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Measurement of the Schrödinger wave of a single particle

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We show that it is possible to measure the Schrödinger wave of a single quantum system. This provides a strong argument for associating physical reality with the quantum state of a single system, and challenges the usual assumption that the quantum state has physical meaning only for an ensemble of identical systems.

While the Schrödinger wave is a basic element of quantum theory, it is generally believed that one cannot associate physical reality to the wave of a single particle. Rather, the Schrödinger wave is often viewed as a mathematical tool for calculating the probabilities of various outcomes of certain experiments when many such experiments are performed on an ensemble of identical systems, all in the same quantum state. Several arguments seem to support this point of view.

(i) We have never seen the quantum state of a single particle in a laboratory. Indeed, while a wave is typically spread over a region of space, we never see a particle simultaneously in several distinct locations.

(ii) If we could see a quantum state, we could presumably distinguish it from any other quantum state, but the unitary time evolution of states in quantum mechanics implies that it is impossible to distinguish between two different nonorthogonal states. Different outcomes of a measurement to distinguish these two states correspond to orthogonal quantum states of the composite system (measuring device plus particle). But, the initial scalar product between the states was not zero and remains nonzero under unitarity time evolution.

(iii) If we associate physical reality with a spread-out wave then the instantaneous "collapse" of the wave to a point during a position measurement seems to conflict with relativity [1].

In this Letter we demonstrate how the density $\rho=\Psi^*\Psi$ and the current

$$j=\frac{\hbar}{2mi}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)$$

of a Schrödinger wave of a single particle can be measured. The usual measurements assumed in argument (i) alter the Schrödinger wave and are not adequate; rather, here we will describe special *protective* measurements allowing us to measure ρ and j without changing the Schrödinger wave. In some cases energy conservation provides protection for the state, while in other cases we need a special protection procedure.

Let us consider a particle in a discrete nondegenerate energy eigenstate $\Psi(x)$. The standard von Neumann procedure for measuring the value of an observable A in this state involves an interaction Hamiltonian,

$$H=g(t)pA, \tag{1}$$

coupling our system to a measuring device, or pointer, with coordinate and momentum denoted respectively by q and p. The time-dependent coupling $g(t)$ is normalized to $\int g(t)\,\mathrm{d}t=1$. The initial state of the pointer is taken to be a Gaussian centered around zero.

In standard impulsive measurments, $g(t)\neq 0$ only for a very short time interval. Thus, the interaction

term dominates the rest of the Hamiltonian, and the time evolution $\exp(-ipA/\hbar)$ leads to a correlated state: eigenstates of A with eigenvalues a_n are correlated to measuring device states in which the pointer is shifted by these values a_n. By contrast, the protective measurements of interest here utilize the opposite limit of extremely slow measurement. We take $g(t)=1/T$ for most of the time T and assume that $g(t)$ goes to zero gradually before and after the period T. We choose the initial state of the measuring device such that the canonical conjugate p (of the pointer variable q) is bounded. For $g(t)$ smooth enough we obtain an adiabatic process in which the particle cannot make transition from one energy eigenstate to another, and, in the limit $T\to\infty$, the interaction Hamiltonian does not change the energy eigenstate. For any value of p, the energy of the eigenstate shifts by an infinitesimal amount given by first order perturbation theory,

$$\delta E=\langle H_{\rm int}\rangle=\langle A\rangle p/T\,. \tag{2}$$

The corresponding time evolution $\exp(-ip\langle A\rangle)$ shifts the pointer by the average value $\langle A\rangle$. (Here and below we will take $\hbar=1$.) This result contrasts with the usual (strong) measurement in which the pointer shifts by one of the eigenvalues of A. By measuring the averages of a sufficiently large number of variables A_n, the full Schrödinger wave $\Psi(x)$ can be reconstructed to any desired precision.

As a specific example we take the A_n to be (normalized) projection operators on small regions V_n having volume v_n,

$$A_n=\frac{1}{v_n}\,,\quad \text{if } x\in\mathrm{V}_n\,,$$
$$=0\,,\quad \text{if } x\notin\mathrm{V}_n\,. \tag{3}$$

The measurement of A_n yields

$$\langle A_n\rangle=\frac{1}{v_n}\int_{\mathrm{V}_n}|\Psi|^2\,\mathrm{d}v=|\Psi_n|^2\,, \tag{4}$$

where $|\Psi_n|^2$ is the average of the density $\rho(x)=|\Psi(x)|^2$ over the small region V_n. Performing measurements in sufficiently many regions V_n we can reconstruct $\rho(x)$ everywhere in space. (Simultaneous measurement of all the variables A_n requires slower and weaker interactions, and thus takes more time.)

For a real state the density $\rho(x)$ is itself enough to reconstruct the Schrödinger wave; we can fix the sign by flipping it across nodal surfaces.

In the general case, however, in addition to measurements of the density $\rho(x)$, we have to measure current density. This time we also adiabatically measure the averages of

$$B_n=\frac{1}{2\mathrm{i}}\,(A_n\nabla+\nabla A_n)\,. \tag{5}$$

Indeed, $\langle B_n\rangle$ are the average values of the current,

$$j=\frac{1}{2\mathrm{i}m}\,(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)$$

in the region V_n. Writing $\Psi(x)=r(x)\exp[\mathrm{i}\theta(x)]$ with $r(x)=\sqrt{\rho(x)}$, we find that

$$mj(x)/\rho(x)=\nabla\theta\,, \tag{6}$$

and the phase $\theta(x)$ can be found by integrating j/ρ.

For a charged particle the density $\rho(x)$ times the charge yields the effective charge density. In particular, it means that an appropriate adiabatic measurement of the Gauss flux out of a certain region must yield the expectation value of the charge inside this region (the integral of the charge density over this region). Likewise, adiabatic measurement of the Ampère contour integral yields the expectation value of the total current flowing through this contour in the stationary case.

Our discussion of the current of the particle is valid only for a Hamiltonian without vector potential. However, the eigenstates of such a Hamiltonian with a nonvanishing current are necessarily degenerate due to time reversal invariance. The method described above is appropriate only for nondegenerate eigenstates and, therefore, we have to consider problems with a vector potential $\boldsymbol{A}$, for which we do have nondegenerate stationary states with nonzero current (e.g. the Aharonov–Bohm effect). Then, the definition of the (electric) current must be modified by the replacement $\nabla\to\nabla-\mathrm{i}e\boldsymbol{A}$. This replacement has to be done also for the definition of the observables B_n (eq. (5)), and it leads to the obvious modification of eq. (6).

We have shown that stationary quantum states can be observed. This is our main argument for associating physical reality with the quantum state of a single particle. Since our measurement lasts a long pe-

riod of time we do not have a method for measuring the Schrödinger wave at a given time. Thus, we have a direct argument for associating physical reality with stationary Schrödinger waves only over a *period* of time. The reader may therefore suspect that our measurements represent time-averaged physical properties of the system. Let us now present a few arguments explaining why, nevertheless, these measurements reflect properties of the Schrödinger wave at any given moment of time during the measurement.

An essential feature of our adiabatic measurement is that the state $|\Psi\rangle$ does not change throughout the experiment. Since the Schrödinger wave yields the complete description of a system and the interaction with the measuring device is constant throughout the measurement, we conclude that the action of the system on the measuring device is the same at any moment during the measurement.

The mathematical description of our measurement tells us the same: for any, even very short, period of time, the measuring device shifts by an amount proportional to $\langle A\rangle$, the expectation value of the measured variable, rather than to one of its eigenvalues a_n. Thus, expectation values, which mathematically characterize Schrödinger waves, can be associated with very short periods of time. In the instantaneous limit, expectation values and, therefore, the quantum state manifest themselves as properties of a quantum system defined at a given time. (Note, however, that pointer shifts during short time intervals are unobservable since they are much smaller than the uncertainty; only the total shift accumulated during the whole period of measurement is much larger than the width of the initial distribution, and therefore observable on a single particle.)

Moreover, suppose that (contrary to standard quantum theory), a system has a complete description that *does* change during the measurement process, and the (constant) Schrödinger wave we measure does not describe the system at a given time but represents only a time average of some hidden variables over the period of the measurement. Consider a model of a hydrogen atom in which the electron performs very fast ergodic motion in the region corresponding to the quantum cloud. The charge density might be either zero (if the electron is not there) or singular (if the electron is inside the infinitesimally small region including the space point in question). In spite of this fact, the measurement we have described will yield outcomes corresponding to a nonsingular charge density cloud. What it measures is the time average of the density, or how long a time the electron spent in a given place.

In order to see that this picture is inappropriate for the quantum case let us consider another example: a particle in a one-dimensional box of length L in the first excited state. The spatial part of the state is

$$\sqrt{2/L}\sin(2\pi x/L)\,.$$

The adiabatic measuring procedure described above will yield the Schrödinger wave density,

$$(2/L)\sin^2(2\pi x/L)\,.$$

In particular, it equals zero at the center of the box. If there is some hidden position of the electron which changes in time such that the measured density is proportional to the amount of time the electron spends there, then half of the time it must be in the left half of the box and half of the time in the right half of the box. But it can spend no time at the center of the box; i.e., it must move at infinite velocity at the center. It is absolutely unclear what such an electron "position" would be. There *is* a theory [2] which introduces a "position" for a particle in addition to its Schrödinger wave; but according to this theory, the "velocity" of the particle in the given energy eigenstate vanishes: it does not move at all. In the quantum picture the eigenstate of the particle in the box can be represented as a superposition of two running waves moving in opposite directions. The zero density at the center of the box is due to destructive interference – the phenomenon which cannot be reproduced in a classical ergodic model of a particle.

The procedure described above cannot measure properties of a state obtained by superposing several nondegenerate energy eigenstates. Applied to such a state, a measurement of A will yield shifts of the pointer corresponding to the expectation values of the variable A in the various energy eigenstates. In general, these values are distinct with differences greater than the initial uncertainty of the pointer position. Thus, after the interaction, the system and the measuring device are entangled. By "looking" at the

measuring device we cause the Schrödinger wave to choose one of the energy eigenstates. Measurement of the Schrödinger wave – namely, measurement of the expectation values of the projection operators – causes collapse. A superposition of nondegenerate energy eigenstates is not protected by energy conservation: unitary evolution during the measurement leads to correlations between energy states and the states of the measuring device without changing the total energy, while collapse changes the energy itself.

Nevertheless, we can measure even a superposition of energy eigenstates by a procedure similar to the one described above. We just have to add an appropriate protection mechanism. The simplest way to protect a time-dependent Schrödinger wave is via dense state-verification measurements that test (and thus protect) the time evolution of the quantum state. If we are interested in all the details of this time-dependent state we cannot use measurements which are too slow. Every measurement of the density and current of a Schrödinger wave must last a period of time which is smaller than the characteristic time of the evolution of the state; and the time intervals between consecutive protections must be even smaller. However, in principle, Schrödinger wave measurement to any desired accuracy is possible: for any desired accuracy there is a density of the state-verification measurements that will protect the state from being changed due to the measurement interaction. Additional protection is necessary also for measurement of stationary but degenerate states; and the scheme of dense projection measurements is applicable here too. Even for dense projective measurements, most of the time the system evolves according to its free Hamiltonian, so we are allowed to say that what we measure is the property of the system and not of the protection procedure.

When measurements involve the above kind of protection, we have to know the state in order to prescribe the proper protection. One might object, therefore, that our measurement yields no new information, since the state is already known. However, we can separate the protection and measurement procedures: one experimentalist provides protection and the other measures the Schrödinger wave itself. Then the second experimentalist does obtain new information. The most important point, however, is that we directly measure properties of the Schrödinger wave of a single system using a standard measuring procedure. Our direct measurements of the density and the current of the Schrödinger wave challenge the commonly accepted notion that quantum states can be fully observed only when the measurement is performed on an ensemble of identical systems.

Consider now an apparent paradox arising from the measurement of the Schrödinger wave. It is well known that even assuming instantaneous "collapse" of a quantum state, one cannot use the collapse for sending signals faster than light. At first, however, the possibility of measuring the value of the Schrödinger wave at a given location seems to allow such superluminal communication. Consider a particle in a superposition $(|1\rangle+|2\rangle)/\sqrt{2}$ of being in two boxes separated by a very large distance. For this particle the expectation value of the projection onto the first box is $\langle P_1\rangle=\frac{1}{2}$. This value must be the outcome of a measurement performed on the first box. If, however, just prior to a measurement of the Schrödinger wave in the first box, someone opens and looks into the second box, causing collapse to a localized state $|1\rangle$ or $|2\rangle$, then the outcome of the measurement of the projection operator in the first box will drastically change: we no longer find $\langle P_1\rangle=\frac{1}{2}$ but rather 0 or 1 (depending on what is found in the second box). It seems, therefore, that measurements on one box can influence measurements on another box located arbitrarily far away.

However, this argument contains a flaw: the state $(|1\rangle+|2\rangle)/\sqrt{2}$ is not a discrete nondegenerate eigenstate. Since there is no overlap between the states $|1\rangle$ and $|2\rangle$, the orthogonal state $(|1\rangle-|2\rangle)/\sqrt{2}$ has the same energy. Thus, there is no natural protection due to the energy conservation, and an additional protection is needed. This protection, however, involves explicitly nonlocal interactions. These nonlocal interactions are the source of the alleged superluminal signal propagation. (A more subtle paradox of this sort is considered in another work [3].)

Let us come back to the three arguments against the realistic view of the Schrödinger wave presented in the beginning of this Letter. First, we have shown that we *can* observe a quantum state. Although our discussion relied on Gedanken experiments, recent

experimental work with so-called "weak links" in quantum circuits shows that slow adiabatic measurements of the Schrödinger wave can be performed in the laboratory [4].

The second argument is a correct statement, but it only implies that there is no single *universal* procedure for observing states. It still allows for the possibility of an appropriate measuring procedure for any given state.

The last argument (iii) is the most serious one. Assume that the Schrödinger wave of a particle is nonvanishing only inside two separate boxes, and we find it in one of them. How did part of the wave move instantaneously from one box to another? We believe that a full answer to this argument requires a new approach to quantum theory [5].

We have shown that expectation values of quantum variables and the quantum state itself have physical meaning, i.e., they are measurable for individual quantum systems. This result stands in sharp contrast to the standard approach in which the Schrödinger wave and expectation values are statistical properties of ensembles of identical systems.

Note added in proof. We have completed, in collaboration with J. Anandan, an extension of this work which includes a detailed analysis of protective measurements of the spin-$\frac{1}{2}$ system [6].

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