The Predictability of the Results of Measurements of Noncommuting Variables

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In the orthodox understanding of quantum mechanics, two variables, A and B, cannot simultaneously have the definite values $A = a$ and $B = b$, if neither $[A, B]|A = a\rangle$ nor $[A, B]|B = b\rangle$ vanishes. Suppose, however, that at time t_1 A is found to have the value a, and that at time t_2 B is found to have the value b. Then, by means of prediction from time t_1, and retrodiction from time t_2, the values of both A and B at time t, where $t_1 < t < t_2$, can obviously be ascertained. We have found that there are situations in which we can ascertain not only the simultaneous values of two noncommuting variables, but rather of an infinite number of noncommuting variables. Roughly, these situations are, in some sense, linear superpositions of the situations wherein we can ascertain (by means of prediction and retrodiction together) the values of the noncommuting variables. The proper way to describe those situations is by using the language of multiple-time states,[1] but we can understand them also if we look on a specific part of a composite system at times between two measurements on the whole system.

We use here a generalization of the formula[2] for the probability of the result of measurements at time t, given the states of the system at times t_1 and t_2, where $t_1 < t < t_2$. The probability of finding $C = c$ at time t, if at time t_1 the state is $|\phi_1\rangle$ and at time t_2 the state is $|\phi_2\rangle$, is

$$P(C = c) = \frac{|\langle\phi_2|\mathbf{P}_{C-c}|\phi_1\rangle|^2}{\Sigma_i|\langle\phi_2|\mathbf{P}_{C-c_i}|\phi_1\rangle|^2},$$

where $\mathbf{P}_{C-c_i}$ is the projection operator on the subspace of states for which $C = c$ and where the sum in the denominator goes over all possible values of C.

We consider the spin-½ particle within a time interval between two measurements on the composite system that consists of this particle and another, "external," spin-½ particle. (The second particle takes the place of the measuring device of the multiple-time measurement.[3]) It was found that there are situations wherein the result $\sigma_\xi = 1$ (i.e., if spin is measured in the direction $\hat{\xi}$, the result is "up" with probability 1) is true simultaneously for a continuum of directions $\hat{\xi}$. It was proven that such a continuum of rays takes the form of a cone. For example, we consider the superposition of the situations: $\sigma_x(t_1) = 1$ and $\sigma_y(t_2) = 1$, $\sigma_y(t_1) = 1$ and $\sigma_x(t_2) = 1$. The spin-½ particle behaves as if it is in this superposition provided that the composite system is prepared at time t_1 in the state

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_{ext}|\uparrow_x\rangle + |\downarrow_z\rangle_{ext}|\uparrow_y\rangle),$$

and is found at time t_2 in the state

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_{ext}|\uparrow_y\rangle + |\downarrow_z\rangle_{ext}|\uparrow_x\rangle).$$

We assume that the system has zero Hamiltonian, and no measurement is performed on the external spin-½ particle between t_1 and t_2. The cone of directions $\hat{\xi}$ for which $\sigma_\xi = 1$ include, in this case, the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$.

REFERENCES

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