Novel Properties of Preselected and Postselected Ensembles[a]

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The present considerations first arose in the context of an ongoing attempt to think carefully about what quantum mechanics allows us to infer about the past. My collaborators and I had for some time been thinking, more particularly, of ensembles of quantum mechanical systems that are defined by both preselection and postselection. Consider, for example, a spin-$\frac{1}{2}$ system (with zero Hamiltonian) measured on Monday to be in the state, $L_z = +\frac{1}{2}$, and which is measured on Friday to be in the state, $L_x = +\frac{1}{2}$. If it so happens that L_z was measured on that system on Wednesday, then, of course, that measurement finds with certainty that $L_z = +\frac{1}{2}$, and if it happens that L_x was measured on that system on Wednesday, then that measurement finds with certainty that $L_x = +\frac{1}{2}$; thus, these two facts seem to amount to saying that for such a system as this, on Wednesday, $L_x = +\frac{1}{2}$ and $L_z = +\frac{1}{2}$, albeit $(L_x, L_z) \neq 0$. On the other hand, it seems, at first sight, to be impossible to give two such assertions as these any experimental meaning at any single time for any single system, because if both L_z and L_x are measured on Wednesday, those two measurements disrupt one another, and the results, $L_x = +\frac{1}{2}$ and $L_z = -\frac{1}{2}$, will no longer be invariably obtained. We began to wonder at a certain point whether those disruptive effects might somehow be controlled or eliminated by somehow reducing the accuracy with which L_x and L_z are measured. It was this wondering that eventually produced what is to follow. We will begin by briefly reviewing what quantum mechanical measurements are.

Von Neumann's famous account of the operations of quantum mechanical measuring devices runs roughly like this: In order to measure some given observable, A, of a quantum mechanical system, S, what is required is that one produce a Hamiltonian of interaction between S and a measuring device, which has the form,

$$H_{\text{int}} = -g(t)qA, \tag{1}$$

where q is some internal variable of the measuring device, and $g(t)$ is a time-dependent coupling function that is nonzero only during some short interval, $t_o < t < t_1$, when the measuring device is "switched on." Then the measurement is accomplished as follows:

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the Heisenberg equation for π, where π is defined to be the canonical momentum conjugate to the canonical coordinate, q, of the measuring device, reads

$$\frac{\partial \pi}{\partial t} = g(t)A; \tag{2}$$

therefore, if π is initially set, say, at zero, and if the value of $\int_{t_o}^{t} g(t)dt$ is known, then the value of A at $t \simeq t_o \simeq t_1$ can be read off from the value of π after t_1 (thus π is often referred to as the "pointer variable").

The fact that any precise measurement of A must necessarily and uncontrollably disturb the values of observables that fail to commute with A can be traced, within this account, to the fact that a precise measurement of A requires that the value of π be precisely fixed prior to t_o. Consequently, this also requires that the uncertainty in q during the measurement interaction described in equation 1 (and hence, as well, the possible strength of that interaction) is unbounded.

On the other hand, it emerges quite clearly within this account that if one is willing to accept uncertainties in the initial value of q, along with the resultant inaccuracies in the measurement of A, then the uncertainties in the value of q during the measurement interaction, and hence, the possible strength of that interaction, and the disturbance caused by it to variables of S that fail to commute with A, can be bounded and controlled. We shall refer here to such a trading-off—specifically, to the sacrificing of the accuracy of measurements of A in order to gain some control of the disturbances caused by such measurements to variables that fail to commute with A—as a weakening of the measurement of A; and our present concern shall be to point out a most extraordinary statistical property of such weakened measurements, which we have recently discovered.

Consider a system of N spin-½ particles (the Hamiltonian of which we shall suppose, for simplicity, to be zero). Suppose that at the time t_i a precise measurement of the total angular momentum of this N-particle system in the x-direction (L_x) is carried out, and that this measurement produces the (largest possible) result $L = \frac{1}{2}N$; furthermore, suppose that at time t_f ($t_f > t_i$) a precise measurement of L_y is carried out on this system, and that this measurement happens to produce the result $L_y = \frac{1}{2}N$ (such pairs of results, when N is large, will of course be rare, but they are nonetheless always possible; we should like to confine our attention here to a system wherein such a pair of results happens to have emerged). If we are later informed that another precise measurement of L_x, say, was carried out at time t_1 with $t_i < t_1 < t_f$, then (as is well known) we could assert with certainty that the result of that measurement must have been $L_x = \frac{1}{2}N$ (because otherwise the result of the measurement at t_i could not have been what it was); similarly, if we are later informed that a precise measurement of L_y was carried out at t_2 with $t_i < t_2 < t_f$, we would be in a position to assert with certainty that the result of that measurement must have been $L_y = \frac{1}{2}N$. Indeed, it is even the case that if we were later informed that a precise measurement of L_x was carried out at t_1 and a precise measurement of L_y was carried out at t_2, with $t_i < t_1 < t_2 < t_f$, then we should be in a position to say with certainty that the result of the measurement of t_1 was $L_x = \frac{1}{2}N$ and that the result of the measurement of t_2 was $L_y = \frac{1}{2}N$. However, it should be carefully noted that in this last case, the time-order of the two intermediate measurements is vitally important. These two measurements, after all, being precise, will uncontrollably disturb one another; thus, in the event that $t_i < t_2 < t_1 < t_f$, there will in general be no correlation whatsoever between the results of the measurements of t_i and t_1, nor between the results of those at t_f and t_2.

Suppose, however, that we were to weaken these two intermediate measurements in such a way as to gain some considerable control over (or even to eliminate) the disturbances they cause to one another. Suppose, more particularly, that the initial state of the measuring device is arranged in such a way as to bound the possible value of q as follows:

$$|q| < \frac{1}{(N/2)^{1/2+\epsilon}}, \tag{3}$$

where ϵ may be an arbitrarily small positive number. In that case, the resulting uncertainty in π will be of the order of $\sqrt{N/2}$, which, if N is taken to be large, is small compared with the maximal possible values of L_x and L_y; therefore, measuring devices prepared in this way can still serve (albeit imperfectly) as reasonably informative indicators of the values of those angular momenta. On the other hand, if we set

$$\int g(t)dt = 1 \tag{4}$$

for each of these devices, then the bound (equation 3) on q will guarantee that measurements of L_x, say, with such devices as these, will change the value of L_y only by amounts of the order of $\sqrt{N/2}$, which is (as we have just seen) within the intrinsic error associated with these measurements. Such weakened measurements of L_x and L_y, then, can be expected, as it were, to "commute"; it can be expected, that is, that two such measurements will verifiably leave one another's results essentially undisturbed.

Reconsider, now, the system of N spins described above, which was measured, precisely, at time t_i to be in the state $L_x = \frac{1}{2}N$ and at t_f to be in the state $L_y = \frac{1}{2}N$. Suppose that we are informed later on that a weak measurement of L_x, of the kind we have just described, was carried out at t_1 ($t_i < t_1 < t_f$). Then, especially if N is large, it can be asserted with a high degree of confidence that the result of this weakened measurement was $L_x = \frac{1}{2}N$ (more precisely, it will be the case that if $\langle L_x \rangle = 0$ before the interaction begins, then it will invariably be the case that $\langle L_x \rangle = \frac{1}{2}N$ after t_1, where π is the pointer variable of the weakened L_x measuring device; furthermore, if N is large, the uncertainties in q, both before and after the experiment, will be very small compared with this displacement in its expectation value); thus, by virtue of the time-reversal-symmetric character of the statistical predictions of quantum theory, the same argument can be made concerning a weak measurement of L_y that may have been carried out at t_2 within that same interval. Clearly, no additional complications are introduced by supposing that both measurements (first the measurement of L_x and then that of L_y) are carried out within that interval, as we did above; however, in the present case, because of the "commutative" behavior of these weak measurements, we also expect that the order in which they are carried out will make no difference. Indeed, it can easily be confirmed by straightforward calculation that whether $t_1 < t_2$ or $t_2 < t_1$, the expectation values of the pointer variables of both the L_x and the L_y measuring devices will, in the circumstances described above, be displaced by precisely (up to corrections of the order of $\sqrt{N/2}$) $\frac{1}{2}N$.

This produces something of a paradox, which runs as follows: Suppose that instead (as above) of measuring the value of L_x of time t_1 and the value of L_y of time t_2, we measure, with a single device, the sum of those two values. Such a measurement can easily be accomplished by means of an interaction Hamiltonian of the form,

$$H_{int} = g_1(t)q\frac{L_x}{\sqrt{2}} + g_2(t)q\frac{L_y}{\sqrt{2}}, \tag{5}$$

where $g_1(t)$ is nonzero only in the vicinity of t_1, and $g_2(t)$ is nonzero only in the vicinity of t_2 (the factors of $1/\sqrt{2}$, as the reader shall presently see, have been inserted for the sake of convenience). Furthermore, if "weak" bounds of the form of equation 3 are imposed on q, and in cases where L_x is precisely measured to be $\frac{1}{2}N$ at t_i and L_y is precisely measured to be $\frac{1}{2}N$ at t_f, the total displacement of the expectation value of π after both t_1 and t_2 will, by the above arguments, always be (up to corrections of the order of $\sqrt{N/2}$) $N/\sqrt{2}$, whether t_1 precedes t_2 or t_2 precedes t_1, or, indeed, $t_1 = t_2$. However, consider this last possibility. In the event that $t_1 = t_2$ [in the event, that is, that $g_1(t) = g_2(t)$], the interaction Hamiltonian of equation 5 reduces to

$$H_{int} = g_1(t)q\,\frac{L_x + L_y}{\sqrt{2}}, \tag{6}$$

which is the Hamiltonian required for a measurement of the projection of the total angular momentum along the α-axis (L_α), where α is the ray that bisects the right angle between x and y. Now, we have just argued that this measurement will (within such intervals as we have just described, and so long as q is bounded in accordance with equation 3) almost invariably produce the result $N/\sqrt{2}$; however, this seems a most paradoxical result, because the particular measurement here in question is (looked at in another way) simply a measurement of L_α, the largest possible eigenvalue of which is the vastly smaller number $\frac{1}{2}N$. How can it be that measurements of L_α, under these circumstances, with such regularity, produce impossible results?

The first thing to do, it would seem, is to verify the result of our argument by more rigorous techniques, and this, happily, is not a particularly difficult task. The state of the composite system consisting of the N spins together with the L_α-measuring apparatus after the L_α-interaction is complete, and supposing that L_x was found to have the value, $N/2$, at t_i, will be

$$e^{iq\,\frac{L_x + L_y}{\sqrt{2}}}\left|L_x = \frac{N}{2}\right\rangle|\pi \simeq 0\rangle, \tag{7}$$

wherein we have supposed, for simplicity, that $g_1(t)$ has the value of 1 whenever the measuring device is "switched on," and wherein $|\pi \simeq 0\rangle$ represents the initial state of that device, which (in accordance with equation 3) will be characterized by some Gaussian distribution of π-values, of width, $\sqrt{N/2}$, and peaked, say, about $\pi = 0$. Now, if it subsequently happens that at t_f, L_y is found to have the value, $N/2$, then the final state of the measuring apparatus (modulo an overall constant of normalization) will be

$$\left\langle L_y = \frac{N}{2}\right|e^{iq\,\frac{L_x + L_y}{\sqrt{2}}}\left|L_x = \frac{N}{2}\right\rangle|\pi \simeq 0\rangle; \tag{8}$$

thus, the time-evolution operator for the measuring apparatus through such a sequence of events will be

$$\left\langle L_y = \frac{N}{2}\right| e^{iq \frac{L_x + L_y}{\sqrt{2}}} \left| L_x = \frac{N}{2}\right\rangle. \tag{9}$$

Also, it can rigorously be shown (without too much trouble) that if q is taken to obey the bound (equation 3), then

$$\left\langle L_y = \frac{N}{2}\right| e^{iq \frac{L_x + L_y}{\sqrt{2}}} \left| L_x = \frac{N}{2}\right\rangle = e^{iqN/\sqrt{2}} + O\left(\frac{1}{N'}\right) \tag{10}$$

as N becomes large. The effect of such a sequence of events, then, in this limit, is invariably to translate the initial $|\pi \simeq 0\rangle$ apparatus state by the impossible (or at least, at first sight, unreasonable) distance of $N/\sqrt{2}$, rather than (what would seem more reasonable) a distance equivalent to any of the eigenvalues of L_α, which is precisely as our earlier (and more intuitive) argument had led us to believe.

What is happening here (albeit the demonstration is quite straightforward) is something of a miracle. The measuring apparatus state is translated, in the course of these events, by a superposition of different distances corresponding to the various possible eigenvalues of L_α; the resultant translated states, in the end, will quantum mechanically interfere with one another in such a way as to produce an effective translation that is larger than any of them. In such sequences of events, everything in the final apparatus states, save the outermost limits of the tails (which must necessarily exist given equation 3) of the translated π-distributions, ends up canceling itself out; the central peaks annihilate one another and disappear, and what remains is a new peak, made up of constructive interferences among the many tails, way out in the middle of nowhere at $N/2$. Moreover (and this is what seems genuinely miraculous), the nature of these anomalous interferences is precisely such as to make the L_x and L_y components of the total angular momentum (both of which have the value, $N/2$) appear, as measured by our weak experiments, to add together in L as if they were components of a classical vector. These, strictly speaking, are of course errors of the measuring apparatus, but they are errors of an astonishingly subtle, systematic, and sensible kind; they are errors that (given initial and final conditions on the spins such as we have postulated here) invariably arise, and that conspire to point an internally consistent picture of a classical (rather than a quantum mechanical) system. However, it ought to be kept constantly in mind that such conspiracies among errors as these necessarily feature exceedingly rare sequences of events; indeed, if such were not the case, such conspiracies would constitute a genuine difficulty for quantum theory.

The effect we have just described, needless to say, is one particular example of what is quite clearly a very broad class of physical phenomena. This broad class of physical phenomena will arise in countless varieties of physical systems, wherein it may well take forms that are yet more striking than the one considered here. Our intent in this present paper has been simply to point out that such a class of phenomena exists; however, we shall have much more to say on the scope and the meaning of these phenomena in forthcoming publications.