## NOTES AND DISCUSSIONS

# Minimum time for the evolution to an orthogonal quantum state

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Although it is well known that the rate of evolution of a quantum state depends on the energy uncertainty, the exact limit for the minimum time for the evolution to an orthogonal quantum state is not easy to find in the literature. This limit is $T = h/4\Delta E$, i.e., for a system with a constant energy uncertainty this is the minimum time for the evolution to an orthogonal state. Recently, Anandan and Aharonov[1] proved it using sophisticated arguments about the geometry in the projective space. Much earlier, Fleming[2] gave a somewhat lengthy proof based on Ersak's equation.[3] In this note I present a simple proof for this limit without invoking any geometrical arguments.

The proof uses a very simple general formula[4] that is interesting by itself. For any Hermitian operator $A$ and any quantum state $|\Psi\rangle$:

$$A|\Psi\rangle = \langle A\rangle|\Psi\rangle + \Delta A|\Psi_\perp\rangle, \tag{1}$$

where $|\Psi_\perp\rangle$ is orthogonal to $|\Psi\rangle$. Indeed, we can always decompose

$$A|\Psi\rangle = \alpha|\Psi\rangle + \beta|\Psi_\perp\rangle$$

with $\beta$ real and non-negative. Then $\langle\Psi|A|\Psi\rangle = \langle\Psi|(\alpha|\Psi\rangle + \beta|\Psi_\perp\rangle)$ yields $\alpha = \langle A\rangle$, and

$$\langle\Psi|A^\dagger A|\Psi\rangle = (\alpha^*\langle\Psi| + \beta^*\langle\Psi_\perp|)(\alpha|\Psi\rangle + \beta|\Psi_\perp\rangle)$$

yields

$$\beta = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \equiv \Delta A.$$

Our task is to find a minimum time $T$, subject to the condition that the system has a constant energy uncertainty $\Delta E$, such that $\langle\Psi(T)|\Psi(0)\rangle = 0$. We shall first calculate the maximum possible vaue for the rate of change of the overlap $|\langle\Psi(t)|\Psi(0)\rangle|$ at a given moment. We shall find that it is a function only of the overlap itself (and the energy uncertainty $\Delta E$). Therefore, the maximum rate of change of the overlap during all of the time evolution ensures the minimum time of the evolution to an orthogonal state.

Let us start the calculations:

$$\frac{d}{dt}|\langle\Psi(t)|\Psi(0)\rangle|^2 = 2\,\mathrm{Re}\left(\langle\Psi(t)|\Psi(0)\rangle\langle\Psi(0)|\frac{d}{dt}|\Psi(t)\rangle\right). \tag{2}$$

Using Eq. (1), we obtain

$$\frac{d}{dt}|\Psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle = -\frac{i}{\hbar}(\langle E\rangle|\Psi(t)\rangle + \Delta E|\Psi_\perp(t)\rangle), \tag{3a}$$

where

$$\langle\Psi(t)|\Psi_\perp(t)\rangle = 0. \tag{3b}$$

Substituting Eq. (3) into Eq. (2) yields

$$\frac{d}{dt}|\langle\Psi(t)|\Psi(0)\rangle|^2 = -\frac{2\Delta E}{\hbar}\mathrm{Re}[i\langle\Psi(t)|\Psi(0)\rangle\langle\Psi(0)|\Psi_\perp(t)\rangle]. \tag{4}$$

In order to obtain the maximum possible absolute value of the right-hand side of Eq. (4) [for given $\Delta E$ and $|\langle\Psi(t)|\Psi(0)\rangle|$] we have to find the maximum possible value of $|\langle\Psi(0)|\Psi_\perp(t)\rangle|$. To this end we expand the initial state $|\Psi(0)\rangle$ as follows:

$$|\Psi(0)\rangle = \langle\Psi(t)|\Psi(0)\rangle|\Psi(t)\rangle + \langle\Psi_\perp(t)|\Psi(0)\rangle|\Psi_\perp(t)\rangle + \alpha|\Psi_{\perp\perp}(t)\rangle, \tag{5}$$

where

$$\langle\Psi(t)|\Psi_{\perp\perp}(t)\rangle = 0 \text{ and } \langle\Psi_\perp(t)|\Psi_{\perp\perp}(t)\rangle = 0.$$

The normalization of the quantum states, then, requires that

$$|\langle\Psi(0)|\Psi_\perp(t)\rangle|^2 = 1 - |\langle\Psi(t)|\Psi(0)\rangle|^2 - |\alpha|^2. \tag{6}$$

Therefore, the maximum value of $|\langle\Psi(0)|\Psi_\perp(t)\rangle|$ is obtained for $\alpha = 0$, and it is equal to $\sqrt{1 - |\langle\Psi(0)|\Psi(t)\rangle|^2}$. Thus, the maximum possible absolute value of the rate of change of the square of the overlap is

$$(2\Delta E/\hbar)|\langle\Psi(0)|\Psi(t)\rangle|\sqrt{1 - |\langle\Psi(0)|\Psi(t)\rangle|^2}. \tag{7}$$

We find that this maximum rate, indeed, depends only on the value of the overlap and on the energy uncertainty. Therefore, the condition for the fastest evolution to an orthogonal state is that during the whole period of the evolution the right-hand side of Eq. (4) is equal to minus Eq. (7):

$$\frac{d}{dt}|\Psi(t)|\Psi(0)\rangle|^2 = -\frac{2\Delta E}{\hbar}|\langle\Psi(0)|\Psi(t)\rangle| \times\sqrt{1 - |\langle\Psi(0)|\Psi(t)\rangle|^2}. \tag{8}$$

After introducing a parameter $\phi$, $\cos\phi = |\langle\Psi(0)|\Psi(t)\rangle|$, Eq. (8) becomes

$$\frac{d}{dt}\phi = \frac{\Delta E}{\hbar}. \tag{9}$$

Finally, since the orthogonal state corresponds to $\phi = \pi/2$, the minimum time is, indeed

$$T = \pi/2\dot\phi = h/4\Delta E, \tag{10}$$

where $\dot\phi$ is the time derivative of $\phi$.

What we have proved is that the time of the evolution to an orthogonal state is not smaller than $h/4\Delta E$. The fact that is *is* the minimum time follows from the existence of a physical system that realizes this limit:[1] a spin-1/2 particle precessing in a magnetic field.


[1] J. Anandan and Y. Aharonov, "Geometry of quantum evolution," Phys. Rev. Lett. 65, 1697–1700 (1990).

[2] G. N. Fleming, Nuovo Cimento A 16, 232–240 (1973).

[3] I. Ersak, "The number of wave functions of an unstable particle," Sov. J. Nucl. Phys. 9, 263–264 (1969).

[4] Y. Aharonov and L. Vaidman, "Properties of a quantum system during the time interval between two measurements," Phys. Rev. A 41, 11–20 (1990).