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Measurement of the Negative Kinetic Energy of Tunnelling Particles

Yakir AHARONOV*, Sandu POPESCU, Daniel ROHRLICH**, Lev VAIDMAN

School of Physics and Astronomy, Tel-Aviv University
Ramat-Aviv, Tel-Aviv 69978 Israel

We consider measurements of kinetic energy performed on particles from a "pre- and post-selected" ensemble. The particles are prepared in a bound state; next, kinetic energy measurements are performed; and finally, we perform a measurement selecting particles which are far inside the tunnelling region. We prove that for all these particles the readings of the kinetic energy measuring device cluster around a negative value! This value is the *weak value* recently introduced by Aharonov, Albert and Vaidman for description of systems in pre- and post-selected ensembles. Tunnelling provides a striking example of a weak value obtained from a measurement which is not "weak".

1. Introduction

The phenomenon of barrier penetration, such as tunnelling through a potential barrier, is an outstanding example of quantum behavior. Quantum particles can be found in regions where a classical particle could never go, since it would have negative kinetic energy. But in quantum theory, too, the eigenvalues of kinetic energy cannot be negative. How, then, can a quantum particle "tunnel"? The apparent paradox is resolved by noting that the wave function of a tunnelling particle only partly overlaps the forbidden region. There is no wave function that represents a particle restricted to a region where its potential energy is larger than its total energy.

Nevertheless, we will show that actual measurements of kinetic energy can yield negative values, and that, under proper conditions, a remarkable consistency appears in these apparent errors. In a model experiment, we measure the kinetic energy of a bound particle to any desired precision. We then attempt to localize the particle within the classically forbidden region. The attempt rarely succeeds, but whenever it does, we find that the kinetic energy measurements gave an "unphysical" negative result; moreover, these results cluster around the appropriate value, the difference between the total and the potential energy. This consistency, which seems to come from nowhere – a background of errors – suggests strongly that the notion of a quantum observable is richer than the one generally accepted. The negative values of kinetic energy realize the recently introduced concept of a *weak value*[1,2] of a quantum variable.

2. Analysis of errors in measurement

We begin by reviewing the standard von Neumann[3] theory of measurement in non-relativistic quantum mechanics. Suppose we wish to measure a dynamical quantity A. We choose a measuring device with an interaction Hamiltonian

$$H_{int} = g(t)PA \quad , \qquad (1)$$

where P is a canonical momentum of the measuring device; the conjugate position Q corresponds to the position of a pointer on the device. The time-dependent coupling constant $g(t)$ is nonzero only for a short time interval corresponding to the measurement, and is normalized so that

$$\int g(t)dt = 1 \quad . \qquad (2)$$

When the time interval is very short, we call the measurement impulsive. For an impulsive measurement, H_{int} dominates the Hamiltonians for the particle and the measuring device. Then, since $\dot{Q} = \frac{i}{\hbar}[H_{int}, Q] = g(t)A$, we obtain (in the Heisenberg representation) the result

$$Q_{fin} - Q_{in} = A \quad , \qquad (3)$$

where Q_{fin} and Q_{in} denote the final and initial settings of the pointer.

In an ideal measurement the initial position of the pointer is precisely defined, say $Q_{in} = 0$, and so from its final position we read the precise value of A. But in practice, measurements involve uncertainty. To make this uncertainty explicit, we can take the initial state of the pointer to be

$$\Phi_{in}(Q) = (\epsilon^2\pi)^{-1/4} e^{-Q^2/2\epsilon^2} \quad . \qquad (4)$$

The uncertainty in the initial position of the pointer produces errors of order ϵ in the determination of A; when $\epsilon \rightarrow 0$ we recover the ideal measurement. Suppose that the system under study is initially in an eigenstate of A with eigenvalue a. Ideal measurements can yield only the result a. But when the pointer itself introduces an uncertainty, other results are possible, indeed a scatter of results, with a spread of about ϵ, and peaked at the eigenvalue a. If the measuring device works as described, then

* also Department of Physics, University of South Carolina, Columbia, SC 29208 U.S.A.
** Racah Institute of Physics, Hebrew University, Jerusalem.
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any measured value is possible, although large errors are exponentially suppressed. There is no mystery in the appearance of such errors; they are expected, given the uncertainty associated with the measuring device. Measurements of a positive definite operator such as p^2 could even yield negative values. Of course, the dial of the measuring device might have a pin preventing negative readings, but let us assume that it does not. Even if the negative values themselves are unphysical, they are part of a distribution representing the measurement of a physical quantity. They should not be thrown out, since they give information about the distribution and contribute to the best estimate of the peak value.

The standard theory of measurement not only allows errors, it also prescribes their interpretation: they constitute scatter around a true physical value which can only be one of the eigenvalues of the operator measured. Since these errors originate in the measuring device, and not in the system under study, it seems that they cannot depend on any property of the system. However, closer analysis of these errors in the context of *sequences* of measurements reveals a pattern which, far from being random, clearly reflects properties of the system under study.

We shall show a correlation between position measurements and prior kinetic energy measurements: nearly all particles found far outside the potential well yielded negative values of kinetic energy. The correlation, however, works one way only. Nearly all particles that yielded negative values of kinetic energy are still found *inside* the well. The latter ensemble is much larger than the former.

3. The weak value

Consider a system which has been pre-selected in a state $|\Psi_{in}\rangle$ and shortly afterwards post-selected in a state $|\Psi_{fin}\rangle$. The weak value of any physical variable A in the time interval between pre-selection and post-selection is defined to be

$$A_w \equiv \frac{\langle\Psi_{fin}|A|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle} \quad . \tag{5}$$

Let us show briefly how these values emerge from a measuring procedure with a sufficiently weak interaction. We consider a sequence of measurements: a pre-selection of $|\Psi_{in}\rangle$, (weak) measurement interaction of the form of Eq.(1), and a (successful) post-selection measurement of the state $|\Psi_{fin}\rangle$. The state of the measuring device after this sequence is given (up to normalization) by

$$\Phi_{fin}(Q) = \langle\Psi_{fin}|e^{-\frac{i}{\hbar}P\,A}|\Psi_{in}\rangle e^{-Q^2/2\epsilon^2} \quad . \tag{6}$$

After simple algebraic manipulation we can rewrite it (in the P-representation) as

$$\tilde{\Phi}_{fin}(P) = \langle\Psi_{fin}|\Psi_{in}\rangle\, e^{-\frac{i}{\hbar}A_w P}\, e^{-\epsilon^2P^2/2\hbar^2}$$
$$+\langle\Psi_{fin}|\Psi_{in}\rangle \sum_{n=2}^{\infty}\frac{(iP/\hbar)^n}{n!}[(A^n)_w-(A_w)^n]e^{-\epsilon^2P^2/2\hbar} \; . \tag{7}$$

If ϵ is large enough (which corresponds to an imprecise measurement), then we can neglect the second term of (7) when we Fourier transform back to the Q-representation. Large ϵ corresponds to weak measurement in the sense that the interaction Hamiltonian (1) is small. Thus, in the limit of weak measurement, the final state of the measuring device (in the Q-representation) is

$$\Phi_{in}(Q) = (\epsilon^2\pi)^{-1/4}e^{-(Q-A_w)^2/2\epsilon^2} \quad . \tag{8}$$

This state represents a measuring device pointing to the result A_w! We have showed that weak enough measurements on pre- and post-selected ensembles yield, instead of eigenvalues, a "weak" value which might lie far outside the range of eigenvalues.

From (7) we can derive the *weakness condition:*[2)]

$$\frac{\Gamma(n/2)}{\epsilon^n(n-2)!}|(A^n)_w-(A_w)^n| \ll 1 \quad . \tag{9}$$

We see that if $(A^n)_w$ and $(A_w)^n$ are approximately equal, then the condition on the weakness of the measurement interaction is not severe, i.e. the uncertainty ϵ need not be very large. In fact, there are examples in which a measurement yields a weak value (5) while the measuring iteraction is not weak at all.[4,5)] As we show next, measurement of the kinetic energy of tunnelling particles is one such example.

4. Negative kinetic energy

Our example may be summarized as follows: we prepare a sufficiently large ensemble of particles bound in a potential well, in an eigenstate of energy, and measure the kinetic energy of each particle to a given precision. Then we measure the position of each particle and select only those cases where the particle is localized within some region "far enough" from the well – with "far enough" depending on how precisely the kinetic energy was measured. It turns out that for almost all such post-selected particles the measured kinetic energy was negative. Not only are the measured values negative, they also cluster around a particular negative value, the weak value of the kinetic energy for this pre- and post-selected ensemble. Also, the spread of the clustering is ϵ, the characteristic spread for kinetic energy measurements with this device.

The initial state is a ground state of a particle in a smooth potential well. The Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{U_0}{\cosh^2(\alpha x)} \quad , \tag{10}$$

where $U_0 = \alpha^2\hbar^2/m$. In order to simplify the following expressions we choose m=1 and use units such that $\hbar = 1$. The ground state then, with energy $E_0 = -\alpha^2/2$, will be

$$\Psi_{in}(x) = |E_0\rangle = \frac{\sqrt{\alpha/2}}{\cosh(\alpha x)} \quad . \tag{11}$$

The final measurement is a post-selection on a Gaussian located far from the potential well:

$$\Psi_{fin}(x) = (\frac{\pi}{\sqrt{2}})^{-1/4}e^{-(x-x_0)^2} \quad , \tag{12}$$

with $x_0 \gg 1, 1/\alpha$. Then, the weak value of the kinetic energy can be obtained immediately:

$$K_w = \frac{\langle \Psi_{fin} | (E - V) | E_0 \rangle}{\langle \Psi_{fin} | E_0 \rangle} \simeq E_0 - V(x_0) \simeq -\frac{\alpha^2}{2} \quad . \quad (13)$$

We can also calculate it using the kinetic energy operator $\hat{K} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$. For $x \gg 1/\alpha$ we have $1/\cosh(\alpha x) \simeq 2e^{-\alpha x}$. Thus,

$$K_w = \frac{\int e^{-(x-x_0)^2} \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) \frac{1}{\cosh(\alpha x)}\, dx}{\int e^{-(x-x_0)^2} \frac{1}{\cosh(\alpha x)}\, dx} \simeq -\frac{\alpha^2}{2} \quad . \quad (14)$$

This method is also appropriate for calculation of $(K^n)_w$:

$$(K^n)_w = \frac{\int e^{-(x-x_0)^2} \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)^n \frac{1}{\cosh(\alpha x)}\, dx}{\int e^{-(x-x_0)^2} \frac{1}{\cosh(\alpha x)}\, dx} \simeq \left(-\frac{\alpha^2}{2}\right)^n \quad . \quad (15)$$

Therefore, for $x_0 >> 1/\alpha$, i.e. for particles found deep in the classically forbidden region, we have

$$(K^n)_w \simeq (K_w)^n \quad . \quad (16)$$

Equation (16) is exactly the condition for normal measurement with good precision to yield (negative!) weak values for kinetic energy. Tunnelling presents a novel situation in which whatever the strength of the measurement, we can find an appropriate post-selection (large enough x_0) such that condition (9) is fulfilled. We see that normal, "strong" measurements can yield weak values. This fact increases the significance of weak values while making the name "weak" less appropriate.

5. Conclusions

We have seen that kinetic energy measurements on a particle in a potential well can yield negative values consistently. From the standpoint of the standard logic of quantum mechanics, all that we have described here is a game of errors of measurement. Ideal measurements of kinetic energy can yield only positive values, since all eigenvalues of the kinetic energy operator are positive. But in practice, measurements are not exact, and even if their precision is very good, sometimes – rarely – they yield negative values. If particles are subsequently found far enough from the potential well, the measured kinetic energy of these particles comes out consistently negative. Regularly, large measurement "errors" appeared, producing a distribution peaked at an "unphysical" negative value. Mathematically, this distribution arises from an unusual interference.[6] What emerges from this interference is an approximate Gaussian centered on the appropriate negative value, with a characteristic spread of the measuring device, i.e. the pointer of the measuring device shows the (negative) weak value.

We have shown that the "game of errors" displays a remarkable consistency; and this consistency allows negative kinetic energies to enter physics in a natural way. The statement, "The kinetic energy of a particle in a classically forbidden region is negative" has a consistent meaning, namely, "The weak value of kinetic energy of a particle in a classically forbidden region is negative". We arrive at a new language for describing quantum processes.

The language of weak values provides us with a new intuition. The operators of total energy, kinetic energy and the potential energy do not commute. Therefore, we do not have an analog of the classical formula $E = K + V$. However, for the weak values, it follows trivially from the definition that $E_w = K_w + V_w$. Thus, weak values allow us to apply a kind of classical intuition in many problems. In fact, we have applied it in calculating the weak value of kinetic energy in Eq.(13). Weak values yield a classical picture in the following sense: In our pre- and post-selected ensemble we could perform several kinetic energy measurements, and even several measurements of total energy and potential energy as well. If the particle is finally found far from the potential well, i.e. if it belongs to our ensemble, and if all these measurements were not too strong, then the kinetic energy measurements yield results clustered around the weak value, the total energy measurements yield values clustered around E_0 ($E_w = E_0$), and the potential energy measurements yield values close to zero ($V_w \simeq 0$).

The new language also provides new intuition about *local* properties of a particle in a quantum state. Intuitively, we can say that negative values of kinetic energy arise as the local value of the kinetic energy operator. In our example, the pre-selected bound state is projected onto a post-selected state localized far from the well. The combination of pre- and post-selection concentrates the measurement on a region outside the potential well. We note a surprising extension of this result: measurements that yield negative kinetic energy, like other impulsive measurements, are independent of the Hamiltonian of the system under study. We could neglect the Hamiltonians for the system and measuring device, and treat only their interaction. It follows that we can observe particles with negative kinetic energy even if there is no binding potential at all. What matters is only the shape of the pre-selected wave function of the particle in the region of overlap with the post-selected wave function.

The example of a particle in a potential well is a limiting case of quantum tunnelling, when the barrier becomes very broad. The interpretation offered here applies to finite barriers, too. However, more precise measurements of kinetic energy require post-selected states farther into the classically forbidden region; for narrow barriers, negative kinetic energies may be hard to observe.

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