab.

4. Realistic Lorentz-invariant Interpretation of Quantum Mechanics

We now turn to the arguments against the possibility of a realistic Lorentz-invariant interpretation of quantum mechanics.[3–6] The starting point of these arguments was the definition of elements of reality and the principle of Lorentz invariance. In contrast to the usual EPR-type argument, no locality principle, forbidding an action at a distance, was assumed. The adopted definitions are:

(i) *Element of reality* (Redhead[16]): "If we can predict with certainty, or at any rate with probability one, the result of measuring a physical quantity at time t, then at the time t, there exists an element of reality corresponding to this physical quantity and having a value equal to the predicted measurement result."

(ii) *The Principle of Lorentz invariance*: "If an element of reality corresponding to some Lorentz-invariant physical quantity exists and has a value within space-time region R with respect to one space-like hypersurface containing R, then it exists and has the same value in R with respect to any other hypersurface containing R."

In the usual EPR argument an element of reality corresponding to an outcome of a measurement is fixed by the mere *possibility* of inferring the outcome from measurements in a causally disconnected region. In contrast, since the present approach does not assume locality, elements of reality are fixed only by *actual* measurements.

5. The Proof of Clifton, Pagonis, and Pitowsky

The argument due to Clifton[4] is based on the modified Greenberger-Horne-Zeilinger[17] (GHZ) setup for demonstrating the nonexistence of local hidden variables. Three spin-1/2 particles, located in the corners of a very large triangle, move fast in directions pointing out of the center of the triangle. At time t_1 (in the rest frame) the particles are prepared in the state

$$|\Psi_1\rangle = |GHZ\rangle = \frac{1}{\sqrt{3}}(|\uparrow_{1z}\uparrow_{2z}\uparrow_{3z}\rangle - \downarrow_{1z}\downarrow_{2z}\downarrow_{3z}\rangle). \tag{10}$$

At time t_2 the spin components in x-direction are measured on all particles and the results $\sigma_{ix} = x_i$ are obtained. Consider now some possible measurements performed on the particles at a time t, $t_1 < t < t_2$. For each of the three observers who perform the σ_{ix} measurements, the measurements on the other particles (at time t in the rest frame) are performed *after* his σ_{ix} measurement, and he can predict (each in his Lorentz frame) the following result with certainty:

$$\sigma_{2y}\sigma_{3y} = x_1 \tag{11a}$$

$$\sigma_{1y}\sigma_{3y} = x_2 \tag{11b}$$

$$\sigma_{1y}\sigma_{2y} = x_3 \tag{11c}$$

Eqs. (11a-c) represent elements of reality in space-time regions corresponding to the respective Lorentz frames. The principle of Lorentz invariance yields that these are also the elements of reality in the rest frame. Multiplying Eqs. (11b) and (11c) we obtain:

$$\sigma_{1y}^2\sigma_{3y}\sigma_{2y} = x_2 x_3 \tag{12}$$

Taking in account that $\sigma_{1y}^2 = 1$, we conclude that $x_1 = x_2 x_3$. This conclusion, however, contradicts quantum mechanics: in the GHZ state $x_1 x_2 x_3 = -1$.

Pitowsky and Clifton *et al.* obtain their elements of reality as *predictions* of different observers, but their argument holds only when they consider the predictions of all observers. However, there is no Lorentz observer for which all the predictions are inferences from the past toward the future: at least some of the inferences must be *retrodictions.* In fact, we have a quantum system on which two complete measurement are performed in succession, and claims about elements of reality apply to times between these two measurements. Here, the GHZ state is prepared initially and measurement of the x-components of spin for all particles determines the final state. The discussion at the beginning of this Letter thus applies.

The state $|\Psi_1\rangle$ is given by Eq. (8), while $|\Psi_2\rangle = |x_1, x_2, x_3\rangle$, i.e., the state with certain x-components of spin. The operators to be considered between these two states are $\sigma_{1y}\sigma_{3y}$, $\sigma_{1y}\sigma_{2y}$, and $\sigma_{2y}\sigma_{3y}$. The formalism, Eq. (6), yields (as it should) the probability 1 for the outcomes given by Eqs. (11a-c). But it also shows that the measurements of commuting operators $\sigma_{1y}\sigma_{3y}$, and $\sigma_{1y}\sigma_{2y}$ disturb each other. Eq. (6) yields that the probability to find both results (11b) and (11c), *when measured together* is just $\frac{1}{4}$. Again, measuring the product differs from measuring both of the operators separately, and the probability of finding $(\sigma_{1y}\sigma_{3y})(\sigma_{1y}\sigma_{2y}) = x_2 x_3$ is zero since the outcome is given by Eq. (11a).

6. The Proof of Hardy

The example of Hardy[5] involves just two particles, an electron and a positron in two entangled setups of the type proposed by Elitzur and Vaidman[18] (EV) for interaction-free measurements. Each EV setup is a Mach-Zehnder interferometer tuned to yield zero counts at a detector D_1 unless a point $\mathcal{P}$ belonging to one arm of the interferometer is not free. The "click" of the detector D_1, after sending just one particle, yields that the point $\mathcal{P}$ is not empty, without disturbing the object at $\mathcal{P}$. In Hardy's example the point $\mathcal{P}$ is common to the two EV setups. One EV device tests the point $\mathcal{P}$ with a single electron, while the other tests the same point $\mathcal{P}$ with a single positron. If both electron and the positron come to the point $\mathcal{P}$ together then they annihilate, and it might happen that both devices yield that the point $\mathcal{P}$ is not empty, i.e., detectors D_1 of both the electron and the positron interferometers "click". Let us assume this outcome. Now, consider a Lorentz frame in which the observer of the electron EV device is the first to obtain a "click". She infers that the *positron* was at $\mathcal{P}$. In fact, she *retrodicts*, since the events she infers were in her absolute past. (Hardy is able to discuss the observer's *predictions* by considering the question: "Is the particle in the arm of the interferometer which includes $\mathcal{P}$?" instead of the question: "Is the particle at $\mathcal{P}$?" See also Ref. (6).) In another Lorentz frame, however, the observer of the positron EV device is the first to obtain the result. He deduces that the *electron* was at $\mathcal{P}$. The principle of Lorentz invariance yields that there are two elements of reality: the electron at $\mathcal{P}$ and the positron at $\mathcal{P}$. The product rule here is very natural: if the electron is at $\mathcal{P}$ and the positron is at $\mathcal{P}$ then the electron and the positron are at $\mathcal{P}$. The latter, however, leads to contradiction: the particles at $\mathcal{P}$ must annihilate and cannot be detected by either observer.

Hardy's example also involves pre- and post-selection. Here, the preselection is the preparation of the electron-positron state, while the post-selection is the detection of electron and positron at detectors D_1. Thus, we can apply the ABL formalism. However, in this case the free Hamiltonian is not zero; it describes the interaction of the electron and the positron with beam splitters and mirrors as well as their annihilation at $\mathcal{P}$. Therefore, the state $|\Psi_1\rangle$ in the formula (6) must be the initial state evolved forward in time until t, the time when one of the particles reaches the point $\mathcal{P}$; while the state $|\Psi_2\rangle$ must be obtained by evolving the final state backward in time until t. Straightforward calculation shows that Eq. (6) reproduces Hardy's result: if one observer tests: "Was the electron at $\mathcal{P}$?" her result must be "yes"; if the other observer looks for the positron at $\mathcal{P}$, his answer must be "yes" too (but if both of them makes these measurements, each observer will obtain "yes" with probability $\frac{1}{3}$, and they will never obtain "yes" together). Here too, the operator considered by Hardy is the product of two projection operators, and its measurement is not equivalent to two simultaneous measurements, one testing for electron at $\mathcal{P}$ and another testing for positron at $\mathcal{P}$. The measurement of the product can be implemented by observing photons due to electron-positron annihilation. Formula (6) yields probability zero to obtain the product equal 1, in contrast to probability one obtained from the product rule.

7. Inconsistency in Applying the Product Rule

We believe that Redhead's definition of elements of reality is a plausible one. It does not lead to contradiction with Lorentz invariance if we do not adopt the product rule. But in the light of the discussion above, it is clear that the product rule is incompatible with Redhead's definition. The elements of reality are inferred on the assumption that there are no measurements disturbing their values. Clifton Pagonis and Pitovsky[4] state explicitly: "For our argument, we shall assume that no such intervening measurements take place." But as we showed, measurements of the operators they consider *do* interfere with each other. So, it is inconsistent with the definition of the elements of reality to apply the product rule. (Note that the product rule and its generalization to any function of commuting operators are widely used in no-hidden-variables theorems.[19] It is valid in all cases when no retrodiction is involved.) If it is an element of reality that $A = a$ and it is an element of reality that $B = b$, it does not follow that $AB = ab$ is an element of reality. It might be that the product AB has a certain value and, therefore, is an element of reality in the Redhead's sense, but it need not equal ab.

In fact, this happens in all the examples we considered. In the first example we have elements of reality $\sigma_{1y} = -1$, $\sigma_{2x} = -1$, and the product is also an element of reality, but $\sigma_{1y}\sigma_{2x} = -1$. In the second example $\mathbf{P}_1 = 1$, $\mathbf{P}_2 = 1$, but $\mathbf{P}_1\mathbf{P}_2 = 0$, where $\mathbf{P}_1$, $\mathbf{P}_2$ are projection operators on the states "the particle in the box 1" and "the particle in the box 2" respectively. In the Pitowsky example the elements of reality are $\sigma_{1y}\sigma_{3y} = x_2$, $\sigma_{1y}\sigma_{2y} = x_3$, and the product, $(\sigma_{1y}\sigma_{3y})(\sigma_{1y}\sigma_{2y}) = \sigma_{2y}\sigma_{3y} = x_1$, but nevertheless $x_2 x_3 \neq x_1$, $(x_2 x_3 = -x_1)$. In Hardy's example $\mathbf{P}_{e^-} = 1$, $\mathbf{P}_{e^+} = 1$, but $\mathbf{P}_{e^-}\mathbf{P}_{e^+} = 0$, where $\mathbf{P}_{e^-}$, $\mathbf{P}_{e^+}$ are projection operators on the states "an electron at $\mathcal{P}$" and "a positron at $\mathcal{P}$" respectively.

Clifton Pagonis and Pitovsky felt that the conclusions about the impossibility of constructing a realistic Lorentz-invariant quantum theory are too strong. They proposed a variety of ways to circumvent these arguments, in particular, by *rejecting* elements of reality corresponding to "incompatible measurement context". It is possible to deal with the failure of the product rule along these lines, but we believe that the most natural way is to give up the product rule. All the examples we considered show that the product rule fails to be true.

8. Elements of Reality of the Pre- and Post-Selected Quantum System

Giving up the product rule allows us to extend the concept of elements of reality. Since we anyway consider circumstances in which retrodictions are involved, we may include retrodictions fully and give them the same status as to predictions. In the examples presented here, predictions were applied to future events as well as to space-like separated events, while retrodictions were applied only to space-like separated events. We propose to apply retrodiction to the past also. The Redhead definition of elements of reality continues to hold, with a minor change of "predict" to "infer." Then, in the case of two spin-1/2 particles, the observer who measures $\sigma_{1x}(t_2) = 1$ not only infers that $\sigma_{2x}(t) = -1$ but also that $\sigma_{1x}(t) = 1$. So we can add to the list of elements of reality at time t also $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$.

According to the definition, the element of reality exists *whether or not* the inference is actually verified. Recently were introduced *weak measurements*[20] which might support this definition. Weak measurements test elements of reality almost without disturbing the quantum system. They refer to ensembles of pre- and post-selected systems. Each system in the ensemble is practically undisturbed by the interaction with the measuring device (which is a standard but very weakly coupled measuring device), but measurement of each system yields almost no information. However, collecting results across the ensemble, we find a result called *weak value*. If the system was pre-selected in a state $|\Psi_1\rangle$ and was post-selected in a state $|\Psi_2\rangle$, then any weak enough measurement of any variable A yields its weak value

$$A_w \equiv \frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}. \tag{12}$$

It has been shown[14] that whenever there exists an element of reality, its value is the weak value. For dichotomic variables an "inverse" theorem[14] is also true: *if the weak value is equal to an eigenvalue, then it is an element of reality* (i.e., a measurement has to yield this value). In all our examples we consider dichotomic variables, so we can obtain our results via the simpler calculation of the weak value (12) rather than via Eq. (6).

We can define *weak elements of reality*. Weak values of physical variables (i.e. the outcomes of weak measurements) are weak elements of reality. The elements of reality of Redhead (with "infer" instead of "predict") are subset of weak elements of reality. In contrast to the reality of Redhead, "weak reality" is defined in all situations. There are numerous situations in which a quantum system has no elements of reality at all in the sense of Redhead (namely, when mixed states are involved).

Another attractive property of weak elements of reality is the sum rule: if $A = B + C$ then $A_w = B_w + C_w$. The sum rule is valid even for noncommuting variables. Despite this parallel with classical physics, weak elements of reality might be very unusual. For example, for the particle in the three boxes there are the following weak elements of reality: there is 1 particle in box 1, there is 1 particle in box 2, there is -1 particle in box 3! If we weakly measure the number of particles in the boxes (using a pre- and post-selected ensemble of triplets of boxes), say, by measuring the pressure on the walls of the boxes, then we will find pressure corresponding to a particle in each of the first two boxes and the negative of the same value in the third box![14]

Although the sum rule holds for weak elements of reality, the product rule fails even for commuting variables: it is easy to see from the definition (12) that $A = BC$ does not imply $A_w = B_w C_w$. It has to be so because there is failure of the product rule at least for the subset of weak elements of reality, Redhead's elements of reality. This is a somewhat surprising result for the example we have considered of the two spin-1/2 particles. Even weak (supposedly undisturbing) measurement of the product $\sigma_{1y}\sigma_{2x}$ will be different from the product of the outcomes of the weak measurements of σ_{1y} and σ_{2x}. Indeed, $(\sigma_{1y})_w = -1$, $(\sigma_{2x})_w = -1$, and $(\sigma_{1y}\sigma_{2x})_w = -1$, therefore, $(\sigma_{1y})_w(\sigma_{2x})_w \neq (\sigma_{1y}\sigma_{2x})_w$.

9. The Failure of the "And" Rule

Closely connected to the failure of the product rule is the failure of the *"and" rule*: if $A = a$ is an element of reality and if $B = b$ is an element of reality, it does not follow that $\{A = a \text{ and } B = b\}$ is an element of reality. Formally, one can consider the projection operator on a space of states characterized by $A = a$ and the projection operator on a space of states characterized by $B = b$; then the product of these projection operators corresponds to the space of states characterized by $\{A = a \text{ and } B = b\}$. The failure of the product rule for this case implies the failure of the "and" rule.

In fact, two of the examples presented are much more transparent when we consider the "and" rule instead of the product rule. In the case of a particle in three boxes we have: {the particle is in box 1} is an element of reality, {the particle is in box 2} is an element of reality, but {the particle is in box 1 and the particle is in box 2} is not an element of reality. In Hardy's example {the electron at $\mathcal{P}$ } is an element of reality, {the positron is at $\mathcal{P}$ } is an element of reality, but {the electron and the positron are at $\mathcal{P}$ } is not an element of reality. The mutual disturbance of the measurements, which exists in a pre-selected and post-selected situation (even for measurements of commuting variables), explains the cause of the failure of the "and" rule.

But is there a failure of the "and" rule for weak elements of reality? Weak elements of reality are defined as weak values, the outcomes of weak measurements, with the basic property that they do not disturb quantum states significantly. Consider our first example: two spin-1/2 particles in the EPR-Bohm state, postselected in a state $|\uparrow_{1x}\uparrow_{2y}\rangle$. We have found that there are elements of reality (which are also weak elements of reality) $\{\sigma_{1y} = -1\}$ and $\{\sigma_{2x} = -1\}$. Clearly, weak simultaneous measurements of σ_{1y} and σ_{2x} will not disturb each other (while strong measurements certainly will). However, this does not mean that the "and" rule holds for weak elements of reality. It only means that weak elements of reality can be measured (on the pre- and post-selected ensemble) simultaneously. Weak measurement of σ_{1y} together with weak measurement of σ_{2x} *are not* equivalent to weak measurement of σ_{1y} and σ_{2x}. The latter, in fact, is not well defined. We have to specify the two-particle operator to be measured, and then to go to the weak limit. For example, the product is one such two-particle operator and since we proved that the product rule fails in this case, the "and" rule must fail too. (One can see an analogy with the necessity of specifying the operator measured in boxes 2 and 3 for defining the number of particles in the box 1 of our three-box example.[15])

Even more clearly, we can see the failure of the "and" rule for weak elements of reality by reconsidering Hardy's example. There are elements of reality {electron at $\mathcal{P}$} and {positron at $\mathcal{P}$}. Weak independent measurements of the number of electrons at $\mathcal{P}$ and the number of positrons at $\mathcal{P}$ will yield the number 1 for both. But weak measurement of {electron and positron at $\mathcal{P}$}, i.e. weak measurement of the number of created photons, will yield 0.

Although the failure of the "and" rule may suggest a version of quantum logic, we do not propose such a resolution. We prefer to keep the standard logic of propositions with no failure of the "and" rule for propositions: if A is true and B is true, then {A and B} is also true. Nature, however, is described by quantum elements of reality which do not obey our classical intuition. Introduction of such elements of reality helps us construct a Lorentz-invariant description of the evolution of quantum systems.[21−22]

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