Novel Properties of Preselected and Postsselected Ensembles

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The present considerations first arose in the context of an ongoing attempt to think carefully about what quantum mechanics allows us to infer about the past. My collaborators and I had for some time been thinking, more particularly, of ensembles of quantum mechanical systems that are defined by both preparation and postselection.

Consider, for example, a spin-1/2 system (with zero Hamiltonian) measured on Monday to be in the state, \( \sigma_z = +\frac{1}{2} \), and which is measured on Friday to be in the state, \( \sigma_z = -\frac{1}{2} \). If it so happens that \( \sigma_z \) was measured on that system on Wednesday, then, of course, that measurement finds with certainty that \( \sigma_z = +\frac{1}{2} \), and if it happens that \( \sigma_z \) was measured on that system on Wednesday, then that measurement finds with certainty that \( \sigma_z = -\frac{1}{2} \); thus, these two facts seem to amount to saying that for such a system as this, on Wednesdays \( \sigma_z = +\frac{1}{2} \) and \( \sigma_z = -\frac{1}{2} \), albeit \( \langle \sigma_z \rangle \neq 0 \). On the other hand, it seems, at first sight, to be impossible to give two such assertions as these any experimental meaning at any single time for any single system, because if both \( \sigma_z \) and \( \sigma_z \) are measured on Wednesday, those two measurements disrupt one another, and the results, \( \sigma_z = +\frac{1}{2} \) and \( \sigma_z = -\frac{1}{2} \), will no longer be invariably obtained. We begin to wonder at a certain point whether those disruptive effects might somehow be controlled or eliminated by somehow reducing the accuracy with which \( \sigma_z \) and \( \sigma_z \) are measured. It turns this wondering that eventually produced what is to follow. We will begin by briefly reviewing what quantum mechanical measurements are.

Von Neumann’s famous account of the operations of quantum mechanical measuring devices runs roughly like this: In order to measure some given observable, \( \mathcal{A} \), of a quantum mechanical system, \( \mathcal{S} \), what is required is that one produce a Hamiltonian of interaction between \( \mathcal{S} \) and a measuring device, which has the form,

\[
H_{\text{meas}} = -g \hat{A} \hat{S},
\]

(1)

where \( g \) is some internal variable of the measuring device, and \( \hat{A}(t) \) is a time-dependent coupling function that is nonzero only during some short interval, \( \tau < \tau \), when the measuring device is “switched on.” Then the measurement is accomplished as follows:

\[6\] This work was supported by NSF Grant No. PHY-8408265.

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the Heisenberg equation for $\psi$, where $\psi$ is defined to be the canonical momentum conjugate to the canonical coordinate $q$, of the measuring device, reads

$$\frac{d\psi}{dt} = i\hbar [A, \psi]$$

(1)

Therefore, if $\psi$ is initially set, say, at zero, and if the value of \int_0^\infty g(t)dt is known, then the value of $A$ at $t = 1$, $\psi(1)$, can be read off from the value of $\psi$ at $t = 0$ (in this context $\psi$ is often referred to as the "pointer variable").

The fact that any precise measurement of $A$ must necessarily and uncontrollably disturb the values of observables that fail to commute with $A$ can be traced, within this account, to the fact that a precise measurement of $A$ requires that the value of $\psi$ be precisely fixed prior to $\psi$. Consequently, this also requires that the uncertainty in $\psi$ during the measurement interaction described in equation (1) (and hence, as well, the possible strength of that interaction) is unabounded.

On the other hand, it emerges quite clearly within this account that if one is willing to accept uncertainties in the initial value of $\psi$, along with the resultant inaccuracies of the measurement of $A$, then the uncertainty in the value of $q$ during the measurement interaction, and hence the possible strength of that interaction, and the disturbance caused by it to variables that fail to commute with $A$, can be bounded and controlled. We shall refer here to such a trade-off—specifically, to the sacrificing of the accuracy of measurements of $A$ in order to gain some control of the disturbances caused by such measurements in variables that fail to commute with $A$—as a weakening of the measurement of $A$. For our present concern shall be to point out a most extraordinary statistical property of such weakened measurements, which we have recently discovered.

Consider a system of $N$ spin-1/2 particles (the Hamiltonian of which we shall suppose, for simplicity, to be $H_{\text{tot}}$). Suppose that at time $t = 0$, a precise measurement of the total angular momentum of this $N$-particle system in the $z$-direction ($L_z$) is carried out, and that this measurement produces the (largest possible) result $L_z = |N|\hbar$; furthermore, suppose that at some time $t < t_0$ a precise measurement of $L_z$ is carried out on this system, and that this measurement happens to produce the result $L_z = \pm N$ (such pairs of results, when $N$ is large, will of course be rare, but they are nonetheless always possible; we should like to confine our attention here to a system wherein such a pair of results happens to have emerged). If we are later informed that another precise measurement of $L_z$, say, was carried out at time $t_1$ with $t_1 < t < t_0$ (then, as is well known) we could assert with certainty that the result of this measurement must have been $L_z = \pm N$ (because otherwise the result of the measurement at $t$, could not have been what it was), similarly, if we are later informed that a precise measurement of $L_z$ was carried out at $t_2$ with $t_2 < t_1 < t_0$ we would be in a position to assert with certainty that the result of this measurement must have been $L_z = \pm N$. Indeed, it is even the case that if we were later informed that a precise measurement of $L_z$ was carried out at $t$, and a precise measurement of $L_z$ was carried out at $t_1$ with $t < t_1 < t_0$, then we should be in a position to say with certainty that the result of the measurement of $t_1$ was $L_z = \pm N$ and that the result of the measurement of $t$ was $L_z = \pm N$. However, it should be carefully noted that in this last case, the time-order of the two intermediate measurements is vitally important. These two measurements, after all, being precise,
will unreasonably disturb one another; thus, in the event that \( t_x < t_i < t_z \), there will in general be no correlation whatsoever between the results of the measurements at \( t_x \) and \( t_z \), nor between the results of those at \( t_x \) and \( t_z \).

Suppose, however, that we were to weaken these two intermediate measurements in such a way as to gain some considerable weight over (or even to eliminate) the disturbances they cause to one another. Suppose, more particularly, that the initial state of the measuring device is arranged in such a way as to bound the possible value of \( q \) as follows:

\[
|q| < \frac{1}{(\sqrt{2})^{N/2}}.
\]

(3)

where \( q \) is an arbitrarily small positive number. In that case, the remaining uncertainty in \( q \) will be of the order of \( (\sqrt{2})^{-N} \), which, if \( N \) is taken to be large, is small compared with the maximal possible values of \( L_x \) and \( L_z \), therefore, measuring devices prepared in this way can still arise (albeit imperfectly) as reasonably informative indicators of the value of these two elements. On the other hand, if we let

\[
\int f(q) dq = 1
\]

(4)

for each of these devices, then the bound (equation 3) on \( q \) will guarantee that measurements of \( L_x \) say, with such devices as these, will change the value of \( L_x \) only by amounts of the order of \( (\sqrt{2})^{-N} \) which is (as we have just seen) within the intrinsic error associated with these measurements. Such weakened measurements of \( L_x \) and \( L_z \), then, can be expected to be, as it were, "commutate"; it can be expected, that is, that two such measurements will veritably have one another's results essentially undisturbed.

Consider, now, the system of \( N \) spins described above, which was measured, precisely, at time \( t_0 \) to be in the state \( L_x = \frac{1}{2} \) and at \( t_1 \) to be in the state \( L_z = \frac{1}{2} \). Suppose that we are informed later on that a weak measurement of \( L_x \) of the kind we have just described, was carried out at \( t_2 (t_2 < t_1 < t_0) \). Then, especially if \( N \) is large, it can be asserted with a high degree of confidence that the result of this weakened measurement was \( L_x = \frac{1}{2} \). More precisely, it will be the case that \( \langle L_x \rangle \approx 0 \) before the interaction begins, then it will invariably be the case that \( \langle L_x \rangle \approx \frac{1}{2} \) after \( t_0 \), where \( \approx \) is the pointer variable of the weakened \( L_x \) measuring device, furthermore, if \( N \) is large, the uncertainty in \( \approx \) both before and after the experiment, will be very small compared with this distinguished in its expected value, and, therefore, the time-reversal symmetric character of the statistical predictions of quantum theory, the present argument can be made concerning a weak measurement of \( L_x \) that may have been carried out at \( t_2 \) within that same interval. Clearly, these additional complications are introduced by supposing that both measurements (first the measurement of \( L_x \) and then that of \( L_z \)) are carried out within that interval, as we did above; however, in the present case, because of the "commutative" behavior of these was measurements, we also expect that the order in which they are carried out will not make a difference. Indeed, it can easily be confirmed by straightforward calculation that whether \( t_x < t_z < t_y \), or \( t_y < t_x < t_z \), the expectation values of the pointer variables of both the \( L_x \) and the \( L_z \) measuring devices will, in the circumstances described above, be displaced by precisely (up to corrections of the order of \( (\sqrt{2})^{-N} \)).

This produces something of a paradox, which runs as follows: Suppose that instead
(as above) of measuring the value of $L_x$ of time $t_1$, and of the value of $L_z$ of time $t_2$, we measure, with a single device, the sum of these two values. Such a measurement can easily be accomplished by means of an interaction Hamiltonian of the form,

$$H_{\text{int}} = \frac{\tilde{g}(\eta)}{\sqrt{2}} L_x + g(\eta) L_z,$$

(5)

where $g(\eta)$ is nonzero only in the vicinity of $t_1$, and $g(\eta)$ is nonzero only in the vicinity of $t_2$. The factors of $1/\sqrt{2}$, as the reader shall presently see, have been inserted for the sake of convenience. Furthermore, if "weak" bounds of the form of equation 3 are imposed on $g(\eta)$ and in cases where $L_x$ is precisely measured at $t_0$, and $L_z$ is precisely measured at $t_0$, the total displacement of the expectation value of $\varphi$ after both $t_1$ and $t_2$ will, by the above arguments, always be of the order of $\sqrt{2}/N$, whether $t_1$ precedes $t_2$ or $t_1$ precedes $t_2$, or, indeed, $t_1 = t_2$. However, consider this last possibility. In the event that $t_1 = t_2$, then (in the event, that is, that $g(\eta) = g(\eta)$), the interaction Hamiltonian of equation 5 reduces to

$$H_{\text{int}} = \frac{\tilde{g}(\eta)}{\sqrt{2}} L_x + L_z,$$

(6)

which is the Hamiltonian required for a measurement of the projection of the total angular momentum along the $z$-axis ($L_z$), where $z$ is the ray that bisects the right angle between $x$ and $y$. Now, we have just argued that the measurement will (within such intervals as we have just described, and as long as $\varphi$ is bounded in accordance with equation 3) almost invariably produce the result $N/2$; however, this seems a most paradoxical result, because the particular measurement here is question is looked at in another way: simply a measurement of $L_z$, the largest possible eigenvector of which is $\tilde{g}(\eta)$, the only smaller number $|N/2|$. How can we be that measurements of $L_z$, under these circumstances, with such regularity, produce impossible results?

The first thing to do, it would seem, is to verify the result of our argument by more rigorous techniques, and this, happily, is not a particularly difficult task. The state of the composite system consisting of the $N$ spins together with the $L_z$ measuring apparatus after the $L_z$ interaction is complete, and imposing that $L_z$ was found to have the value $N/2$, at $t_2$, will be

$$\varphi = L_z = \frac{N}{2} |\varphi = \frac{N}{2}|$$

(7)

wherein we have supposed, for simplicity, that $g(\eta)$ has the value of 1 whenever the measuring device is "switched on," and wherein $|\varphi = 0\rangle$ represents the initial state of that device, which (in accordance with equation 3) will be characterized by some Gaussian distribution of $\varphi$-values, of width, $\sqrt{N/2}$ and peaked, say, about $\varphi = 0$. Now, if it subsequently happens that at $t_1$, $L_z$ is found to have the value $N/2$, then the first state of the measuring apparatus (modulo an overall constant of normalization) will be

$$\varphi = L_z = \frac{N}{2} |\varphi = \frac{N}{2}|$$

(8)

thus, the time-evolution operator for the measuring apparatus through such a sequence
of events will be

\[ L = \frac{N}{2} \left( \frac{L_+ + L_-}{\sqrt{2}} \right) - \frac{N}{2} \left( \frac{L_+ - L_-}{\sqrt{2}} \right) \]

(9)

Also, it can rigorously be shown (without too much trouble) that if \( q \) is taken to obey the bound (equation 3), then

\[ L = \frac{N}{2} \left( \frac{L_+ + L_-}{\sqrt{2}} \right) - \frac{N}{2} \left( \frac{L_+ - L_-}{\sqrt{2}} \right) - \frac{N}{2} \left( \frac{L_+ + L_-}{\sqrt{2}} \right) \]

(10)

as \( N \) becomes large. The effect of such a sequence of events, thus, in this limit, is

irreversibly to translate the initial \( q = 0 \) apparatus state by the immeasurable (at least,

at first sight, unobservable) distance of \( N/\sqrt{2} \), rather than what would seem more

reasonable) a distance equivalent to any of the eigenvalues of \( L \), which is precisely as

we earlier (and more intuitive) argument had led us to believe.

What is happening here (albeit the demonstration is quite straightforward) is

something of a miracle. The measuring apparatus state is translated, in the course

of these events, by a superposition of different distances corresponding to the various

possible eigenvalues of \( L \); the resultant translated states, in the end, will quantum

mechanically interfere with one another in such a way as to produce an effective

translation that is larger than any of them. In such sequences of events, everything in

the final apparatus state, save the outermost limits of the tails (which must necessarily

exist given equation 3) of the translated + distributions, ends up canceling itself out;

the central peaks annihilate one another and disappear, and what remains is a new

peak, made up of constructive interferences among the many tails, way out in the

middle of nowhere at \( N/\sqrt{2} \). Moreover (and this is what seems genuinely miraculous),

the nature of these a priori interferes irreversibly such as to make the \( L_+ \) and \( L_- 

components of the total angular momentum (both of which have the value, \( N/2 \))

appear, as measured by our weak experiments, to add together in \( L \) as if they were

components of a classical vector. These, strictly speaking, are of course errors of the

measuring apparatus, but they are errors of an astonishingly subtle, systematic, and

infallible kind; they are errors that (given initial and final conditions on the spins such as

we have posited here) invariably arise, and that compels us to point an internally

consistent picture of a classical (rather than a quantum mechanical) system. However,

it ought to be kept constantly in mind that such conspiracies among errors as these

necessarily feature exceedingly rare sequences of events, if such were not the case, such

conspiracies would constitute a genuine difficulty for quantum theory.

The effect we have just described, needless to say, is one particular example of what

is quite clearly a very broad class of physical phenomena. This broad class of physical

phenomena will arise in countless varieties of physical systems, where it may well

take forms that are yet more striking than the one considered here. Our intent in this

present paper has been simply to point out that such a class of phenomena exists;

however, we shall have much more to say on the scope and the meaning of these

phenomena in forthcoming publications.