Lorentz-Invariant "Elements of Reality" and the Joint Measurability of Commuting Observables

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It is shown that joint measurements of certain commuting operators, performed on pre- and postselected quantum systems, invariably disturb each other. This result is applied to recent assertions that quantum theory has no realistic Lorentz-invariant interpretation (even without requiring locality).

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Recently, a few authors [1–4], adopting a very plausible definition of "elements of reality," claimed to show that a realistic Lorentz-invariant interpretation of quantum mechanics is not possible. They presented gedanken experiments for which quantum predictions contradict Lorentz invariance. In contrast to the Einstein-Podolsky-Rosen (EPR) argument, their proof does not include a locality assumption. In this Letter we will show that the contradiction disappears when we abandon a "product rule" for elements of reality, even for elements of reality corresponding to commuting operators. We will show that the product rule has to be abandoned because joint measurements of commuting operators in the considered situations invariably disturb each other.

The organization of this Letter is as follows: We start with a discussion of measurements performed on a pre- and postselected quantum system. We will show that even commuting operators cannot be measured on such systems without disturbing each other. Then we briefly review the arguments, due to Pitowsky [1], Clifton and co-workers [2,4], and Hardy [3], against a realistic Lorentz-invariant interpretation of quantum mechanics, and explain their usage of the "product rule." We conclude with a brief discussion of an extension of the concept of elements of reality which makes them Lorentz invariant.

In every textbook of quantum mechanics we can find a condition for simultaneous measurability of variables A and B; the corresponding operators must commute:

$$[A,B]=0\,. \tag{1}$$

Commutativity of the operators A and B is a strong sufficient condition; for a given quantum state $|\Psi\rangle$, it is sufficient to have commutativity with respect to that state:

$$[A,B]|\psi\rangle=0\,. \tag{2}$$

The commutativity condition (2) is a necessary and sufficient condition for simultaneous measurability of A and B. If the operators A and B do not commute, the measurement of one disturbs the outcome of the other. For example, consider a standard measuring procedure [5] with an interaction Hamiltonian given by

$$H=g(t)pA\,. \tag{3}$$

Here p is a canonical momentum of the measuring device; the conjugate position q corresponds to the position of a pointer on the device. The coupling $g(t)$ is nonzero for a short time interval, and during the measurement we obtain (in the Heisenberg picture)

$$\frac{dB}{dt}=i[H,B]=ig(t)p[A,B]\,. \tag{4}$$

Thus, commuting operators are measurable without mutual disturbance, while noncommuting operators disturb one another.

If A and B commute, and if at a given moment we know that a measurement of A must yield $A=a$ while a measurement of B must yield $B=b$, we can safely claim that the product AB is also known and equal to ab. We repeat this well-known fact because, surprisingly, it is not true when we consider a pre- and postselected quantum system.

To define a pre- and postselected quantum system, we consider a quantum system at time t. For simplicity we let the free Hamiltonian be zero. At time $t_1<t$ the system is prepared in a quantum state $|\Psi_1\rangle$, and at a time $t_2>t$ a measurement is performed and the system is found in the state $|\Psi_2\rangle$. We ask about possible measurements at time t. Suppose A is measured at time t. If either $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate of A, then clearly the outcome of the measurement is determined [6]; it is the corresponding eigenvalue of A. Measuring the commuting operator B before, after, or even during the measurement of A does not, in principle, disturb the measurement of A. However, for a pre- and postselected quantum system it might be that the result of measuring A is certain, even if neither $|\Psi_1\rangle$ nor $|\Psi_2\rangle$ is an eigenstate of A. In this case a measurement at any time between t_1 and t_2 of certain operators commuting with *A invariably disturbs the A measurement.*

A simple example is the setup proposed by Bohm for analyzing the EPR argument: two separate spin-$\frac{1}{2}$ particles prepared, at time t_1, in a singlet state

$$|\Psi_1\rangle=\frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle-|\downarrow_1\uparrow_2\rangle)\,. \tag{5}$$

At time t_2 measurements of σ_{1x} and σ_{2y} are performed and certain results are obtained. If at time t, $t_1<t<t_2$,

a measurement of σ_{1y} is performed (and if this is the only measurement performed between t_1 and t_2), then the outcome of the measurement is known with certainty: $\sigma_{1y}(t) = -\sigma_{2y}(t_2)$. If, instead, only a measurement of σ_{2x} is performed at time t, the result of the measurement is also certain: $\sigma_{2x}(t) = -\sigma_{1x}(t_2)$. The operators σ_{1y} and σ_{2x} obviously commute, but nevertheless, measuring $\sigma_{2x}(t)$ clearly disturbs the outcome of the measurement of $\sigma_{1y}(t)$: It is not certain anymore.

Measuring the product $\sigma_{1y}\sigma_{2x}$ is, in principle, different from the measurement of both σ_{1y} and σ_{2x} separately. In our example the outcome of the measurement of the product *is* certain, but it does not equal the product of the results which must come out of the measurements of σ_{1y} and σ_{2x} when either one of them is performed without the other [7].

We demonstrate this point by applying the generalization of the formula of Aharonov, Bergmann, and Lebowitz [8] (ABL) for calculating probabilities for the results of an intermediate measurement performed on a pre- and postselected system. If the initial state is $|\Psi_1\rangle$ and the postselected state is $|\Psi_2\rangle$, then the probability for an intermediate measurement of A to yield $A = a_n$ is given by [9]

$$\text{prob}(A = a_n) = \frac{|\langle\Psi_2|\hat{P}_{A=a_n}|\Psi_1\rangle|^2}{\sum_k |\langle\Psi_2|\hat{P}_{A=a_k}|\Psi_1\rangle|^2}, \tag{6}$$

where the sum is over all eigenvalues of A and $\hat{P}_{A=a_k}$ is the projection operator onto the subspace with eigenvalue a_k. The formula immediately yields probability 1 when $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate, but it also can yield 1 when neither of the states is an eigenstate, as we now show.

The state $|\Psi_1\rangle$ is given by Eq. (5). Suppose the results of the postselection measurements are $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$. Then the state $|\Psi_2\rangle = |\uparrow_{1x}\uparrow_{2y}\rangle$. To predict the outcome of a measurement of σ_{1y} we have to use the projection operators $\hat{P}_{[\sigma_{1y}=1]} = |\uparrow_{1y}\rangle\langle\uparrow_{1y}|$ and $\hat{P}_{[\sigma_{1y}=-1]} = |\downarrow_{1y}\rangle\langle\downarrow_{1y}|$. Applying formula (6), we indeed obtain $\text{prob}(\sigma_{1y} = -1) = 1$. In the same way we obtain $\text{prob}(\sigma_{2x} = -1) = 1$. For calculation of the probabilities of the measurement of the product $\sigma_{1y}\sigma_{2x}$ we use the projection operators

$$\begin{aligned} \hat{P}_{[\sigma_{1y}\sigma_{2x}=1]} &= |\uparrow_{1y}\uparrow_{2x}\rangle\langle\uparrow_{1y}\uparrow_{2x}| + |\downarrow_{1y}\downarrow_{2x}\rangle\langle\downarrow_{1y}\downarrow_{2x}|, \\ \hat{P}_{[\sigma_{1y}\sigma_{2x}=-1]} &= |\uparrow_{1y}\downarrow_{2x}\rangle\langle\uparrow_{1y}\downarrow_{2x}| + |\downarrow_{1y}\uparrow_{2x}\rangle\langle\downarrow_{1y}\uparrow_{2x}|. \end{aligned} \tag{7}$$

Then Eq. (6) yields $\text{prob}(\sigma_{1y}\sigma_{2x} = 1) = 0$, contrary to the product rule, which requires $\sigma_{1y}\sigma_{2x} = 1$ with probability 1. It follows that the value of the product $\sigma_{1y}\sigma_{2x}$ is certain, but it equals -1.

We have shown that for a pre- and postselected quantum system it might happen that the operators corresponding to two observables A and B commute, $[A,B] = 0$, but measuring B invariably disturbs the results of the measurement of A. Therefore, for a pre- and postselected quantum system one cannot apply a "product rule" that asserts that if measurements of A and B yield $A = a$ and $B = b$ with certainty, then a measurement of AB yields ab. In fact, the value of AB might also be certain, but not equal to ab.

We now review the arguments against the possibility of a realistic Lorentz-invariant interpretation of quantum mechanics [1-4]. The starting point of these arguments was the definition of elements of reality and the principle of Lorentz invariance. In contrast to the usual EPR-type argument, no locality principle, forbidding an action at a distance, was assumed. The adopted definitions are as follows.

(i) Element of reality (Redhead [10]): "If we can predict with certainty, or at any rate with probability one, the result of measuring a physical quantity at time t, then at the time t, there exists an element of reality corresponding to this physical quantity and having a value equal to the predicted measurement result."

(ii) The Principle of Lorentz invariance: "If an element of reality corresponding to some Lorentz-invariant physical quantity exists and has a value within space-time region R with respect to one spacelike hypersurface containing R, then it exists and has the same value in R with respect to any other hypersurface containing R."

In the usual EPR argument an element of reality corresponding to an outcome of a measurement is fixed by the mere *possibility* of inferring the outcome from measurements in a causally disconnected region. In contrast, since the present approach does not assume locality, elements of reality are fixed only by *actual* measurements.

The argument due to Clifton, Pagonis, and Pitowsky [2] is based on the modified Greenberger-Horne-Zeilinger [11] (GHZ) setup for demonstrating the nonexistence of local hidden variables. Three spin-$\frac{1}{2}$ particles, located in the corners of a very large triangle, move fast in directions pointing out of the center of the triangle. At time t_1 (in the rest frame) the particles are prepared in the state

$$|\Psi_1\rangle = |\text{GHZ}\rangle = \frac{1}{\sqrt{3}}(|\uparrow_{1z}\uparrow_{2z}\uparrow_{3z}\rangle - |\downarrow_{1z}\downarrow_{2z}\downarrow_{3z}\rangle). \tag{8}$$

At time t_2 the spin components in the x direction are measured on all particles and the results $\sigma_{ix} = x_i$ are obtained. Consider now some possible measurements performed on the particles at a time t, $t_1 < t < t_2$. For each of the three observers who perform the σ_{ix} measurements, the measurements on the other particles (at time t in the rest frame) are performed *after* his σ_{ix} measurement, and he can predict (each in his Lorentz frame) the following result with certainty:

$$\sigma_{2y}\sigma_{3y} = x_1, \tag{9a}$$

$$\sigma_{1y}\sigma_{3y} = x_2, \tag{9b}$$

$$\sigma_{1y}\sigma_{2y} = x_3. \tag{9c}$$

Equations (9a)–(9c) represent elements of reality in space-time regions corresponding to the respective Lorentz frames. The principle of Lorentz invariance yields that these are also the elements of reality in the rest frame. By multiplying Eqs. (9b) and (9c) we obtain

$$\sigma_{1y}^2\sigma_{3y}\sigma_{2y}=x_2x_3\,. \tag{10}$$

Taking into account that $\sigma_{1y}^2=1$, we conclude that $x_1=x_2x_3$. This conclusion, however, contradicts quantum mechanics: in the GHZ state $x_1x_2x_3=-1$.

Pitowsky and Clifton, Pagonis, and Pitowsky obtain their elements of reality as *predictions* of different observers, but their argument holds only when they consider the predictions of all observers. However, there is no Lorentz observer for which all the predictions are inferences from the past toward the future; at least some of the inferences must be *retrodictions.* In fact, we have a quantum system on which two complete measurements are performed in succession, and claims about elements of reality apply to times between these two measurements. Here, the GHZ state is prepared initially and measurement of the x components of spin for all particles determines the final state. The discussion at the beginning of this Letter thus applies.

The state $|\Psi_1\rangle$ is given by Eq. (8), while $|\Psi_2\rangle=|x_1,x_2,x_3\rangle$, i.e., the state with certain x components of spin. The operators to be considered between these two states are $\sigma_{1y}\sigma_{3y},\sigma_{1y}\sigma_{2y}$, and $\sigma_{2y}\sigma_{3y}$. The formalism, Eq. (6), yields (as it should) the probability 1 for the outcomes given by Eqs. (9a)–(9c). But it also shows that the measurements of commuting operators $\sigma_{1y}\sigma_{3y}$ and $\sigma_{1y}\sigma_{2y}$ disturb each other. Equation (6) yields that the probability to find both results (9b) and (9c) *when measured together* is just $\frac{1}{4}$. Again, measuring the product differs from measuring both of the operators separately, and the probability of finding $(\sigma_{1y}\sigma_{3y})(\sigma_{1y}\sigma_{2y})=x_2x_3$ is zero since the outcome is given by Eq. (9a).

The example of Hardy [3] involves just two particles, an electron and a positron in two entangled setups of the type proposed by Elitzur and Vaidman [12] (EV) for interaction-free measurements [13]. Each EV setup is a Mach-Zehnder interferometer tuned to yield zero counts at a detector D_1 unless a point $\mathcal{P}$ belonging to one arm of the interferometer is not free. The "click" of the detector D_1, after sending just one particle, yields that the point $\mathcal{P}$ is not empty, without disturbing the object at $\mathcal{P}$. In Hardy's example the point $\mathcal{P}$ is common to the two EV setups. One EV device tests the point $\mathcal{P}$ with a single electron, while the other tests the same point $\mathcal{P}$ with a single positron. If both the electron and the positron come to the point $\mathcal{P}$ together then they annihilate, and it might happen that both devices yield that the point $\mathcal{P}$ is not empty, i.e., detectors D_1 of both the electron and the positron interferometers "click." Let us assume this outcome. Now, consider a Lorentz frame in which the observer of the electron EV device is the first to obtain a "click." She infers [14] that the *positron* was at $\mathcal{P}$. In another Lorentz frame, however, the observer of the positron EV device is the first to obtain the result. He deduces that the *electron* was at $\mathcal{P}$. The principle of Lorentz invariance yields that there are two elements of reality: the electron at $\mathcal{P}$ and the positron at $\mathcal{P}$. The product rule here is very natural: If the electron is at $\mathcal{P}$ and the positron is at $\mathcal{P}$ then the electron and the positron are at $\mathcal{P}$. The latter, however, leads to contradiction: The particles at $\mathcal{P}$ must annihilate and cannot be detected by either observer.

Hardy's example also involves pre- and postselection. Here, the preselection is the preparation of the electron-positron state, while the postselection is the detection of electron and positron at detectors D_1. Thus, we can apply the ABL formalism. However, in this case the free Hamiltonian is not zero; it describes the interaction of the electron and the positron with beam splitters and mirrors as well as their annihilation at $\mathcal{P}$. Therefore, the state $|\Psi_1\rangle$ in the formula (6) must be the initial state evolved forward in time until t, the time when one of the particles reaches the point $\mathcal{P}$, while the state $|\Psi_2\rangle$ must be obtained by evolving the final state backward in time until t. Straightforward calculation shows that Eq. (6) reproduces Hardy's result: If one observer tests, "Was the electron at $\mathcal{P}$?" her result must be "yes"; if the other observer looks for the positron at $\mathcal{P}$, his answer must be "yes" too (but if both of them make these measurements, each observer will obtain "yes" with probability $\frac{1}{3}$, and they will never obtain "yes" together). Here, too, the operator considered by Hardy is the product of two projection operators, and its measurement is not equivalent to two simultaneous measurements, one testing for an electron at $\mathcal{P}$ and another testing for a positron at $\mathcal{P}$. The measurement of the product can be implemented by observing photons due to electron-positron annihilation. Formula (6) yields probability zero to obtain the product equal to 1, in contrast to probability 1 obtained from the product rule.

We believe that Redhead's definition of elements of reality is a plausible one. It does not lead to contradiction with Lorentz invariance if we do not adopt the product rule. But in the light of the discussion above, it is clear that the product rule is incompatible with Redhead's definition. The elements of reality are inferred on the assumption that there are no measurements disturbing their values. Clifton, Pagonis, and Pitowsky [2] state explicitly: "For our argument, we shall assume that no such intervening measurements take place." But as we showed, measurements of the operators they consider *do* interfere with each other. So, it is inconsistent with the definition of the elements of reality to apply the product rule [15]. If it is an element of reality that $A=a$ and it is an element of reality that $B=b$, it does not follow that $AB=ab$ is an element of reality. It might be that the product AB has a certain value and, therefore, is an element of reality in the Redhead's sense, but it need not equal *ab*.

In fact, this happens in all the examples we considered. In the first example we have elements of reality $\sigma_{1y} = -1, \sigma_{2x} = -1$, and the product is also an element of reality, but $\sigma_{1y}\sigma_{2x} = -1$. In the Pitowsky example the elements of reality are $\sigma_{1y}\sigma_{3y} = x_2, \sigma_{1y}\sigma_{2y} = x_3$, and the product, $(\sigma_{1y}\sigma_{3y})(\sigma_{1y}\sigma_{2y}) = \sigma_{2y}\sigma_{3y} = x_1$, but nevertheless $x_2x_3 \neq x_1$ $(x_2x_3 = -x_1)$. In Hardy's example $\hat{P}_{e^-} = 1$, $\hat{P}_{e^+} = 1$, but $\hat{P}_{e^-}\hat{P}_{e^+} = 0$, where $\hat{P}_{e^-}, \hat{P}_{e^+}$ are projection operators on the states "an electron at $\mathcal{P}$" and "a positron at $\mathcal{P}$," respectively.

Clifton, Pagonis, and Pitowsky felt that the conclusions about the impossibility of constructing a realistic Lorentz-invariant quantum theory are too strong. They proposed a variety of ways to circumvent these arguments, in particular, by *rejecting* elements of reality corresponding to "incompatible measurement context." It is possible to deal with the failure of the product rule along these lines, but we prefer another possibility.

We give up the product rule and extend the concept of elements of reality. We consider elements of reality defined by *both* prediction and retrodiction. The Redhead definition of elements of reality continues to hold, with a minor change of "predict" to "infer." Thus, in the case of two spin-$\frac{1}{2}$ particles, the observer who measures $\sigma_{1x}(t_2) = 1$ not only infers that $\sigma_{2x}(t) = -1$ but also that $\sigma_{1x}(t) = 1$. So we add to the list of elements of reality at time t also $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$.

One reason for this proposal is that in the relevant circumstances retrodictions are involved anyway. In the examples presented here, predictions were applied to future events as well as to spacelike separated events, while retrodictions were applied only to spacelike separated events. It is natural to apply retrodiction to the past also, making the approach time symmetric. Elements of reality defined by both prediction and retrodiction yield a Lorentz-invariant history of quantum systems [16].

Further support for this approach comes from recently introduced *weak measurements* [17] which allow us to overcome an obvious objection to defining elements of reality which cannot be measured. Weak measurements performed on pre- and postselected ensembles can test even elements of reality corresponding to otherwise incompatible measurement contexts [18].

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[6] Note that "determined" has two different meanings: if $|\Psi_1\rangle$ is the eigenstate, then the outcome is *predicted*, but if $|\Psi_2\rangle$ is the eigenstate, then the outcome is *retrodicted*. It is possible and we prefer to include retrodictions fully and give them the same status as predictions.

[7] To measure the product $\sigma_{1y}\sigma_{2x}$ we may write it as a modular sum, $\sigma_{1y}\sigma_{2x} = (\sigma_{1y} + \sigma_{2x})\text{mod}4 - 1$. It has been shown [Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. D **34**, 1805 (1986)] that nonlocal operators such as $(\sigma_{1y} + \sigma_{2x})\text{mod}4$ can be measured using solely local interactions.

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[13] Critical discussion of Hardy's idea and its variation using Stern-Gerlach devices rather than interferometers appear in Ref. [4].

[14] She *retrodicts*, since the events she infers were in her absolute past. Hardy is able to discuss the observer's *predictions* by considering the question "Is the particle in the arm of the interferometer which includes $\mathcal{P}$?" instead of the question "Is the particle at $\mathcal{P}$?" See also Ref. [4].

[15] The product rule and its generalization to any function of commuting operators are widely used in no-hidden-variables theorems [N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990)]. It is valid in all cases when no retrodiction is involved.

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[17] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).

[18] The outcomes of weak measurements are *weak values* given by a simple formula $A_w \equiv \langle\Psi_2|A|\Psi_1\rangle/\langle\Psi_2|\Psi_1\rangle$. It has been shown [9] that whenever there exists an element of reality, its value is equal to the weak value. For dichotomic variables an "inverse" theorem is also true: If the weak value is equal to an eigenvalue, then it is an element of reality. In all our examples we consider dichotomic variables, so we can obtain our results via the simpler calculation of the weak value A_w rather than the calculation via Eq. (6).