Chapter 26
Derivations of the Born Rule

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Abstract The Born rule, a cornerstone of quantum theory usually taken as a postulate, continues to attract numerous attempts for its derivation. A critical review of these derivations, from early attempts to very recent results, is presented. It is argued that the Born rule cannot be derived from the other postulates of quantum theory without some additional assumptions.

Keywords Quantum probability · Born rule · Many-worlds interpretation · Relativistic causality

26.1 Introduction

My attempt to derive the Born rule appeared in the first memorial book for Itamar Pitowsky (Vaidman 2012). I can only guess Itamar's view on my derivation from reading his papers (Pitowsky 1989, 2003, 2006; Hemmo and Pitowsky 2007). It seems that we agree which quantum features are important, although our conclusions might be different. In this paper I provide an overview of various derivations of the Born rule. In numerous papers on the subject I find in depth analyses of particular approaches and here I try to consider a wider context that should clarify the status of the derivation of the Born rule in quantum theory. I hope that it will trigger more general analyses which finally will lead to a consensus needed for putting foundations of quantum theory on solid ground.

The Born rule was born at the birth of quantum mechanics. It plays a crucial role in explaining experimental results which could not be explained by classical physics. The Born rule is known as a statement about the probability of an outcome of a quantum measurement. This is an operational meaning which corresponds to

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numerous very different statements about ontology in various interpretations of quantum theory. Von Neumann's description of quantum measurement includes, at some stage, a collapse of the quantum state corresponding to a particular result of the measurement and the Born rule provides the probability of getting this result. In this framework there is no definition of when exactly it happens (where is the quantum-classical boundary?), so the Copenhagen interpretation and its recent development in the form of QBism (Caves et al. 2002) do not specify the ontology, leaving the definition of the principle on the operational level. In the framework of the Bohmian interpretation (Bohm 1952), which is a deterministic theory given the initial Bohmian positions of all particles, the Born rule is a postulate about the probability distribution which governs these random initial Bohmian positions. It is a postulate about the genuinely random stochastic collapse process in the framework of physical collapse theories (Ghirardi et al. 1986). In Aharonov's solution to the measurement problem (Aharonov et al. 2014), it is a postulate about the particular form of the backward evolving wavefunciton. In the framework of the many-worlds interpretation (MWI) (Everett III 1957), it is a postulate about experiences of an observer in a particular world (Vaidman 2002). So, in all interpretations, the Born rule is postulated, but the question of the possibility of its derivation is considered to be of interest, and it is still open (Landsman 2009).

A rarely emphasized important fact about the Born rule is that it might be even more relevant for explaining physics phenomena in which the probability is not explicitly manifested. Quantum statistical mechanics which leads to quantum thermodynamics heavily uses the Born rule for explaining everyday observed phenomena. There is nothing random when we observe a blue sky or reddish sun at sunset. The explanation includes scattered photons of various colors absorbed by cones in the eye with their color depended efficiency of the absorption. The ratio of the number of events of photon absorption in different cones corresponds to different experiences of the color of the sky and the sun, and we have to use the Born rule to explain our visual experience (Li et al. 2014). In this explanation we consider the cone photoreceptor in an eye as a single-photon detector and light from the sun scattered by molecules of air as a collection of photons. The quantum nature of light coming from modern artificial light sources is even more obvious. An observer looks on a short flash of a fluorescent soft white bulb and announces the color she has seen. The spectrum of the light from this bulb consists of red, green and blue photons, but nobody would say that she saw red light or that she saw green light from the fluorescent bulb. The Born Rule is needed to calculate the ratio of the signals from the cones. The large number of events of photon detection by cones explains why nobody would say they saw a color different from white, since the Born rule provides an almost vanishing probability for such event. The Born rule also tells us that there is an astronomically small probability that we will see a red sky and a blue sun, but it is not different from other quantum tiny tails which we neglect when we explain the classical world we observe through underlying quantum reality.

26.2 Frequentist Approach

One of the early approaches relied on consideration of infinite series of repeated measurements. In the frequentist approach to probability, we consider the ratio of particular outcomes to the total number of measurements. The probability acquires its meaning only in the infinite limit. The important milestones were the works of Hartle (1968) and Farhi et al. (1989). Then the program was extended by replacing infinite tensor products of Hilbert spaces by continuous fields of C*-algebras (Van Wesep 2006; Landsman 2008). The core feature of these arguments involves taking a limit of an infinite number of quantum systems. Aharonov and his collaborators (Aharonov and Reznik 2002; Aharonov et al. 2017) presented, in my view, the simplest and the most elegant argument based on this type of infinite limit.

Consider a large number N of identical systems all prepared in the same state

$$|\Psi\rangle = \sum_i \alpha_i |a_i\rangle, \tag{26.1}$$

which is a superposition of nondegenerate eigenstates of a variable A. Consider the "average" variable $\bar{A} \equiv \frac{1}{N}\sum_{n=1}^{N} A_n$. Applying the universal formula (Aharonov and Vaidman 1990)

$$A|\Psi\rangle = \langle\Psi|A|\Phi\rangle\, |\Psi\rangle + \Delta A|\Psi_\perp\rangle, \tag{26.2}$$

where $\langle\Psi|\Psi_\perp\rangle = 0$, we obtain

$$\bar{A}\prod_{n=1}^{N} |\Psi\rangle_n = \langle\Psi|A|\Psi\rangle \prod_{n=1}^{N} |\Psi\rangle_n + \frac{\Delta A}{N}\sum_{k=1}^{N}\prod_{n\neq k} |\Psi\rangle_n |\Psi_\perp\rangle_k. \tag{26.3}$$

The amplitude of the first term in the right hand side of the equation is of order 1 while the amplitude of the second term (the sum) is proportional to $\frac{1}{\sqrt{N}}$, so in the limit as N tends to infinity, the second term can be neglected and the product state $\prod_{n=1}^{N} |\Psi\rangle_n$ can be considered an eigenstate of the variable $\bar{A}$ with eigenvalue $\langle\Psi|A|\Psi\rangle$.

Now consider the measurement of $\bar{A}$ followed by measurements of A of each of the individual systems. N_i is the number of outcomes $A = a_i$. The probability of outcome a_i is defined as the limit $p_i \equiv \lim_{N\to\infty} \frac{N_i}{N}$. To derive the Born rule we consider the shift of the pointer of the measuring device measuring $\bar{A}$, in two ways. First, since in the limit, the state is an eigenstate with eigenvalue $\langle\Psi|A|\Psi\rangle$, the pointer is shifted by this value. Second, consider the evolution backward in time given that we have the results of individual measurements of variable A of each system. Then the shift has to be $\sum_{i=1}^{N} \frac{a_i N_i}{N}$. In the limit we obtain $\langle\Psi|A|\Psi\rangle = \sum_i |\alpha_i|^2 a_i = \sum_{i=1}^{N} a_i p_i$. This equation can be generally true only if $p_i = |\alpha_i|^2$ for all eigenvalues a_i. This proves the Born rule.

The legitimacy of going to the limit $N \to \infty$ in the earlier proofs was questioned in Squires (1990), Buniy et al. (2006) and Aharonov's approach was analyzed in Finkelstein (2003). I am also skeptical about the possibility of arguments relying on the existence of infinities to shed light on Nature. Surely, the infinitesimal analysis is very helpful, but infinities lead to numerous very peculiar sophisticated features which we do not observe. I see no need for infinities to explain our experience. Very large numbers can mimic everything and are infinitely simpler than infinity. The human eye cannot distinguish between 24 pictures per second and continuous motion, but infinite information is required to describe the latter. There is no need for infinities to explain all what we see around.

Another reason for my skepticism about possibility to understand Nature by neglecting vanishing terms in the infinite limit is the following example in which these terms are crucial for providing common sense explanation. In the modification of the interaction-free measurement (Elitzur and Vaidman 1993) based on the Zeno effect (Kwiat et al. 1995) we get information about the presence of an object without being near it. The probability of success can be made as close to 1 as we wish by changing the parameters. Together with this, there is an increasing number of moments of time at which the particle can be absorbed by the object with decreasing probability of each absorption. In the limit, the sum of the probabilities of absorption at all different moments goes to zero, but without these cases the success of the interaction-free measurement seems to contradict common sense. These are the cases in which there is an interaction. Taking the limit in proving the Born rule is analogous to neglecting these cases.

The main reason why I think that this approach cannot be the solution is that I do not see what is the additional assumption from which we derive the Born rule. Consider a counter example. Instead of the Born rule, the Nature has "Equal rule". Every outcome a_i of a measurement of A has the same probability given that it is possible, i.e., $\alpha_i \neq 0$. Of course, this model contradicts experimental results, but it does not contradict the unitary evolution part of the formalism of quantum mechanics. I do not see how making the number of experiments infinite can rule out Equal rule. Note that an additional assumption ruling out this model is hinted in Aharonov and Reznik (2002) "the results of physical experiments are stable against small perturbations". A very small change of the amplitude can make a finite change in the probability in the proposed model. (This type of continuity assumption is present in some other approaches too.)

26.3 The Born Rule and the Measuring Procedure

The Born rule is intimately connected to the measurement problem of quantum mechanics. Today there is no consensus about its solution. The Schrödinger equation cannot explain definite outcomes of quantum measurements. So, if it does not explain the existence of a (unique) outcome, how can it explain its probability? It is the collapse process (which is not explained by the Schrödinger equation) that provides the unique outcome, so it seems hopeless to look for an explanation of the Born rule based on the Schrödinger equation.

What might be possible are consistency arguments. If we accept the Hibert space structure of quantum mechanics and we accept that there is probability for an outcome of a quantum measurement, what might this probability measure be? Itamar Pitowsky suggested to take it as the basis and showed how Gleason's theorem (Gleason 1957) (which has its own assumptions) leads to the Born rule (Pitowsky 1998). He was aware of "two conceptual assumptions, or perhaps dogmas. The first is J. S. Bell's dictum that the concept of measurement should not be taken as fundamental, but should rather be defined in terms of more basic processes. The second assumption is that the quantum state is a real physical entity, and that denying its reality turns quantum theory into a mere instrument for predictions" (Bub and Pitowsky 2010). In what followed, he recognized the problem as I do: "This last assumption runs very quickly into the measurement problem. Hence, one is forced either to adopt an essentially non-relativistic alternative to quantum mechanics (e.g. Bohm without collapse, GRW with it); or to adopt the baroque many worlds interpretation which has no collapse and assumes that all measurement outcomes are realized." The difference between us is that he viewed "the baroque many worlds interpretation" as unacceptable (Hemmo and Pitowsky 2007), while I learned to live with it (Vaidman 2002).

Maybe more importantly, we disagree about the first dogma. I am not ready to accept "measurement" as a primitive. Physics has to explain all our experiences, from observing results of quantum measurements to observing the color of the sun and the sky at sunset. I do believe in the ontology of the wave function (Vaidman 2016, 2019) and I am looking for a direct correspondence between the wave function and my experience considering quantum observables only as tools for helping to find this correspondence. I avoid attaching ontological meaning to the values of these observables. It does not mean that I cannot discuss the Born rule. The measurement situation is a well defined procedure and our experiences of this procedure (results of measurements) have to be explained.

The basic requirement of the measurement procedure is that if the initial state is an eigenstate of the measured variable, it should provide the corresponding eigenvalue with certainty. Any procedure fulfilling this property is a legitimate measuring procedure. The Born rule states that the probability it provides should be correct for any legitimate procedure, and this is a part of what has to be proved, but let us assume that the fact that all legitimate procedures provide the same probabilities is given. I will construct then a particular measurement procedure (which fulfills the property of probability 1 for eigenstates) which will allow me to explain the probability formula of the Born rule.

Consider a measurement of variable A performed on a system prepared in the state (26.1). The measurement procedure has to include coupling to the measuring device and the amplification part in which the result is written in numerous quantum systems providing a robust record. Until this happens, there is no point in discussing the probability, since outcomes were not created yet. So the measurement process is:

$$|\Psi\rangle \prod_m |r\rangle_m^{MD} \rightarrow \sum_i \alpha_i |a_i\rangle \prod_{m\in S_i} |\checkmark\rangle_m^{MD} \prod_{m\notin S_i} |r\rangle_m^{MD}, \quad {}_m^{MD}\langle r|\checkmark\rangle_m^{MD} = 0 \ \forall m \tag{26.4}$$

where "ready" states $|r\rangle_m^{MD}$ of the numerous parts m of the measuring device, $m \in S_i$, are changed to macroscopically different (and thus orthogonal) states $|\checkmark\rangle_m^{MD}$, in correspondence with the eigenvalue a_i. For all possible outcomes a_i, the set S_i of subsystems of the measuring device which change their states to orthogonal states has to be large enough. This part of the process takes place according to all interpretations. In collapse interpretations, at some stage, the state collapses to one term in the sum.

This schematic description is not too far from reality. Such a situation appears in a Stern-Gerlach experiment in which the atom, the spin component of which is measured, leaves a mark on a screen by exciting numerous atoms of the screen. But I want to consider a modified measurement procedure. Instead of a screen in which the hitting atom excites many atoms, we put arrays of single-atom detectors in the places corresponding to particular outcomes. The arrays cover the areas of the quantum uncertainty of the hitting atom. The arrays are different, as they have a different number N_i of single-atom detectors which we arrange according to equation $N_i = |\alpha_i|^2 N$. (We assume that we know the initial state of the system.)

In the first stage of the modified measuring procedure an entangled state of the atom and sensors of the single-atom detectors is created:

$$|\Psi\rangle \prod_n |r\rangle_n^{sen} \rightarrow \sum_i \frac{\alpha_i}{\sqrt{N_i}} |a_i\rangle \sum_{k_i} |\checkmark\rangle_{k_i}^{sen} \prod_{n\neq k_i} |r\rangle_n^{sen}, \tag{26.5}$$

where $|r\rangle_n^{sen}$ represents an unexcited state of the sensor with label n running over sensors of all arrays of single-photon detectors. N is the total number of detectors. For each eigenvalue a_i there is one array of N_i detectors with sensors in a superposition of entangled states in which one of the sensors k_i has an excited state $|\checkmark\rangle_{k_i}^{sen}$. At this stage the measurement has not yet taken place. The number of sensors with changed quantum state might be large, but no "branches" with many systems in excited states has been created. We need also the amplification process which consists of excitation of a large number of subsystems of individual detectors. In the modified measurement, instead of a multiple recording of an event specified by the detection of a_i, we record activation of every sensor k_i by excitation of a large (not necessarily the same) number of quantum subsystems m belonging to the set S_{i_k}. Including in our description these subsystems, the description of the measurement process is:

$$|\Psi\rangle \prod_n |r\rangle_n^{sen} \prod_m |r\rangle_m^{MD} \rightarrow \frac{1}{\sqrt{N}} \sum_i |a_i\rangle \sum_{k_i} |\checkmark\rangle_{k_i}^{sen} \prod_{n\neq k_i} |r\rangle_n^{sen} \prod_{m\in S_{i_k}} |\checkmark\rangle_m^{MD} \prod_{m\notin S_{i_k}} |r\rangle_m^{MD}. \tag{26.6}$$

Here we also redefined the states $|\checkmark\rangle_{k_i}^{sen}$ to absorb the phase of α_i to see explicitly that all terms in the superposition have the same amplitude. Every term in the superposition has macroscopic number of subsystems of detectors with states which are orthogonal to the states appearing in other terms. This makes all the terms separate. We have N different options. They consist of sets according to all different possible eigenvalues when the set corresponding to eigenvalue a_i has N_i elements. Assuming that all options are equiprobable, we obtain the Born rule. The probability of a reading corresponding to eigenvalue a_i is $p_i = \frac{N_i}{N} = |\alpha_i|^2$. And this procedure is a good measurement according to our basic requirement: If the initial state is an eigenstate, we will know it with certainty.

In Fig. 26.1, we demonstrate such a situation for a modified Stern-Gerlach experiment with the initial state $|\Psi\rangle = \sqrt{0}.4|\uparrow\rangle + \sqrt{0}.6|\downarrow\rangle$. There are $N = 5$ single-photon detectors. The description of the measurement process (represented for a general case by [26.6]) is now:

$$\left(\sqrt{0}.4|\uparrow\rangle + \sqrt{0}.6|\downarrow\rangle\right) \prod_{n=1}^{5} |r\rangle_n^{sen} \prod_m |r\rangle_m^{MD} \rightarrow$$
$$\frac{1}{\sqrt{5}}\left(|\uparrow\rangle|\checkmark\rangle_1^{sen}\prod_1 + |\uparrow\rangle|\checkmark\rangle_2^{sen}\prod_2 + |\downarrow\rangle|\checkmark\rangle_3^{sen}\prod_3 + |\downarrow\rangle|\checkmark\rangle_4^{sen}\prod_4 + |\downarrow\rangle|\checkmark\rangle_5^{sen}\prod_5\right), \tag{26.7}$$

where

$$\prod_i \equiv \prod_{n\neq i} |r\rangle_n^{sen} \prod_{m\in S_i} |\checkmark\rangle_m^{MD} \prod_{m\notin S_i} |r\rangle_m^{MD}.$$

We obtain a superposition of five equal-amplitude states, each corresponding to one detector clicks and others are not. It is natural to accept equal probabilities for clicks of all these detectors and since there are two detectors corresponding to the outcome ‘up’ and three detectors corresponding to the outcome ‘down’ we obtain the Born rule probabilities for our example.

An immediate question is: how can I claim to derive $p_i = |\alpha_i|^2$ when in my procedure I put in by hand $\frac{N_i}{N} = |\alpha_i|^2$? The answer is that making another choice would not lead to a superposition of orthogonal terms with equal amplitudes, so with another choice the derivation does not go through.

This derivation makes a strong assumption that in the experiment, the firing of each sensor has the same probability. It is arranged that all these events correspond to terms in the superposition with the same amplitude, so the assumption is that equal amplitudes correspond to equal probabilities. It is this fact that is considered to be the main part in the derivation of the Born rule. I doubt that the formalism of quantum mechanics by itself is enough to provide a proof for this statement, see also Barrett (2017). In the next section I will try to identify the assumptions added in various proofs of the Born rule.

Without the proof of the connection between amplitudes and probabilities, the analysis of the experiment I presented above is more of an explanation of the Born rule than its derivation. We also use an assumption that all valid measurement

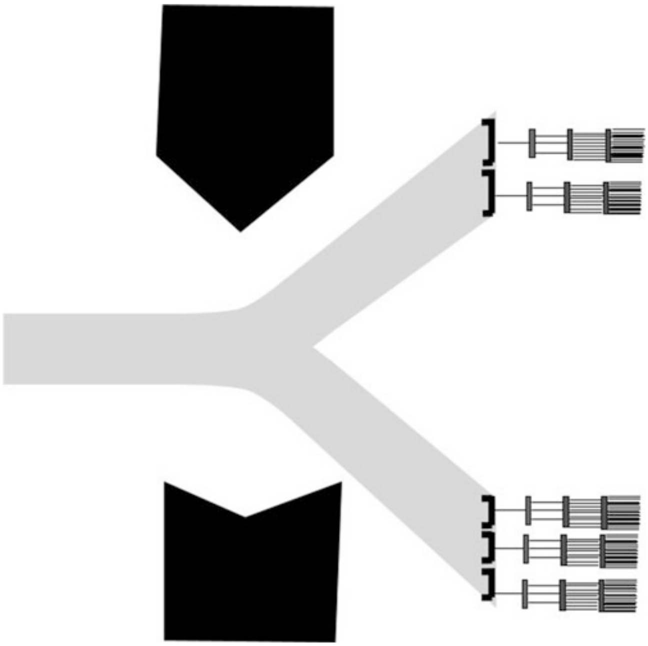

Fig. 26.1 Modified Stern-Gerlach experiment specially tailored for the state $\sqrt{0}.4|\uparrow\rangle + \sqrt{0}.6|\downarrow\rangle$. There are two detectors in the location corresponding to the result 'up' and three detectors in the location corresponding to the result 'down'

experimental procedures provide the same probabilities for outcomes. The modified procedure has a very natural combinatorial counting meaning of probability. It can be applied to the collapse interpretations when we count possible outcomes and in the MWI where we count worlds. The objection that the number of worlds is not a well defined concept (Wallace 2010) is answered when we put weights (measures of existence) on the worlds (Vaidman 1998; Greaves 2004; Groisman et al. 2013).

26.4 Symmetry Arguments

In various derivations of the Born rule, the statement that equal amplitudes lead to equal probabilities relies on symmetry arguments. The starting point is the simplest (sometimes named pivotal) case

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + |a_2\rangle). \tag{26.8}$$

The pioneer in attempting to solve this problem was Deutsch (1999) whose work was followed by extensive development by Wallace (2007), a derivation by Zurek

(2005), and some other attempts such as Sebens and Carroll (2016) and also my contribution with McQueen (Vaidman 2012; McQueen and Vaidman 2018). The key element of these derivations is the symmetry under permutation between $|a_1\rangle$ and $|a_2\rangle$. It is a very controversial topic, with numerous accusations of circularity for some of the proofs (Hemmo and Pitowsky 2007; Barnum et al. 2000; Saunders 2004; Gill 2005; Schlosshauer and Fine 2005; Lewis 2010; Rae 2009; Dawid and Thébault 2014).

In all these approaches there is a tacit assumption (which I also used above) that the probability of an outcome of a measurement of A does not depend on the procedure we use to perform this measurement. Another assumption is that probability depends only on the quantum state. In the Deutsch-Wallace approach some manipulations, swapping, and erasures are performed to eliminate the difference between $|a_1\rangle$ and $|a_2\rangle$, leading to probability half due to symmetry. If the eigenstates do not have internal structure except for being orthogonal states, then symmetry can be established, but it seems to me that these manipulations do not provide the proof for important realistic cases in which the states are different in many respects. It seems that what we need is a proof that all properties, except for amplitudes, are irrelevant. I am not optimistic about the existence of such a proof without adding some assumptions. Indeed, what might rule out the "Equal rule", a naive rule according to which probabilities for all outcomes corresponding to nonzero amplitudes are equal, introduced above?

The assumption of continuity of probabilities as functions of time rules Equal rule out, but this is an additional assumption. The Deutsch-Wallace proof is in the framework of the MWI, i.e. that the physical theory is just unitary evolution which is, of course, continuous, but it is about amplitudes as functions of time. The experience, including the probability of self-location of an observer, supervenes on the quantum state specified by these amplitudes, but the continuity of this supervenience rule is not granted.

Zurek made a new twist in the derivation of the Born rule (Zurek 2005). His key idea is to consider entangled systems and rely on "envariance" symmetry. A unitary evolution of a system which can be undone by the unitary evolution of the system it is entangled with. For the pivotal case, the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle|1\rangle + |a_2\rangle|2\rangle), \tag{26.9}$$

where $|1\rangle$, $|2\rangle$ are orthogonal states of the environment. The unitary swap $|a_1\rangle \leftrightarrow |a_2\rangle$ followed by the unitary swap of the entangled system $|1\rangle \leftrightarrow |2\rangle$ brings us back to the original state which, by assumption, corresponds to the original probabilities. Other Zurek's assumptions are that a manipulation of the second system does not change the probability of the measurement on the system, while the swap of the states of the systems swaps the probabilities for the two outcomes. This proves that the probabilities for the outcomes in the pivotal example must be equal.

In my view, the weak point is the claim that swapping the states of the system swaps the probabilities of the outcomes. This property follows from the quantum formalism when the initial state is an eigenstate, but in our case, when the mechanism for the choice of the outcome is unknown, we also do not know how it is affected by unitary operations. Note, that it is not true for the Stern-Gerlach experiment in the framework of Bohmian mechanics.

Zurek, see also Wallace (2010), Baker (2007), and Boge (2019), emphasises the importance of decoherence: entanglement of environment with eigenstates of the system. Indeed, decoherence is almost always present in quantum measurements and its presence might speed up the moment we can declare that the measurement has been completed, but, as far as I understand, decoherence is neither necessary, nor sufficient for completing a quantum measurement. It is not necessary, because it is not practically possible to perform an interference experiment with a macroscopic detector in macroscopically different states even if it is isolated from the environment. It is not sufficient, because decoherence does not ensure collapse and does not ensure splitting of a world. For a proper measurement, the measuring device must be macroscopic. It is true that an interaction of the system with an environment, instead of a macroscopic measuring device, might lead to a state similar to (26.4) with macroscopic number of microscopic systems of environment "recording" the eigenvalue of the observable. It, however, does not ensure that the measurement happens. It is not clear that macroscopic number of excited microsystem causes a collapse, see analysis of a such situation in the framework of the physical collapse model (Ghirardi et al. 1986) in Albert and Vaidman (1989) and Aicardi et al. (1991). In the framework of the many-worlds interpretation we need splitting of worlds. The moment of splitting does not have a rigorous definition, but a standard definition (Vaidman 2002) is that macroscopic objects must have macroscopically different states. Decoherence might well happen due to a change of states of air molecules which do not represent any macroscopic object.

What I view as the most problematic "symmetry argument proof" of probability half for the pivotal example is the analysis of Sebens and Carroll (2016), see also Kent (2015). Sebens and Caroll considered the measurement in the framework of the MWI and apply the uncertainty of self-location in a particular world as a meaning of probability (Vaidman 1998). However, in my understanding of the example they consider, this uncertainty does not exist (McQueen and Vaidman 2018). In their scenario, a measurement of A on a system in state (26.8) is performed on a remote planet. Sebens and Caroll consider a question: What is the probability of an observer who is here, i.e., far away from the planet, to be in a world with a particular outcome? This question is illegitimate, because he is certainly present in both worlds, there is no uncertainty here. This conclusion is unavoidable in the MWI as I understand it (Vaidman 2002), which is a fully deterministic theory without any room for uncertainty. However, uncertainty in the MWI is considered in Saunders (2004) and Saunders and Wallace (2008), so if this program succeeds (see however Lewis 2007), then the Sebens-Caroll proof might make sense. Another way to make sense of the Sebens-Caroll proof was proposed by Tappenden (2017) based on his unitary interpretation of mind, but I have difficulty accepting this metaphysical picture.

A scenario in which an observer is moved to different locations according to an outcome of a quantum measurement without getting information about this outcome (Vaidman 1998), allows us to consider the probability based on observer's ignorance about self-location and without uncertainty in the theory. This by itself, however, does not prove the probability half for the pivotal case. The proof (McQueen and Vaidman 2018), which is applicable to all interpretations, has two basic assumptions. First, it is assumed that space in Nature has symmetry, so we can construct the pivotal case with symmetry between the states $|a_1\rangle$ and $|a_2\rangle$. We do not rely on permutation of states, we rely on the symmetry of physical space and construct a symmetric state with identical wave packets in remote locations 1 and 2. The second assumption is that everything fulfills the postulate of the theory of special relativity according to which we cannot send signals faster than light. Changing probability by a remote action is sending signals. This proves that changing the shape or even splitting a remote state will not change the probability of finding a_1 provided its amplitude was not changed.

26.5 Other Approaches

Itamar Pitowsky's analysis of the Born rule on the basis of Gleason's theorem (Pitowsky 1998) was taken further to the case of generalized measurements (Caves et al. 2004). Galley and Masanes (2017) continued research which singles out the Born rule from other alternatives. Note that they also used symmetry ("bit symmetry") to single out the Born rule. Together with Muller, they extended their analysis (Masanes et al. 2019) and claimed to prove *everything* just from some "natural" properties of measurements which are primitive elements in their theory. So, people walked very far on the road paved by Itamars's pioneering works. I have to admit that I am not sympathetic to this direction. The authors of Masanes et al. (2019) conclude "Finally, having cleared up unnecessary postulates in the formulation of QM, we find ourselves closer to its core message." For me it seems that they go away from physics. Quantum mechanics was born to explain physical phenomena that classical physics could not. It was not a probability theory. It was not a theory of measurements, and I hope it will not end as such. "Measurements" should not be primitives, they are physical processes as any other, and physics should explain all of them.

Similarly, I cannot make much sense of claims that the Born rule appears even in classical systems presented in the Hilbert space formalism (Brumer and Gong 2006; Deumens 2019). Note that in the quantum domain, the Born rule appears even outside the framework of Hilbert spaces in the work of Saunders (2004), who strongly relies on operational assumptions such as a continuity assumption: "sufficiently small variations in the state-preparation device, and hence of the initial state, should yield small variations in expectation value." This assumption is much more physical than postulates of general probabilistic theories.

The dynamical derivation in the framework of the Bohmian interpretation championed by Valentini (Valentini and Westman 2005; Towler et al. 2011) who argued that under some (not too strong) requirements of complexity, the Born distribution arises similarly to thermal probabilities in ordinary statistical mechanics. See extensive discussion in Callender (2007) and recent analysis in Norsen (2018) which brings also similar ideas from Dürr et al. (1992). The fact that for some initial conditions of some systems relaxation to Born statistics does not happen is a serious weakness of this approach. What I find more to the point as a proof of the Born rule is that the Born statistical distribution remains invariant under time evolution in *all* situations. And that, under some very natural assumptions, is the only distribution with this strong property (Goldstein and Struyve 2007).

Wallace (2010) and Saunders (2010) advocate analyzing the issue of probability in the framework of the consistent histories approach. It provides formal expressions which fit the probability calculus axioms. However, I have difficulty seeing what these expressions might mean. I failed to see any ontological meaning for the main concept of the approach "the probability of a history", and it also has no operational meaning apart from the conditional probability of an actually performed experiment (Aharonov et al. 1964), while the approach is supposed to be general enough to describe evolution of systems which were not measured at the intermediate time.

26.6 Summary of My View

I feel that there is a lot of confusion in the discussions of the subject and it is important to make the picture much more clear. Even if definite answers might not be available now, the question: What are the open problems? can be clarified. First, it is important to specify the framework: collapse theory, hidden variables approach or noncollapse theory. Although in many cases the "derivation of the Born rule" uses similar structure and arguments in all frameworks, the conceptual task is very different. I believe that in all frameworks there is no way to prove the Born rule from other axioms of standard quantum mechanics. The correctly posed question is: What are the additional assumptions needed to derive the Born rule?

Standard quantum mechanics tells us that the evolution is unitary, until it leads to a superposition of quantum states corresponding to macroscopically different classical pictures. There is no precise definition of "macroscopically different classical pictures" and this is a very important part of the measurement problem, but discussions of the Born rule assume that this ambiguity is somehow solved, or proven irrelevant. The discussions analyze a quantitative property of the nonunitary process which happens when we reach this stage assuming that the fact that it happens is given. I see no possibility to derive the quantitative law of this nonunitary process from laws of unitary evolution. It is usually assumed that the process depends solely on the quantum state, i.e. that the probability of an outcome of a measurement of an observable does not depend on some hidden variables and does not depend on the way the observable is measured. The process also should not alter the unitary evolution when a superposition of states corresponding to macroscopically different classical pictures was not created. This, however, is not enough to rule out proposals different from the Born rule, e.g., Equal rule described above. We have to add something to derive the Born rule. We are not supposed to rely on experimental results, they do single out the Born rule, but this is not a "derivation". Instead, if we take some features of observed results as the basis, it is considered as a derivation. I am not sure that it is really better, unless these features are considered not as properties of Nature, but as a basic reason for Nature to be as it is. Then the Born rule derivations become a part of the program to get quantum mechanics from simple axioms (Popescu and Rohrlich 1994; Hardy 2001; Chiribella et al. 2011). In these derivations, quantum mechanics is usually considered as a general probability theory and the main task is to derive the Born rule.

In McQueen and Vaidman (2018) the program is more modest. Unitary quantum mechanics is assumed and two physical postulates are added. First, that there are symmetries in space and second that there is no superluminal signalling. The first principle allows us to construct a pivotal example described by (26.8) in which there is symmetry between states $|a_1\rangle$ and $|a_2\rangle$. The second principle allows us to change one of the eigenstates in the pivotal state without changing the probability to find the other eigenvalue. This is the beginning of the procedure, first shown by Deutsch in (1999), who pioneered these types of derivations.

The situation in the framework of the MWI is conceptually different. The physical essence of the MWI is: unitary evolution of a quantum state of the universe is all that there is. There is no additional process of collapse behaviour which should be postulated. So it seems that here there is no room for additional assumptions and that the Born rule must be derived just from the unitary evolution.

However, the MWI has a problem with probability even before we discuss the quantitative formula of the Born rule. The standard approach to the probability of an event requires that there to be a matter of fact about whether this event and not the other takes place, but in the MWI all events take place. On the other hand, we do have experience of one particular outcome when we perform a measurement. My resolution of this problem (Vaidman 1998) is that indeed, there is no way to ask what is the probability of what will happen, because all outcomes will be actual. The "probability" rule is still needed to explain statistics of observed measurements in the past. There are worlds with all possible statistics, but we happen to observe Born rule statistics. The "probability" explaining these statistics is the probability of self-location in a particular world. In Vaidman (1998) I constructed a scenario with quantum measurements in which the observer is split (and together with him, his world) according to the outcome of the measurement without being aware of the result of the measurement. This provides the ignorance probability of the observer about the world specified by the outcome of the measurement he is a part of. Tappenden (2010) argues that merely considering such a construction allows us to discuss the Born rule. These are supporting arguments of the solution: there is no probabilistic process in Nature: with certainty all possible outcomes of a quantum measurement will be realized, but an observer, living by definition in one of the worlds, can consider the question of probability of being located in a particular world.

All that is in Nature, according to MWI, is a unitary evolving quantum state of the universe and observers correspond to parts of this wave function (Vaidman 2016, 2019). So, there is a hope that the experience of observers, after constructing the theory of observers (chemistry, biology, psychology, decision theory, etc.) can be, in principle, explained solely from the evolution of the quantum state. Apparently, experiences of an observer can be learned from his behavior which is described by the evolution of the wave function. Then, it seems that the Born rule should be derivable from the laws of quantum mechanics. However, I believe that this is not true.

Consider Alice and Bob at separate locations and they have a particle in a state (26.8) where $|a_1\rangle$ corresponds to a particle being at Alice's site and $|a_2\rangle$ corresponds to a particle being at Bob's site. Now assume that instead of the Born rule, which states that the probability of self-location in a world is proportional to the square of the amplitude, Nature has the Equal rule which yields the same probability of self location in all the worlds, i.e. probability $\frac{1}{N}$, where N is the number of worlds. Equal rule allows superluminal signaling. Alice and Bob agree that at a particular time t Alice measures the presence of the particle at her site, i.e. she measures the projection on state $|a_1\rangle$. To send bit 0, Bob does nothing. Alice's measurement splits the world into two worlds: the one in which she finds the particle and the other, in which she does not. Then she has equal probability to find herself in each of the worlds, so she has probability half to find herself in the world in which she finds the particle. For sending bit 1, just before time t, Bob performs a unitary operation on the part of the wave at his site splitting it to a hundred orthogonal states

$$|a_2\rangle \rightarrow \sum_{k=1}^{100} |b_k\rangle, \tag{26.10}$$

and immediately measures operator B which tells him the eigenvalue b_k. This operation splits the initial single world with the particle in a superposition into hundred and one worlds: hundred worlds with one of Bob's detectors finding the particle, and one world in which the particle was not found by Bob's detectors. Prior to her measurement, Alice is present in all these worlds. Her measurement tests if she is in one particular world, so she has only probability of $\frac{1}{101}$ to find the particle at her site.

Bob's unitary operation and measurement change the probability of Alice's outcome. With measurements on a single particle, the communication is not very reliable, but using an ensemble will lead to only very rare cases of an error. The Equal rule will ensure that Alice and Bob meeting in the future will (most probably) verify correctness of the bit Bob has sent.

We know that unitary evolution does not allow superluminal communication. (When we consider a relativistic generalisation of the Schrödinger equation.) Can, a supertechnology, capable of observing superposition of Alice's and Bob's worlds, given that the actual probability rule of self-location is the Equal rule (and not the Born rule), use the above procedure for sending superluminal signals? No! Only Alice and Bob, inside their worlds, have the ability of superluminal communication. It does not contradict relativistic properties of physics describing unitary evolution of all worlds together.

What I argue here, is that the situation in the framework of MWI is not different from collapse theories. There is a need for an independent probability postulate. In collapse theory it is a physical postulate telling us about the dynamics of the ontology, dynamics of the quantum state of the universe describing the (single) world. In MWI, the postulate belongs to the part connecting observer's experiences with the ontology. In the MWI, as in a collapse theory, the experiences supervene on the ontology, the quantum state. The supervenience rule is the same when the quantum state corresponds to a single world, but it has an additional part regarding the probability of self-location, when the quantum state of the universe corresponds to more than world. The postulate describes this supervenience rule.

We can justify the Born rule postulate of self-location by experimental evidence, or by requiring the relativistic constraint of superluminal signaling also within worlds. I find a convincing explanation in the concept of the measure of existence of a world (Vaidman 1998; Groisman et al. 2013). While there is no reason to postulate that the probability of self-location in every world is the same, it is natural to postulate that the probability of self-location in worlds of equal existence (equal square of the amplitude) is the same. Adding another natural assumption that probability of self-location in a particular world should be equal to the sum of the probability of self-location in all the worlds which split from the original one, provides the Born rule.

My main conclusion is that there is no way to derive the Born rule without additional assumptions. It is true both in the framework of collapse theories and, more surprisingly, in the framework of the MWI. The main open question is not the validity of various proofs, but what are the most natural assumptions we should add for proving the Born rule.

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