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The Two-State Vector Formalism: An Updated Review

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13.1 Introduction

The two-state vector formalism of quantum mechanics is a time-symmetrized approach to standard quantum theory particularly helpful for the analysis of experiments performed on pre- and post-selected ensembles. It allows to see numerous peculiar effects which naturally arise in this approach. In particular, the concepts of "weak measurements" (standard measurements with weakening of the interaction) and "weak values" (the outcomes of weak measurements) reveal a very unusual but consistent picture. Recently, more and more effects are viewed as manifestations of weak measurements and more and more weak measurement experiments have been performed. The polemic about the validity of the approach and the meaning of its concepts never stopped. The number of papers written on the subject almost doubled since publication of the first version of the review. The current review does not explain in details the new results, but it puts the development of the approach in the proper context and provides citations for further reading.

13.2 Descriptions of Quantum Systems

13.2.1 The Quantum State

In the standard quantum mechanics, a system at a given time t is described completely by a quantum state

$$|\Psi\rangle\,, \tag{13.1}$$

defined by the results of measurements performed on the system in the past relative to the time t. (It might be that the system at time t is not described

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by a pure quantum state, but by a mixed state (density matrix). However, we can always assume that there is a composite system including this system which is in a pure state.) The status of a quantum state is controversial: there are many papers on reality of a quantum state and numerous interpretations of this "reality." However, it is noncontroversial to say that the quantum state yields maximal information about how this system can affect other systems (in particular, measuring devices) interacting with it at time t. Of course, the results of all measurements in the past, or just the results of the last complete measurement, also have this information, but these results include other facts too, so the quantum state is the most concise information about how the quantum system can affect other systems at time t.

The concept of a quantum state is time-asymmetric: it is defined by the results of measurements in the *past*. This fact by itself is not enough for the asymmetry: in classical physics, the state of a system at time t defined by the results of the complete set of measurements in the past is not different from the state defined by the complete measurements in the future. This is because for a classical system the results of measurements in the future are defined by the results of measurements in the past (and vice versa). In quantum mechanics this is not so: the results of measurements in the future are only partially constrained by the results of measurements in the past. Thus, the concept of a quantum state is genuinely time-asymmetric. The question arises: does the asymmetry of a quantum state reflects the time asymmetry of quantum mechanics, or it can be removed by reformulation of quantum mechanics in a time-symmetric manner?

13.2.2 The Two-state Vector

The two-state vector formalism of quantum mechanics (TSVF) originated in a seminal work of Aharonov, Bergmann, and Lebowitz (ABL) [1] removes this asymmetry. It provides a time-symmetric formulation of quantum mechanics. A system at a given time t is described completely by a *two-state vector*

$$\langle \Phi | \ | \Psi \rangle \ , \tag{13.2}$$

which consists of a quantum state $|\Psi\rangle$ defined by the results of measurements performed on the system in the past relative to the time t and of a backward evolving quantum state $\langle \Phi|$ defined by the results of measurements performed on this system after the time t. Again, the status of the two-state vector might be interpreted in different ways, but a noncontroversial fact is that it yields maximal information about how this system can affect other systems (in particular, measuring devices) interacting with it at time t.

The description of the system with the two-state vector (13.2) is clearly different from the description with a single quantum state (13.1), but in both cases we claim that "it yields maximal information about how this system can affect other systems (in particular, measuring devices) interacting with it

at time t." Does it mean that the TSVF has different predictions than the standard quantum approach? No, the two formalisms describe the same theory with the same predictions. The difference is that the standard approach is time asymmetric and it is assumed that only the results of the measurements in the past exist. With this constraint, $|\Psi\rangle$ indeed contains maximal information about the system at time t. The rational for this approach is that if the results of the future measurements relative to the time t exist too, then "now" is after time t and we cannot return back in time to perform measurements at t. Therefore, taking into account results of future measurements is not useful. In contrast, the TSVF approach is time symmetric. There is no preference to the results of measurements in the past relative to the results of measurements in the future: both are taken into account. Then, there is more information about the system at time t. The maximal information (without constraints) is contained in the two-state vector $\langle\Phi|\ |\Psi\rangle$.

If the TSVF has the same predictions as standard quantum mechanics, what is the reason to consider it? And what about the argument that when the results of future measurements are known it is already too late to make measurements at time t? How might the two-state vector be useful? The answer to the first question is that it is important to understand the time symmetry of nature (described by quantum mechanics). The time asymmetry of the standard approach might be solely due to the usage of time-asymmetric concepts. The answer to the second question is that there are many situations in which we want to know how a system affected other systems in the past. The TSVF proved to be particularly useful after introduction of *weak measurements* [2, 3, 4] which allowed to see that systems described by some two-state vectors can affect other system at time t in a very peculiar way. This has led to the discovery of numerous bizarre effects [5, 6, 7, 8]. It is very difficult to understand these effects in the framework of standard quantum mechanics; some of them can be explained via a miraculous interference phenomenon known as *super-oscillations* [9, 10].

13.2.3 How to Create Quantum Systems Corresponding to Various Complete Descriptions?

The maximal complete description of a quantum system at time t is a two-state vector (13.2). We will name the system which has such a description as *pre- and postselected*. (Again, it might be that at time t the system is not described by a "pure" two-state vector. However, we can assume that there is a composite system including this system which is described by a two-state vector.) In some circumstances, the system might have only a partial description. For example, if time t is "present" and the results of the future measurements do not exist yet, then at that time, the system is described only by a usual forward evolving quantum state (13.1): the *preselected* system. Later, when the results of the future measurements will be obtained, the

description will be completed to the form (13.2). It is also possible to arrange a situation in which, until some measurements in the future, the complete description of the system at time t is the backward evolving quantum state $\langle\Phi|$: the *postselected* system. We will now explain how all these situations can be achieved.

Single Forward-Evolving Quantum State

In order to have *now* a system the complete description of which at time t is a single quantum state (13.1), there should be a complete measurement in the past of time t and no measurement on the system after time t, see Fig. 13.1 a. The system in the state $|\Psi\rangle$ is obtained when a measurement of an observable A at time t_1 is performed, $t_1 < t$, obtaining a specific outcome $A = a$ such that the created state $|a\rangle$ performs unitary evolution between t_1 and t governed by the Hamiltonian H,

$$U(t_1, t) \;=\; e^{-i\int_{t_1}^{t} H dt}\;, \tag{13.3}$$

to the desired state:

$$|\Psi\rangle \;=\; U(t_1, t)\;|a\rangle\;. \tag{13.4}$$

The time "now," t_{now} should either be equal to the time t, or it should be known that during the time period $[t, t_{now}]$ no measurements have been performed on the system. The state $|\Psi\rangle$ remains to be the complete description of the system at time t until the future measurements on the system will be performed yielding additional information.

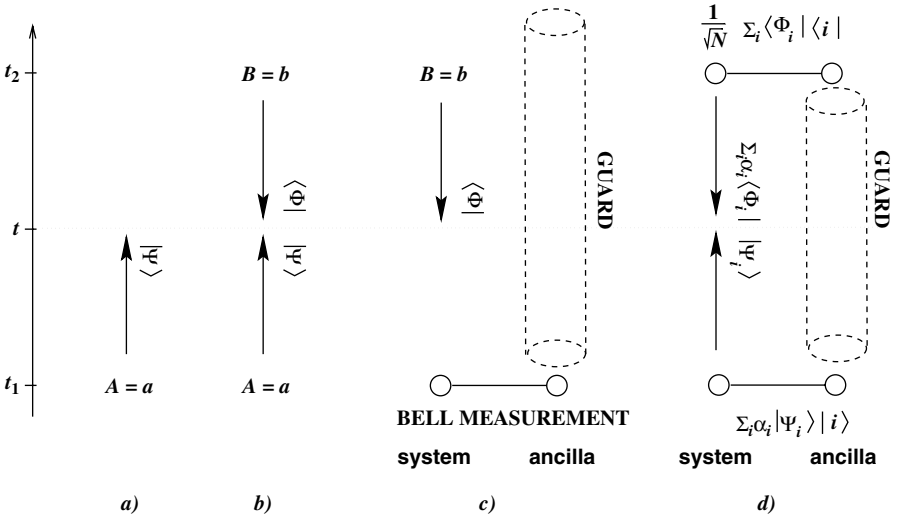

Fig. 13.1. Description of quantum systems: (**a**) pre-selected, (**b**) pre- and postselected, (**c**) postselected, and (**d**) generalized pre- and postselected

The Two-State Vector

In order to have *now* a system the complete description of which at time t is a two-state vector (13.2), there should be a complete measurement in the past of time t and a complete measurement after the time t, see Fig. 13.1b. In addition to the measurement $A = a$ at time t_1, there should be a complete measurement at t_2, $t_2 > t$, obtaining a specific outcome $B = b$ such that the backward time evolution from t_2 to t leads to the desired state

$$\langle \Phi| \;=\; \langle b|\, U^{\dagger}(t, t_2)\,. \tag{13.5}$$

The time "now", t_{now} is clearly larger than t_2. The two-state vector $\langle \Phi|\,|\Psi\rangle$ is the complete description of the system at time t starting from the time t_2 and forever.

A Single Backward-Evolving Quantum State

We have presented above a description of quantum systems by a single forward-evolving quantum state (13.1) and by a two-state vector (13.2). It is natural to ask: Are there systems described by a single backward-evolving quantum state? The notation for such a state is

$$\langle \Phi|\,. \tag{13.6}$$

A measurement of B at time t_2, even in the case it yields the desired outcome $B = b$, is not enough. The difference between preparation of (13.1) and (13.2) is that at present, t, the future of a quantum system does not exist (the future measurements have not been performed yet), but the past of a quantum system exists: it seems that even if *we* do not know it, there is a quantum state of the system evolving towards the future defined by the results of measurements in the past. Therefore, in order to obtain a quantum system described by a backward-evolving quantum state (13.2), in addition to the postselection measurement performed after time t, we have to *erase* the past.

How to erase the past of a quantum system? A complete measurement before the time t certainly partially erases the information which the system had before the measurement, but it also creates the new information: the result of this measurement. It creates another quantum state evolving forward in time, and this is, really, what we need to erase. We have to achieve the situation in which no information arrives from the past. It seems impossible given the assumption that all the past is known. However, if we perform a measurement on a composite system containing our system and an auxiliary system, *an ancilla*, then it can be done, see Fig. 13.1c. Performing a Bell-type measurement results in one of a completely correlated states of the system and the ancilla (the Einstein-Podolsky-Rosen (EPR)-type state). In such a state, each system has equal probability to be found in any state. However, the measurement on one system fixes the state of the other, so, in addition to the Bell-type measurement we need to "guard" the ancilla such that no measurement could be performed on it until now. Again, the complete description

of a quantum system by a single (this time backward-evolving) quantum state can be achieved only for a period of time until the measurements on the ancilla would fix the forward-evolving quantum state for the system.

The backward evolving state is a premise not only of the two-state vector formalism, but also of "retrodictive" quantum mechanics [11, 12, 13, 14, 15], which deals with the analysis of quantum systems based on a quantum measurement performed in the future relative to the time in question. It is also relevant to "consistent histories" and "decoherent histories" approaches [16, 17].

13.2.4 The Generalized Two-State Vector

The descriptions we described above correspond to an "ideal" case. We have assumed that complete measurements have been performed on the system in the past, or in the future or both. The philosophical question is this: can we assume that going sufficiently far away to the past, far away to the future and far in the sense of considering composite systems larger and larger, at the end there always be a complete description in the form of a two-state vector. Usually we do put constraints on how far we go (at least regarding the future and the size of the system). In constructing the situation in which a system is described by a backward-evolving quantum state only, we already limited ourselves to a particular system instead of being satisfied by the correct claim that our system is a part of a composite system (which includes also the ancilla) which does have forward-evolving quantum state. As in the standard approach, limiting our analysis to a particular system leads to descriptions with *mixed* states. There are situations in which the forward-evolving state is a mixed state (the system is correlated to an ancilla) and the backward-evolving state is another mixed state (the system correlated to another ancilla). Although the generalization to the mixed states is straightforward, it is not obvious what is its most convenient form. For a powerful, but somewhat cumbersome formalism, see [18]. However, there is a particular case which is not too difficult to describe. It corresponds to another "pure" two-state vector description: *generalized two-state vector*.

Generalized two-state vector [4] is the name for the superposition of two-state vectors

$$\sum_i \alpha_i \langle \Phi_i| \ |\Psi_i\rangle \ . \tag{13.7}$$

In general, the sets $\{|\Psi_i\rangle\}$, $\{\langle\Phi_i|\}$ need not be orthogonal. Then, the normalization should be chosen consistently, although it is not very important since in main applications of this concept the normalization does not affect anything.

For simplicity, we will consider the case of zero free Hamiltonian for the system and for the ancilla. In order to obtain the generalized two-state vector (13.7) we have to prepare at t_1 the system and the ancilla in a correlated state $\sum_i \alpha_i |\Psi_i\rangle|i\rangle$, where $\{|i\rangle\}$ is a set of orthonormal states of the ancilla. Then we have to "guard" the ancilla such that there will be no measurements or any

other interactions performed on the ancilla until the postselection measurement of a projection on the correlated state $1/\sqrt{N}\sum_i |\Phi_i\rangle|i\rangle$, see Fig. 13.1d. If we obtain the desired outcome, then the system is described at time t by the generalized two-state vector (13.7).

13.3 Ideal Quantum Measurements

13.3.1 Von Neumann Measurements

In this section I shall discuss how a quantum system characterized by a certain description interacts with other systems. Some particular types of interactions are named *measurements* and the effect of these interactions characterized as the results of these measurements. The basic concept is an *ideal quantum measurement* of an observable C. This operation is defined for preselected quantum systems in the following way:

> If the state of a quantum system before the measurement was an eigenstate of C with an eigenvalue c_n then the outcome of the measurement is c_n and the quantum state of the system is not changed.

The standard implementation of the ideal quantum measurement is modeled by the von Neumann Hamiltonian [19]:

$$H = g(t)PC \ , \tag{13.8}$$

where P is the momentum conjugate to the pointer variable Q, and the normalized coupling function $g(t)$ specifies the time of the measurement interaction. The outcome of the measurement is the shift of the pointer variable during the interaction. In an ideal measurement the function $g(t)$ is nonzero only during a very short period of time, and the free Hamiltonian during this period of time can be neglected.

13.3.2 The Aharonov–Bergmann–Lebowitz Rule

For a quantum system described by the two-state vector (13.2), the probability for an outcome c_n of an ideal measurement of an observable C is given by [1, 4]

$$\mathrm{Prob}(c_n) = \frac{|\langle\Phi|\mathbf{P}_{C=c_n}|\Psi\rangle|^2}{\sum_j |\langle\Phi|\mathbf{P}_{C=c_j}|\Psi\rangle|^2} \ . \tag{13.9}$$

For a quantum system described by a *generalized two-state vector* (13.7) the probability for an outcome c_n is given by [4]

$$\mathrm{Prob}(c_n) = \frac{|\sum_i \alpha_i\langle\Phi_i|\mathbf{P}_{C=c_n}|\Psi_i\rangle|^2}{\sum_j |\sum_i \alpha_i\langle\Phi_i|\mathbf{P}_{C=c_j}|\Psi_i\rangle|^2} \ . \tag{13.10}$$

Another important generalization of the formula (13.9) is for the case in which the postselection measurement is not complete and therefore it does not specify a single postselection state $\langle\Phi|$. Such an example was recently considered by Cohen [20] in an (unsuccessful [21]) attempt to find constraints to the applicability of the ABL formula. In this case, the postselection measurement is a projection on a *degenerate* eigenvalue of an observable $B = b$. The modified ABL formula is [21]:

$$\mathrm{Prob}(c_n) = \frac{\|\mathbf{P}_{B=b}\mathbf{P}_{C=c_n}|\Psi\rangle\|^2}{\sum_j \|\mathbf{P}_{B=b}\mathbf{P}_{C=c_j}|\Psi\rangle\|^2} \ . \tag{13.11}$$

This form of the ABL formula allows to connect it to the standard formalism of quantum theory in which there is no post-selection. In the limiting case when the projection operator $\mathbf{P}_{B=b}$ is just the unity operator $\mathbf{I}$, we obtain the usual expression:

$$\mathrm{Prob}(c_n) = ||\mathbf{P}_{C=c_n}|\Psi\rangle||^2 \ . \tag{13.12}$$

13.3.3 Three-Boxes Example

Consider a particle which can be located in one out of three boxes. We denote the state of the particle when it is in box i by $|i\rangle$. At time t_1 the particle is prepared in the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \ . \tag{13.13}$$

At time t_2 the particle is found to be in the state

$$|\Phi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle) \ . \tag{13.14}$$

We assume that in the time interval $[t_1, t_2]$ the Hamiltonian is zero. Therefore, at time t, $t_1 < t < t_2$, the particle is described by the two-state vector

$$\langle\Phi|\ |\Psi\rangle = \frac{1}{3}(\langle 1| + \langle 2| - \langle 3|)\ (|1\rangle + |2\rangle + |3\rangle) \ . \tag{13.15}$$

Probably the most peculiar fact about this single particle is that it can be found with certainty in two boxes [4]. Indeed, if at time t we open box 1, we are certain to find the particle in box 1; and if we open box 2 instead, we are certain to find the particle in box 2. These results can be obtained by straightforward application of the ABL formula (13.9). Opening box i corresponds to measuring the projection operator $\mathbf{P}_i = |i\rangle\langle i|$. The corresponding operators appearing in (13.9) are

$$\mathbf{P}_{\mathbf{P}_i=1} = |i\rangle\langle i|, \qquad \mathbf{P}_{\mathbf{P}_i=0} = \sum_{j\neq i} |j\rangle\langle j| \tag{13.16}$$

Therefore, the calculation of the probability to find the particle in box 1 yields:

$$\mathrm{Prob}(\mathbf{P}_1 = 1) = \frac{|\langle\Phi|1\rangle\langle 1|\Psi\rangle|^2}{|\langle\Phi|1\rangle\langle 1|\Psi\rangle|^2 + |\langle\Phi|2\rangle\langle 2|\Psi\rangle + \langle\Phi|3\rangle\langle 3|\Psi\rangle|^2} = \frac{|\frac{1}{3}|^2}{|\frac{1}{3}|^2 + |0|^2} = 1 \ . \tag{13.17}$$

Similarly, we obtain $\mathrm{Prob}(\mathbf{P}_2 = 1) = 1$. Note, that if we open both box 1 and box 2, we might not see the particle at all.

This example can be generalized to the case of a large number of boxes N. A single particle described by a two-state vector

$$\frac{1}{N}(\langle 1| + \langle 2| + ... - \sqrt{N-2}\langle N|) \ (|1\rangle + |2\rangle + ... + \sqrt{N-2}|N\rangle) \ . \tag{13.18}$$

This single particle is, in some sense, simultaneously in $N-1$ boxes: whatever box is opened (except the last one) we are certain to find the particle there.

Recently, we found that the particle is simultaneously in several boxes even in a more robust sense [22]. We cannot find it simultaneously in all boxes if we look at all of them, but a single photon can! We found that a photon will scatter from our pre- and postselected particle, as if there were particles in all boxes.

The analysis of the three-boxes example has interesting features also in the framework of the consistent histories approach [23, 24, 25]. On the other hand, it generated significant controversies. The legitimacy of counterfactual statements were contested, see discussion in Sect 5.4, the Kastner criticism [26] and Vaidman's reply [27], and it was claimed by Kirkpatrick [28] that the three-boxes example does not exhibit genuine quantum paradoxical feature because it has a classical counterpart. Very recently Ravon and Vaidman [29] showed that Kirkpatrick's proposal fails to mimic quantum behavior and that the three-box example is one of not too many classical tasks which can be done better using quantum tools. (We could not see a refutation of this statement in Kirkpatrick's reply [30].) This is the paradoxical feature of the three-box experiment which was overlooked by Leavens et al. [31] who considered variations of the three-box experiment with modified pre- and postselected states.

Recently, a setup equivalent to the three-box example was presented as a novel *counterfactual computation* method [32]. The analysis of this proposal in the framework of the two-state vector formalism [33] shows that one cannot claim that the computer yields the result of computation without actually performing the computation and therefore, the proposal fails to provide counterfactual computation for all possible outcomes as it was originally claimed.

13.3.4 The Failure of the Product Rule

An important difference between pre- and postselected systems and preselected systems only is that the *product rule* does not hold [34]. The product rule, which does hold for preselected quantum systems is that if $A = a$ and

$B = b$ with certainty, then it is certain that $AB = ab$. In the three-boxes case we know with certainty that $\mathbf{P}_1 = 1$, $\mathbf{P}_2 = 1$. However, $\mathbf{P}_1\mathbf{P}_2 = 0$.

Another example of this kind in a which measurement in one place affects the outcome of a measurement in another place is a pre- and postselected pair of separate spin-$\frac{1}{2}$ particles [35]. The particles are prepared, at time t_1, in a singlet state. At time t_2 measurements of σ_{1x} and σ_{2y} are performed and certain results are obtained, say $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$, i.e., the pair is described at time t, $t_1 < t < t_2$, by the two-state vector

$$\frac{1}{\sqrt{2}} \langle\uparrow_x | \langle\uparrow_y | (|\uparrow_z\rangle|\downarrow_z\rangle - |\downarrow_z\rangle|\uparrow_z\rangle) . \tag{13.19}$$

If at time t a measurement of σ_{1y} is performed (and if this is the only measurement performed between t_1 and t_2), then the outcome of the measurement is known with certainty: $\sigma_{1y}(t) = -1$. If, instead, only a measurement of σ_{2x} is performed at time t, the result of the measurement is also certain: $\sigma_{2x}(t) = -1$. The operators σ_{1y} and σ_{2x} obviously commute, but nevertheless, measuring $\sigma_{2x}(t)$ clearly disturbs the outcome of the measurement of $\sigma_{1y}(t)$: it is not certain anymore.

Measuring the product $\sigma_{1y}\sigma_{2x}$, is, in principle, different from the measurement of both σ_{1y} and σ_{2x} separately. In our example, the outcome of the measurement of the product *is* certain, the ABL formula (13.9) yields $\sigma_{1y}\sigma_{2x} = -1$. Nevertheless, it does not equal the product of the results which must come out of the measurements of σ_{1y} and σ_{2x} when every one of them is performed without the other.

Note measurability of the product $\sigma_{1y}\sigma_{2x}$ using only local interactions. Indeed, we may write the product as a modular sum, $\sigma_{1y}\sigma_{2x} = (\sigma_{1y} + \sigma_{2x})\mathrm{mod}4 - 1$. It has been shown [36] that nonlocal operators such as $(\sigma_{1y} + \sigma_{2x})\mathrm{mod}4$ can be measured using solely local interactions.

Hardy [37] analyzed another very spectacular example in which an electron and a positron are found with certainty if searched for in a particular place, but, nevertheless, if both are searched simultaneously, there is certainty *not* to find them together. Again, the failure of the product rule explains this counterintuitive situation and the far reaching conclusions of Hardy's paper seem not to be warranted [34].

The two spin-$\frac{1}{2}$ particles example with a small modification of omitting the measurement at time t_2 performed on a second particle, but instead, "guarding" it starting from time t_1 against any measurement, is a demonstration of obtaining a quantum system described only by a backward-evolving quantum state $\langle\uparrow_x |$. The probability distribution for outcomes of spin-component measurements performed at time t is identical to that of a particle in a preselected state $|\uparrow_x\rangle$. Note that for quantum systems which are postselected only, the product rule does hold.

Recently [38] it has been shown that pre- and postselection allows another related peculiar feature: "a posteriori" realization of super-correlations

maximally violating the CHSH bound, which have been termed as Popescu–Rohrlich boxes [39].

13.3.5 Ideal Measurements Performed on a System Described by Generalized Two-State Vector

Another modification, replacing the measurements at t_2 on two particles by measurement of a nonlocal variable such as a Bell operator on both particles and guarding the second particle between t_1 and t_2 produces a *generalized two-state vector* for the first particle. Such particles might have a peculiar feature that the outcome of spin component measurements is certain in a continuum of directions. This is a surprising result because the preselected particle might have definite value of spin component at most in one direction and the particle described by two-state vector will have definite results of spin component measurements in two directions: one defined by preselection and one defined by postselection (the directions might coincide). For example [4], the particle described by a generalized two-state vector

$$\cos\chi\langle\uparrow_z|\,|\uparrow_z\rangle - \sin\chi\langle\downarrow_z|\,|\downarrow_z\rangle\,, \qquad \chi\in\left(0,\frac{\pi}{2}\right)\,, \tag{13.20}$$

will yield the outcome $\sigma_\eta = 1$ for the cone of directions $\hat{\eta}$ making angle θ with the z axes such that $\theta = 4\arctan\sqrt{\tan\chi}$. This can be verified directly using the formula (13.10), but we will bring another argument for this result below.

The generalized two-state vector is obtained when there is a particular result of the nonlocal measurement at time t_2. It is interesting that we can construct a particular measurement at time t_2 such that whatever the outcome will be there will be a cone of directions in which the spin has a definite value. These cones intersect in general in four lines. It can be arranged that they will "touch" on, say x-axis and intersect in y- and z-axes. Then, in all cases we will be able to ascertain the value of σ_x, σ_y, and σ_z of a single particle [5].

The problem was also analyzed in the framework of the standard approach [40, 41] and after coining the name "The Mean King Problem" continued to be a topic of an extensive analysis. It has been generalized to the spin-1 particle [42] and to a higher dimentional case [43, 44]. The research continues until today [45, 46, 47, 48, 49, 50]. Moreover, today's technology converted from gedanken quantum game to a real experiment. Schulz et al. [51] performed this experiment with polarized photons (instead of spin-$\frac{1}{2}$ particles).

13.4 Weak Measurements

13.4.1 Introduction

The most interesting phenomena which can be seen in the framework of the TSVF are related to *weak measurements* [3]. A weak measurement is a standard measuring procedure (described by the Hamiltonian (13.8)) with weakened coupling. In an ideal measurement, the initial position of the pointer

Q is well localized around zero and therefore the conjugate momentum P has a very large uncertainty which leads to a very large uncertainty in the Hamiltonian of the measurement (13.8). In a weak measurement, the initial state of the measuring device is such that P is localized around zero with small uncertainty. This leads, of course, to a large uncertainty in Q and therefore the measurement becomes imprecise. However, by performing the weak measurement on an ensemble of N identical systems we improve the precision by a factor $\sqrt{N}$ and in some special cases we can obtain good precision even in a measurement performed on a single system [2].

The idea of weak measurements is to make the coupling with the measuring device sufficiently weak so that the change of the quantum state due to the measurements can be neglected. In fact, we require that the two-state vector is not significantly disturbed, i.e., neither the usual, forward-evolving quantum state, nor the backward-evolving quantum state is changed significantly. Then, the outcome of the measurement should be affected by both states. Indeed, the outcome of a weak measurement of a variable C performed on a system described by the two-state vector $\langle\Phi|\ |\Psi\rangle$ is the *weak value* of C:

$$C_w \equiv \frac{\langle\Phi|C|\Psi\rangle}{\langle\Phi|\Psi\rangle}\ . \tag{13.21}$$

Strictly speaking, the readings of the pointer of the measuring device will cluster around $\mathrm{Re}(C_w)$. In order to find $\mathrm{Im}(C_w)$ one should measure the shift in P [3].

The weak value for a system described by a generalized two-state vector (13.7) is [4]:

$$C_w = \frac{\sum_i \alpha_i \langle\Phi_i|C|\Psi_i\rangle}{\sum_i \alpha_i \langle\Phi_i|\Psi_i\rangle}\ . \tag{13.22}$$

Next, let us give the expression for the weak value when the postselection measurement is not complete. Consider a system preselected in the state $|\Psi\rangle$ and postselected by the measurement of a degenerate eigenvalue b of a variable B. The weak value of C in this case is:

$$C_w = \frac{\langle\Psi|\mathbf{P}_{B=b}C|\Psi\rangle}{\langle\Psi|\mathbf{P}_{B=b}|\Psi\rangle}\ . \tag{13.23}$$

This formula allows us to find the outcome of a weak measurement performed on a preselected (only) system. Replacing $\mathbf{P}_{B=b}$ by the unity operator yields the result that the weak value of a preselected system in the state $|\Psi\rangle$ is the expectation value:

$$C_w = \langle\Psi|C|\Psi\rangle\ . \tag{13.24}$$

Let us show how the weak values emerge as the outcomes of weak measurements. We will limit ourselves to two cases: first, the weak value of the preselected state only (13.24) and then, the weak value of the system described by the two-state vector (13.21).

In the weak measurement, as in the standard von Neumann measurement, the Hamiltonian of the interaction with the measuring device is given by (13.8). The weakness of the interaction is achieved by preparing the initial state of the measuring device in such a way that the conjugate momentum of the pointer variable, P, is small, and thus the interaction Hamiltonian (13.8) is small. The initial state of the pointer variable is modeled by a Gaussian centered at zero:

$$\Psi_{in}^{MD}(Q) = (\Delta^2\pi)^{-1/4} e^{-Q^2/2\Delta^2} \ . \tag{13.25}$$

The pointer is in the "zero" position before the measurement, i.e., its initial probability distribution is

$$\text{Prob}(Q) = (\Delta^2\pi)^{-1/2} e^{-Q^2/\Delta^2} \ . \tag{13.26}$$

If the initial state of the system is a superposition $|\Psi\rangle = \Sigma\alpha_i|c_i\rangle$, then after the interaction (13.8) the state of the system and the measuring device is:

$$(\Delta^2\pi)^{-1/4}\Sigma\alpha_i|c_i\rangle e^{-(Q-c_i)^2/2\Delta^2} \ . \tag{13.27}$$

The probability distribution of the pointer variable corresponding to the state (13.27) is:

$$\text{Prob}(Q) = (\Delta^2\pi)^{-1/2}\Sigma|\alpha_i|^2 e^{-(Q-c_i)^2/\Delta^2} \ . \tag{13.28}$$

In case of the ideal measurement, this is a weighted sum of the initial probability distribution localized around various eigenvalues. Therefore, the reading of the pointer variable in the end of the measurement almost always yields the value close to one of the eigenvalues. The limit of weak measurement corresponds to $\Delta \gg c_i$ for all eigenvalues c_i. Then, we can perform the Taylor expansion of the sum (13.28) around $Q = 0$ up to the first order and rewrite the probability distribution of the pointer in the following way:

$$\begin{aligned} &\text{Prob}(Q) = (\Delta^2\pi)^{-1/2}\Sigma|\alpha_i|^2 e^{-(Q-c_i)^2/\Delta^2} = \\ &(\Delta^2\pi)^{-1/2}\Sigma|\alpha_i|^2(1-(Q-c_i)^2/\Delta^2) = (\Delta^2\pi)^{-1/2} e^{-(Q-\Sigma|\alpha_i|^2 c_i)^2/\Delta^2} \end{aligned} \tag{13.29}$$

But this is exactly the initial distribution shifted by the value $\Sigma|\alpha_i|^2 c_i$. This is the outcome of the measurement, in this case the weak value is the expectation value:

$$C_w = \Sigma|\alpha_i|^2 c_i = \langle\Psi|C|\Psi\rangle \ . \tag{13.30}$$

The weak value is obtained from statistical analysis of the readings of the measuring devices of the measurements on an ensemble of identical quantum systems. But it is different conceptually from the standard definition of expectation value which is a mathematical concept defined from the statistical analysis of the *ideal* measurements of the variable C all of which yield one of the eigenvalues c_i.

Now let us turn to the system described by the two-state vector (13.2). As usual, the free Hamiltonian is assumed to be zero so it can be obtained by preselection of $|\Psi\rangle$ at t_1 and postselection of $|\Phi\rangle$ at t_2. The (weak) measurement interaction of the form (13.8) takes place at time t, $t_1 < t < t_2$. The state of the measuring device after this sequence of measurements is given (up to normalization) by

$$\Psi^{MD}(Q) = \langle\Phi|e^{-iPC}|\Psi\rangle e^{-Q^2/2\Delta^2} \ . \tag{13.31}$$

After simple algebraic manipulations we can rewrite it (in the P-representation) as

$$\begin{aligned}\tilde{\Psi}^{MD}(P) = \langle\Phi|\Psi\rangle \ e^{-iC_wP} \ e^{-\Delta^2P^2/2} \\ +\langle\Phi|\Psi\rangle \sum_{n=2}^{\infty} \frac{(iP)^n}{n!}[(C^n)_w - (C_w)^n]e^{-\Delta^2P^2/2} \ .\end{aligned} \tag{13.32}$$

If Δ is sufficiently large, we can neglect the second term of (13.32) when we Fourier transform back to the Q-representation. Large Δ corresponds to weak measurement in the sense that the interaction Hamiltonian (13.8) is small. Thus, in the limit of weak measurement, the final state of the measuring device (in the Q-representation) is

$$\Psi^{MD}(Q) = (\Delta^2\pi)^{-1/4}e^{-(Q-C_w)^2/2\Delta^2} \ . \tag{13.33}$$

This state represents a measuring device pointing to the weak value (13.21).

Weak measurements on pre- and postselected ensembles yield, instead of eigenvalues, a value which might lie far outside the range of the eigenvalues. Although we have shown this result for a specific von Neumann model of measurements, the result is completely general: any coupling of a pre- and postselected system to a variable C, provided the coupling is sufficiently weak, results in effective coupling to C_w. This weak coupling between a single system and the measuring device will not, in most cases, lead to a distinguishable shift of the pointer variable, but collecting the results of measurements on an ensemble of pre- and postselected systems will yield the weak values of a measured variable to any desired precision.

When the strength of the coupling to the measuring device goes to zero, the outcomes of the measurement invariably yield the weak value. To be more precise, a measurement yields the real part of the weak value. Indeed, the weak value is, in general, a complex number, but its imaginary part will contribute only a (position dependent) phase to the wave function of the measuring device in the position representation of the pointer. Therefore, the imaginary part will not affect the probability distribution of the pointer position which is what we see in a usual measurement. However, the imaginary part of the weak value also has physical meaning. It is equal to the shift of the Gaussian wave function of the measuring device in the momentum representation. Thus, measuring the shift of the momentum of the pointer will yield the imaginary part of the weak value.

The research of weak measurements continues until today. Recently, Botero [52] noted that in some cases the pointer of the weak measurements in some cases has narrower distribution after the weak measurement interaction than it has before. Note also recent different ways of the analysis of the weak measurement effect [53, 54, 55, 56, 57, 58, 59, 60].

13.4.2 Examples: Measurements of Spin Components

Let us consider a simple Stern–Gerlach experiment: measurement of a spin component of a spin-$\frac{1}{2}$ particle. We shall consider a particle prepared in the initial state spin "up" in the $\hat{x}$ direction and postselected to be "up" in the $\hat{y}$ direction. At the intermediate time we measure, weakly, the spin component in the $\hat{\xi}$ direction which is bisector of $\hat{x}$ and $\hat{y}$, i.e., $\sigma_\xi = (\sigma_x + \sigma_y)/\sqrt{2}$. Thus $|\Psi\rangle = |\uparrow_x\rangle$, $|\Phi\rangle = |\uparrow_y\rangle$, and the weak value of σ_ξ in this case is

$$(\sigma_\xi)_w = \frac{\langle\uparrow_y|\sigma_\xi|\uparrow_x\rangle}{\langle\uparrow_y|\uparrow_x\rangle} = \frac{1}{\sqrt{2}}\frac{\langle\uparrow_y|(\sigma_x+\sigma_y)|\uparrow_x\rangle}{\langle\uparrow_y|\uparrow_x\rangle} = \sqrt{2}\,. \tag{13.34}$$

This value is, of course, "forbidden" in the standard interpretation where a spin component can obtain the (eigen)values ± 1 only.

An effective Hamiltonian for measuring σ_ξ is

$$H = g(t)P\sigma_\xi\,. \tag{13.35}$$

Writing the initial state of the particle in the σ_ξ representation, and assuming the initial state (13.25) for the measuring device, we obtain that after the measuring interaction the quantum state of the system and the pointer of the measuring device is

$$\cos(\pi/8)|\uparrow_\xi\rangle e^{-(Q-1)^2/2\Delta^2} + i\sin(\pi/8)|\downarrow_\xi\rangle e^{-(Q+1)^2/2\Delta^2}\,. \tag{13.36}$$

The probability distribution of the pointer position, if it is observed now without postselection, is the sum of the distributions for each spin value. It is, up to normalization,

$$\mathrm{Prob}(Q) = \cos^2(\pi/8)e^{-(Q-1)^2/\Delta^2} + \sin^2(\pi/8)e^{-(Q+1)^2/\Delta^2}\,. \tag{13.37}$$

In the usual strong measurement, $\Delta \ll 1$. In this case, as shown on Fig. 13.2a, the probability distribution of the pointer is localized around -1 and $+1$ and it is strongly correlated to the values of the spin, $\sigma_z = \pm 1$.

Weak measurements correspond to a Δ which is much larger than the range of the eigenvalues, i.e., $\Delta \gg 1$. Figure 13.2b shows that the pointer distribution has a large uncertainty, but it is peaked between the eigenvalues, more precisely, at the expectation value $\langle\uparrow_x|\sigma_\xi|\uparrow_x\rangle = 1/\sqrt{2}$. An outcome of an individual measurement usually will not be close to this number, but it can be found from an ensemble of such measurements, see Fig. 13.2*c*. Note, that we have not yet considered the postselection.

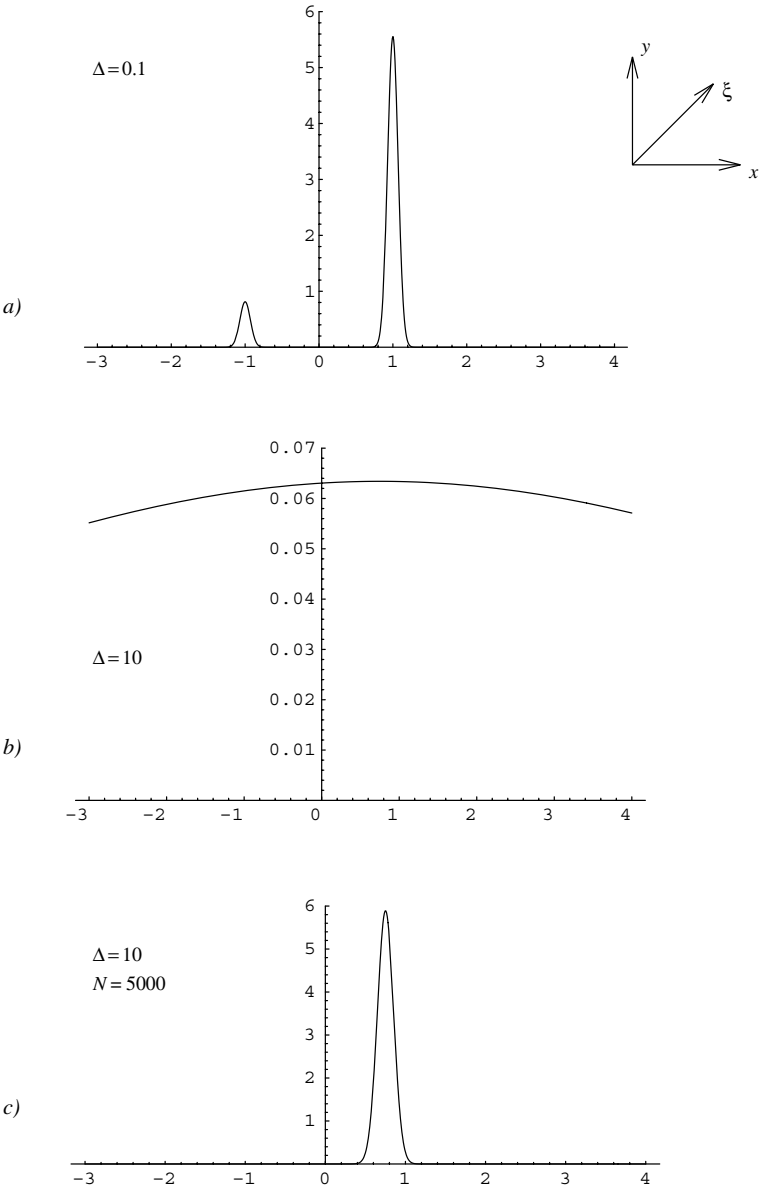

Fig. 13.2. Spin component measurement without post-selection: Probability distribution of the pointer variable for measurement of σ_ξ when the particle is preselected in the state $|\uparrow_x\rangle$. (**a**) Strong measurement, $\Delta = 0.1$. (**b**) Weak measurement, $\Delta = 10$. (**c**) Weak measurement on the ensemble of 5000 particles. The original width of the peak, 10, is reduced to $10/\sqrt{5000} \simeq 0.14$. In the strong measurement (**a**) the pointer is localized around the eigenvalues ± 1, while in the weak measurements (**b**) and (**c**) the peak is located in the expectation value $\langle\uparrow_x|\sigma_\xi|\uparrow_x\rangle = 1/\sqrt{2}$

In order to simplify the analysis of measurements on the pre- and postselected ensemble, let us assume that we first make the postselection of the spin of the particle and only then look at the pointer of the device that weakly measures σ_ξ. We must get the same result as if we first look at the outcome of the weak measurement, make the postselection, and discard all readings of the weak measurement corresponding to the cases in which the result is not $\sigma_y = 1$. The postselected state of the particle in the σ_ξ representation is $\langle\uparrow_y| = \cos(\pi/8)\langle\uparrow_\xi| - i\sin(\pi/8)\langle\downarrow_\xi|$. The state of the measuring device after the postselection of the spin state is obtained by projection of (13.36) onto the postselected spin state:

$$\Phi(Q) = \mathcal{N}\left(\cos^2(\pi/8)e^{-(Q-1)^2/2\Delta^2} - \sin^2(\pi/8)e^{-(Q+1)^2/2\Delta^2}\right), \quad (13.38)$$

where $\mathcal{N}$ is a normalization factor. The probability distribution of the pointer variable is given by

$$\mathrm{Prob}(Q) = \mathcal{N}^2\left(\cos^2(\pi/8)e^{-(Q-1)^2/2\Delta^2} - \sin^2(\pi/8)e^{-(Q+1)^2/2\Delta^2}\right)^2. \quad (13.39)$$

If the measuring interaction is strong, $\Delta \ll 1$, then the distribution is localized around the eigenvalues ± 1 (mostly around 1 since the pre- and postselected probability to find $\sigma_\xi = 1$ is more than 85%), see Fig. 13.3*a* and *b*. But when the strength of the coupling is weakened, i.e., Δ is increased, the distribution gradually changes to a single broad peak around $\sqrt{2}$, the weak value, see Fig. 13.3c–e.

The width of the peak is large and therefore each individual reading of the pointer usually will be far from $\sqrt{2}$. The physical meaning of the weak value can, in this case, be associated only with an ensemble of pre- and postselected particles. The accuracy of defining the center of the distribution goes as $1/\sqrt{N}$, so increasing N, the number of particles in the ensemble, we can find the weak value with any desired precision, see Fig. 13.3f.

In our example, the weak value of the spin component is $\sqrt{2}$, which is only slightly more than the maximal eigenvalue, 1. By appropriate choice of the pre- and postselected states we can get pre- and postselected ensembles with arbitrarily large weak value of a spin component. One of our first proposals [6] was to obtain $(\sigma_\xi)_w = 100$. In this case the postselected state is nearly orthogonal to the preselected state and, therefore, the probability to obtain appropriate postselection becomes very small. While in the case of $(\sigma_\xi)_w = \sqrt{2}$ the pre- and postselected ensemble was about half of the preselected ensemble, in the case of $(\sigma_\xi)_w = 100$ the postselected ensemble will be smaller than the original ensemble by the factor of $\sim 10^{-4}$.

13.4.3 Weak Measurements Which Are not Really Weak

We have shown that weak measurements can yield very surprising values which are far from the range of the eigenvalues. However, the uncertainty of a single weak measurement (i.e., performed on a single system) in the above

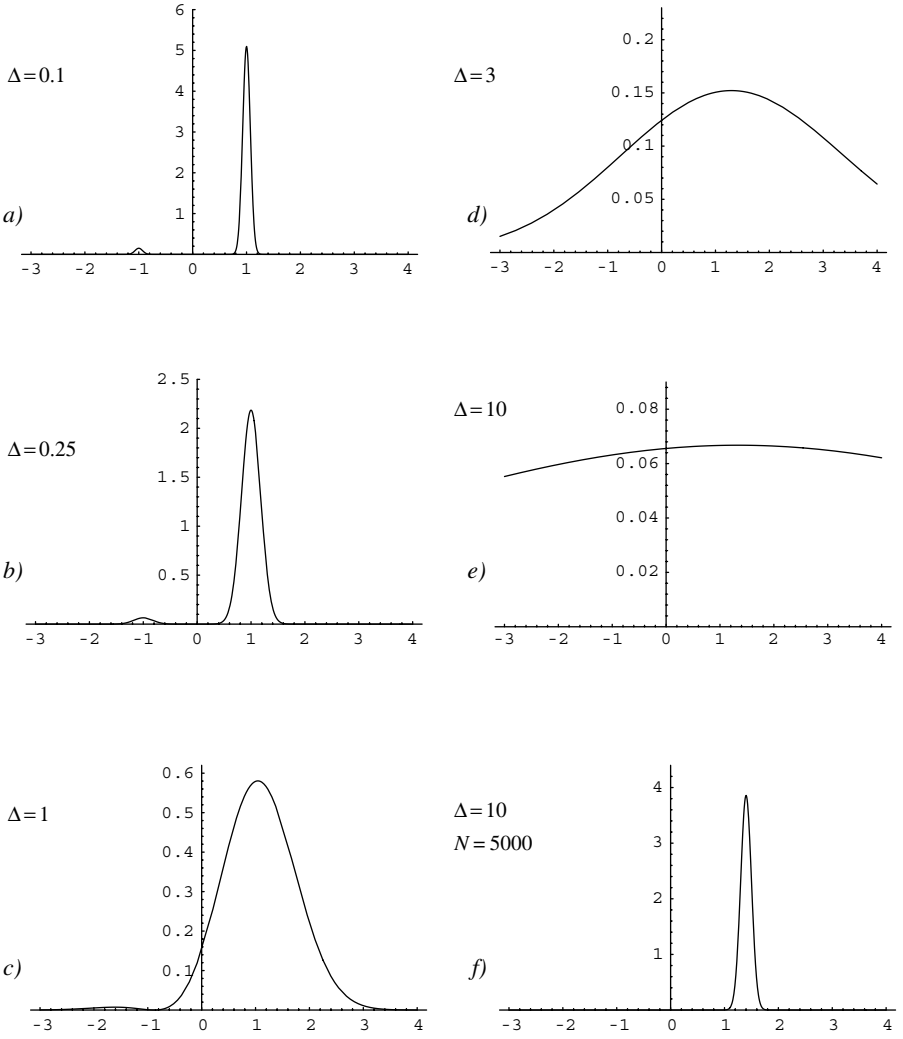

Fig. 13.3. Measurement on pre- and postselected ensemble: Probability distribution of the pointer variable for measurement of σ_ξ when the particle is preselected in the state $|\uparrow_x\rangle$ and postselected in the state $|\uparrow_y\rangle$. The strength of the measurement is parameterized by the width of the distribution Δ. (**a**) $\Delta = 0.1$; (**b**) $\Delta = 0.25$; (**c**) $\Delta = 1$; (**d**) $\Delta = 3$; (**e**) $\Delta = 10$. (**f**) Weak measurement on the ensemble of 5000 particles; the original width of the peak, $\Delta = 10$, is reduced to $10/\sqrt{5}000 \simeq 0.14$. In the strong measurements (**a**)–(**b**) the pointer is localized around the eigenvalues ± 1, while in the weak measurements (**d**)–(**f**) the peak of the distribution is located in the weak value $(\sigma_\xi)_w = \langle\uparrow_y|\sigma_\xi|\uparrow_x\rangle/\langle\uparrow_y|\uparrow_x\rangle = \sqrt{2}$. The outcomes of the weak measurement on the ensemble of 5000 pre- and postselected particles, (f), are clearly outside the range of the eigenvalues, (-1,1)

example is larger than the deviation from the range of the eigenvalues. Each single measurement separately yields almost no information and the weak value arises only from the statistical average on the ensemble. The weakness and the uncertainty of the measurement goes together. Weak measurement corresponds to small value of P in the Hamiltonian (13.8) and, therefore, the uncertainty in P has to be small. This requires large Δ, the uncertainty of the pointer variable. Of course, we can construct measurement with large uncertainty which is not weak at all, for example, by preparing the measuring device in a mixed state instead of a Gaussian, but no precise measurement with weak coupling is possible. So, usually, a weak measurement on a single system will not yield the weak value with a good precision. However, there are special cases when it is not so. Usual strength measurement on a single pre- and postselected system can yield "unusual" (very different from the eigenvalues) weak value with a good precision. Good precision means that the uncertainty is much smaller than the deviation from the range of the eigenvalues.

Our example above was not such a case. The weak value $(\sigma_\xi)_w = \sqrt{2}$ is larger than the highest eigenvalue, 1, only by ~ 0.4, while the uncertainty, 1, is not sufficiently large for obtaining the peak of the distribution near the weak value, see Fig. 13.3c. Let us modify our experiment in such a way that a single experiment will yield meaningful surprising result. We consider a system of N spin-$\frac{1}{2}$ particles all prepared in the state $|\uparrow_x\rangle$ and postselected in the state $|\uparrow_y\rangle$, i.e., $|\Psi\rangle = \prod_{i=1}^N |\uparrow_x\rangle_i$ and $\langle\Phi| = \prod_{i=1}^N \langle\uparrow_y|_i$. The variable which is measured at the intermediate time is $C \equiv (\sum_{i=1}^N (\sigma_i)_\xi)/N$. The operator C has $N+1$ eigenvalues equally spaced between -1 and $+1$, but the weak value of C is

$$C_w = \frac{\prod_{k=1}^N \langle\uparrow_y|_k \ \sum_{i=1}^N ((\sigma_i)_x + (\sigma_i)_y) \ \prod_{j=1}^N |\uparrow_x\rangle_j}{\sqrt{2}\ N(\langle\uparrow_y|\uparrow_x\rangle)^N} = \sqrt{2}\,. \tag{13.40}$$

The interaction Hamiltonian is

$$H = \frac{g(t)}{N} P \sum_{i=1}^N (\sigma_i)_\xi\ . \tag{13.41}$$

The initial state of the measuring device defines the precision of the measurement. When we take it to be the Gaussian (6), it is characterized by the width Δ. For a meaningful experiment we have to take Δ small. Small Δ corresponds to large uncertain P, but now, the strength of the coupling to each individual spin is reduced by the factor $1/N$. Therefore, for large N, both the forward-evolving state and the backward-evolving state are essentially not changed by the coupling to the measuring device. Thus, this single measurement yields the weak value. In [7] it is proven that if the measured observable is an average on a large set of systems, $C = \left(\sum_i^N C_i\right)/N$, then we can always construct a single, good precision measurement of the weak value. Here let us present just numerical calculations of the probability distribution of the measuring device for N pre- and postselected spin-$\frac{1}{2}$ particles. The state of the pointer after the postselection for this case is

$$\mathcal{N}\sum_{i=0}^{N}\frac{(-1)^i}{(i!(N-i)!)}\left(\cos^2(\pi/8)\right)^{N-i}\left(\sin^2(\pi/8)\right)^{i}e^{-(Q-\frac{(2N-i)}{N})^2/2\Delta^2}\ . \quad (13.42)$$

The probability distribution for the pointer variable Q is

$$prob(Q)=\mathcal{N}^2\Big(\sum_{i=0}^{N}\frac{(-1)^i}{(i!(N-i)!)}\left(\cos^2(\pi/8)\right)^{N-i}\left(\sin^2(\pi/8)\right)^{i}e^{-(Q-\frac{(2N-i)}{N})^2/2\Delta^2}\Big)^2. \quad (13.43)$$

The results for $N = 20$ and different values of Δ are presented in Fig. 13.4. We see that for $\Delta = 0.25$ and larger, the obtained results are very good: the final probability distribution of the pointer is peaked at the weak value, $\left((\sum_{i=1}^{N}(\sigma_i)_\xi)/N\right)_w = \sqrt{2}$. This distribution is very close to that of a measuring device measuring operator O on a system in an eigenstate $|O{=}\sqrt{2}\rangle$. For N large, the relative uncertainty can be decreased almost by a factor $1/\sqrt{N}$ without changing the fact that the peak of the distribution points to the weak value.

Although our set of particles preselected in one state and postselected in another state is considered as one system, it looks like an ensemble. In quantum theory, measurement of the sum does not necessarily yield the same result as the sum of the results of the separate measurements, so conceptually our measurement on the set of particles differs from the measurement on an ensemble of pre- and postselected particles. However, in our example of weak measurements, the results are the same.

A less ambiguous case is the example considered in the first work on weak measurements [2]. In this work a single system of a large spin N is considered. The system is preselected in the state $|\Psi\rangle = |S_x{=}N\rangle$ and postselected in the state $|\Phi\rangle = |S_y{=}N\rangle$. At an intermediate time the spin component S_ξ is weakly measured and again the "forbidden" value $\sqrt{2}N$ is obtained. The uncertainty has to be only slightly larger than $\sqrt{N}$. The probability distribution of the results is centered around $\sqrt{2}N$, and for large N it lies clearly outside the range of the eigenvalues, $(-N, N)$. Unruh [61] made computer calculations of the distribution of the pointer variable for this case and got results which are very similar to what is presented in Fig.13.4.

An even more dramatic example is a measurement of the kinetic energy of a tunneling particle [8]. We consider a particle preselected in a bound state of a potential well which has negative potential near the origin and vanishing potential far from the origin; $|\Psi\rangle = |E{=}E_0\rangle$. Shortly later, the particle is postselected to be far from the well, inside a classically forbidden tunneling region; this state can be characterized by vanishing potential $|\Phi\rangle = |U{=}0\rangle$. At an intermediate time, a measurement of the kinetic energy is performed. The weak value of the kinetic energy in this case is

$$K_w = \frac{\langle U{=}0|K|E{=}E_0\rangle}{\langle U{=}0|E{=}E_0\rangle} = \frac{\langle U{=}0|E-U|E{=}E_0\rangle}{\langle U{=}0|E{=}E_0\rangle} = E_0\ . \quad (13.44)$$

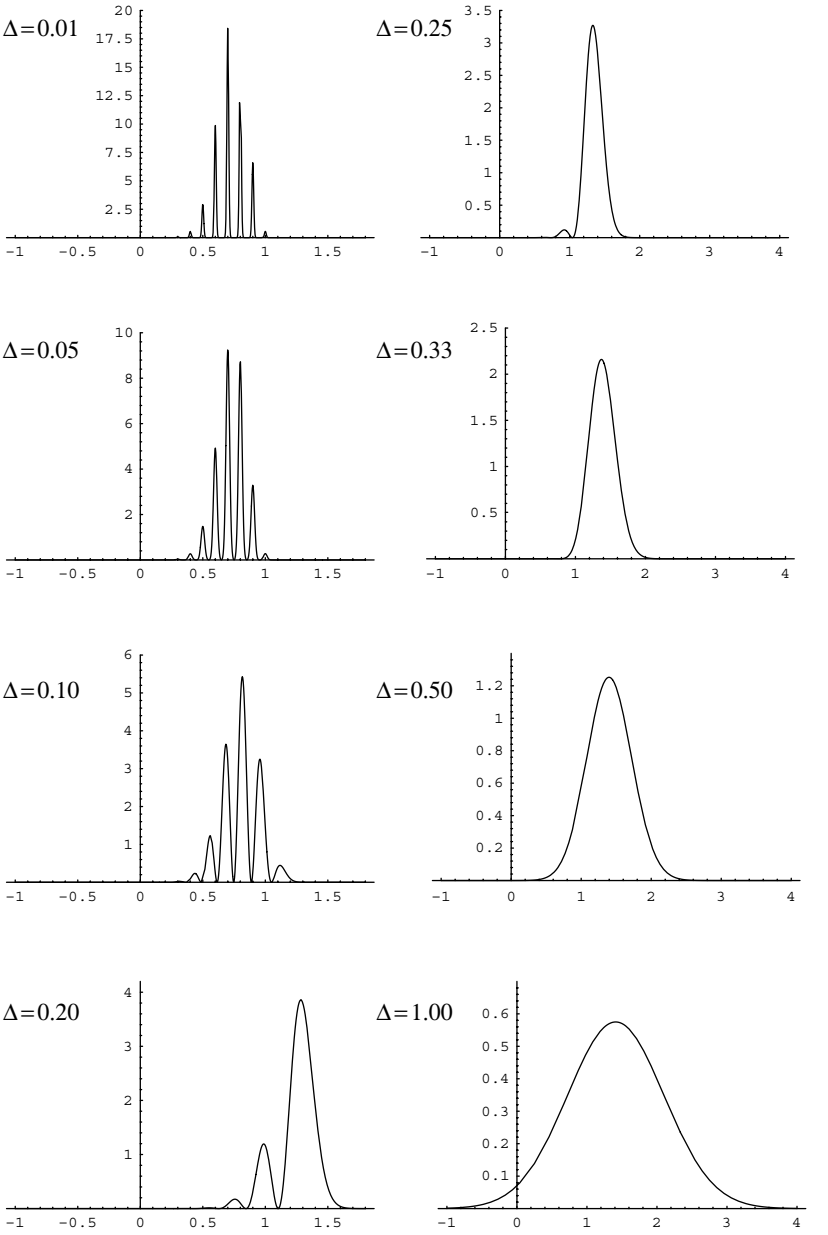

Fig. 13.4. Measurement on a single system: Probability distribution of the pointer variable for the measurement of $A = (\sum_{i=1}^{20}(\sigma_i)_\xi)/20$ when the system of 20 spin-$\frac{1}{2}$ particles is preselected in the state $|\Psi_1\rangle = \prod_{i=1}^{20}|\uparrow_x\rangle_i$ and postselected in the state $|\Psi_2\rangle = \prod_{i=1}^{20}|\uparrow_y\rangle_i$. While in the very strong measurements, $\Delta = 0.01$–0.05, the peaks of the distribution located at the eigenvalues, starting from $\Delta = 0.25$ there is essentially a single peak at the location of the weak value, $A_w = \sqrt{2}$

The energy of the bound state, E_0, is negative, so the weak value of the kinetic energy is negative. In order to obtain this negative value the coupling to the measuring device need not be too weak. In fact, for any finite strength of the measurement we can choose the postselected state sufficiently far from the well to ensure the negative value. Therefore, for appropriate postselection, the usual *strong* measurement of a positive definite operator invariably yields a negative result! This weak value predicted by the two-state vector formalism demonstrates a remarkable consistency: the value obtained is exactly the value that we would expect a particle to have when the particle is characterized in the intermediate times by the two wave functions, one in a ground state, and the other localized outside the well. Indeed, we obtain this result precisely when we postselect the particle far enough from the well that it could not have been kicked there as a result of the intermediate measurement. A peculiar interference effect of the pointer takes place: destructive interference in the whole "allowed" region and constructive interference of the tails in the "forbidden" negative region. The initial state of the measuring device $\Phi(Q)$, due to the measuring interaction and the postselection, transforms into a superposition of shifted wave functions. The shifts are by the (possibly small) eigenvalues, but the superposition is approximately equal to the original wave function shifted by a (large and/or forbidden) weak value:

$$\sum_n \alpha_n \Psi^{MD}(Q - c_n) \simeq \Psi^{MD}(Q - C_w) \; . \tag{13.45}$$

These surprising, even paradoxical effects are really only gedanken experiments. The reason is that, unlike weak measurements on an ensemble, these are extremely rare events. For yielding an unusual weak value, a single preselected system needs an extremely improbable outcome of the postselection measurement. Let us compare this with a weak measurement on an ensemble. In order to get N particles in a pre- and postselected ensemble which yield $(\sigma_\xi)_w = 100$, we need $\sim N10^4$ particles in the preselected ensemble. But, in order to get a single system of N particles yielding $(S_\xi)_w = 100N$, we need $\sim 10^{4N}$ systems of N preselected particles. In fact, the probability to obtain an unusual value by error is much larger than the probability to obtain the proper postselected state. What makes these rare effects interesting is that there is a strong (although only one-way) correlation: for example, every time we find in the postselection measurement the particle sufficiently far from the well, we know that the result of the kinetic energy is negative, and not just negative: it is equal to the weak value, $K_w = E_0$, with a good precision.

13.4.4 Relations Between Weak and Strong Measurements

In general, weak and strong measurements do not yield the same outcomes. The outcomes of strong measurements are always the eigenvalues while the outcomes of weak measurements, the weak values, might be very different from the eigenvalues. However, there are two important relations between them [4].

(i) If the description of a quantum system is such that a particular eigenvalue of a variable is obtained with certainty in case it is measured strongly, then the weak value of this variable is equal to this eigenvalue. This is correct in all cases, i.e., if the system described by a corresponding single (forward or backward evolving) eigenstate, or if it is described by a two-state vector, or even if it is described by a generalized two-state vector.

(ii) The inverse of this theorem is true for dichotomic variables such as projection operators of spin components of spin-$\frac{1}{2}$ particles. The proofs of both statements are given in [4].

Let us apply the theorem (i) for the example of three boxes when we have a large number of particles all pre- and postselected in the two-state vector (13.15). The *actual* story is as follows: A macroscopic number N of particles (gas) were all prepared at t_1 in a superposition of being in three separated boxes (13.13). At later time t_2 all the particles were found in another superposition (13.14) (this is an extremely rare event). In between, at time t, *weak measurements* of a number of particles in each box, which are, essentially, usual measurements of pressure in each box, have been performed. The readings of the measuring devices for the pressure in the boxes 1, 2, and 3 were

$$\begin{aligned} p_1 &= p \,, \\ p_2 &= p \,, \\ p_3 &= -p \,, \end{aligned} \tag{13.46}$$

where p is the pressure which is expected to be in a box with N particles.

We are pretty certain that this "actual" story never took place because the probability for the successful postselection is of the order of 3^{-N}; for a macroscopic number N it is too small for any real chance to see it happening. However, given that the postselection does happen, we are safe to claim that the results (13.46) are correct, i.e., the measurements of pressure at the intermediate time with very high probability have shown these results.

Indeed, the system of all particles at time t (signified by index i) is described by the two-state vector

$$\langle \Phi| \, |\Psi\rangle = \frac{1}{3^N} \prod_{i=1}^{i=N} (\langle 1|_i + \langle 2|_i - \langle 3|_i) \prod_{i=1}^{i=N} (|1\rangle_i + |2\rangle_i + |3\rangle_i) \,. \tag{13.47}$$

Then, intermediate measurements yield, for each particle, probability 1 for the the following outcomes of measurements:

$$\begin{aligned} \mathbf{P}_1 &= 1 \,, \\ \mathbf{P}_2 &= 1 \,, \\ \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 &= 1 \,, \end{aligned} \tag{13.48}$$

where $\mathbf{P}_1$ is the projection operator on the state of the particle in box 1, etc. Thus, from (13.48) and theorem (i) it follows:

$$\begin{aligned} (\mathbf{P}_1)_w &= 1 \ , \\ (\mathbf{P}_2)_w &= 1 \ , \\ (\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3)_w &= 1 \ . \end{aligned} \tag{13.49}$$

Since for any variables, $(X+Y)_w = X_w + Y_w$ we can deduce that $(\mathbf{P}_3)_w = -1$.

Similarly, for the "number operators" such as $\mathcal{N}_1 \equiv \Sigma_{i=1}^{N} \mathbf{P}_1^{(i)}$, where $\mathbf{P}_1^{(i)}$ is the projection operator on the box 1 for a particle i, we obtain:

$$\begin{aligned} (\mathcal{N}_1)_w &= N \ , \\ (\mathcal{N}_2)_w &= N \ , \\ (\mathcal{N}_3)_w &= -N \ . \end{aligned} \tag{13.50}$$

In this rare situation the "weak measurement" need not be very weak: a usual measurement of pressure is a weak measurement of the number operator. Thus, the time-symmetrized formalism yields the surprising result (13.46): the result of the pressure measurement in box 3 is negative! It equals minus the pressure measured in the boxes 1 and 2.

Of course, the negative pressure was not measured in a real laboratory (it requires an extremely improbable postselection), but a nonrobust weak measurement for three-box experiment has been performed in a laboratory [62].

Another example of relation between strong and weak measurements is Hardy's paradox [37]. The analysis of strong measurements appears in [34] and the weak measurements are analyzed in detail in [63]. See also discussions of a realistic experimental proposals [64, 65, 66, 67].

An application of the inverse theorem yields an alternative proof of the results regarding strong measurements of spin components of a spin-$\frac{1}{2}$ particle described by the generalized two-state vector (13.20). Indeed, the linearity property of weak measurements yields a "geometrical picture" for weak values of spin components of a spin-$\frac{1}{2}$ particle. The operators σ_x, σ_y, and σ_z are a complete set of spin operators and they yield a geometry in the familiar three-dimensional space. Each generalized two-state vector of a spin-$\frac{1}{2}$ particle corresponds to a vector in this three-dimensional space with components equal to the weak values of σ_x, σ_y, and σ_z. We call it "weak vector." The weak value of a spin component in an arbitrary direction, then, is given by the projection of the weak vector on this direction. If the weak vector is real and its value larger than 1, then there is a cone of directions the projection on which is equal 1. This yields an alternative proof that in some situations there is a continuum of directions forming a cone in which the result of a spin-component measurements are known with certainty, see Sect. 13.3.5.

13.4.5 Experimental Realizations of Weak Measurements

Realistic weak measurements (on an ensemble) involve preparation of a large preselection ensemble, coupling to the measuring devices of each element of

the ensemble, postselection measurement which, in all interesting cases, selects only a small fraction of the original ensemble, selection of corresponding measuring devices, and statistical analysis of their outcomes. In order to obtain good precision, this selected ensemble of the measuring devices has to be sufficiently large. Although there are significant technological developments in "marking" particles running in an experiment, clearly the most effective solution is that the particles themselves serve as measuring devices. The information about the measured variable is stored, after the weak measuring interaction, in their other degree of freedom. In this case, the postselection of the required final state of the particles automatically yields the selection of the corresponding measuring devices . The requirement for the postselection measurement is, then, that there is no coupling between the variable in which the result of the weak measurement is stored and the postselection device.

An example of such a case is the Stern–Gerlach experiment where the shift in the momentum of a particle, translated into a spatial shift, yields the outcome of the spin measurement. Postselection measurement of a spin component in a certain direction can be implemented by another (this time strong) Stern–Gerlach coupling which splits the beam of the particles. The beam corresponding to the desired value of the spin is then analyzed for the result of the weak measurement. The requirement of nondisturbance of the results of the weak measurement by postselection can be fulfilled by arranging the shifts due to the two Stern–Gerlach devices to be orthogonal to each other. The details are spelled out in [6].

An analysis of a realistic experiment which can yield large weak value Q_w appears in [68]. Duck, Stevenson, and Sudarshan [69] proposed a slightly different optical realization which uses a birefringent plate instead of a prism. In this case the measured information is stored directly in the spatial shift of the beam without being generated by the shift in the momentum. Ritchie, Story, and Hulet [70] adopted this scheme and performed the first successful experiment measuring the weak value of the polarization operator. Their results are in very good agreement with theoretical predictions. They obtained weak values which are very far from the range of the eigenvalues, $(-1, 1)$, their highest reported result is $Q_w = 100$. The discrepancy between calculated and observed weak value was 1%. The RMS deviation from the mean of 16 trials was 4.7%. The width of the probability distribution was $\Delta = 1000$ and the number of pre- and postselected photons was $N \sim 10^8$, so the theoretical and experimental uncertainties were of the same order of magnitude. Their other run, for which they showed experimental data on graphs (which fitted very nicely theoretical graphs), has the following characteristics: $Q_w = 31.6$, discrepancy with calculated value 4%, the RMS deviation 16%, $\Delta = 100$, $N \sim 10^5$. A similar optical experiment has been successfully performed several years ago [71].

Recently, optical weak measurement experiments moved to the field of fiber optics [72, 73, 74]. Another step prevents now any sceptic to argue that the unusual outcomes of weak measurement are a classical effect because

macroscopic number of photons are involved in these experiments. The weak measurement of photon polarization have been performed with single particles [75]. Note also a more controversial issue of measurement of "time of arrival" [76] for which weak measurement technique were also applied [77, 78, 79].

Already at 1990 [3] we gave an example of a gedanken experiment in which pre- and postselection lead to a superluminal propagation of light. Steinberg and Chiao [80, 81] connected this to superluminal effect observed for tunneling particles. The issue was analyzed recently by Aharonov et al. [82] and Sokolovsky et al. [83]. Rohrlich and Aharonov [84] also predicted that there is really a physical meaning for this superluminal propagation: we should expect Cherenkov radiation in such experiment.

Note also proposals for *weak nonlocal* measurements [85, 86]. In these works it was pointed out that observation of correlations between outcomes of local weak measurements can yield values of nonlocal variables. However, these methods are very inefficient, and the methods of efficient nonlocal measurements [36] require conditions which contradict conditions of weak measurements, so we doubt that there will be efficient weak nonlocal measurement proposals suitable for realization in a laboratory.

13.5 The Quantum Time-Translation Machine

13.5.1 Introduction

To avoid possible misinterpretations due to the name "time machine," let us explain from the outset what our machine [7] can do and how it differs from the familiar concept of "time machine." Our device is not for time travel. All that it can accomplish is to change the rate of time flow for a closed quantum system. Classically, one can slow down the time flow of a system relative to an external observer, e.g., by fast travel. Our quantum time machine is able to change the rate of time flow of a system for a given period by an arbitrary, even *negative*, factor. Therefore, our machine, contrary to any classical device, is capable of moving the system to its "past." In that case, at the moment the machine completes its operation the system is in a state in which it was some time *before the beginning* of the operation of the time machine. Our machine can also move the system to the future, i.e., at the end of the operation of the time machine the system is in a state corresponding to some later time of the undisturbed evolution.

A central role in the operation of our time machine is played by a peculiar mathematical identity which we discuss in Sect. 13.5.2. In order to obtain different time evolutions of the system we use the gravitational time dilation effect which is discussed in Sect. 13.5.3. In Sect. 13.5.4 we describe the design and the operation of our time machine. The success of the operation of our time machine depends on obtaining a specific outcome in the postselection quantum measurement. The probability of the successful postselection

measurement is analyzed in Sect. 13.5.5. The concluding discussion of the limitations and the advantages of our time machine appear in Sect. 13.5.6.

13.5.2 A Peculiar Mathematical Identity

The peculiar interference effect of weak measurements (13.45), that a particular superposition of identical Gaussians shifted by small values yields the Gaussian shifted by a large value occurs not just for Gaussians, but for a large class of functions. Consider now that the system is described by such a wave function and the shifts are due to the time evolutions for various periods of time. Then, this effect can be a basis of a (gedanken) time machine. A specific superposition of time evolutions for short periods of time δt_n yields a time evolution for a large period of time Δt

$$\sum_{n=0}^{N} \alpha_n U(\delta t_n)|\Psi\rangle \sim U(\Delta t)|\Psi\rangle \ . \tag{13.51}$$

This approximate equality holds (with the same δt_n and Δt) for a large class of states $|\Psi\rangle$ of the quantum system, and in some cases even for all states of the system.

In order to obtain different time evolutions $U(\delta t_n)$ we use the gravitational time dilation effect. For finding the appropriate δt_n and α_n we will rely on

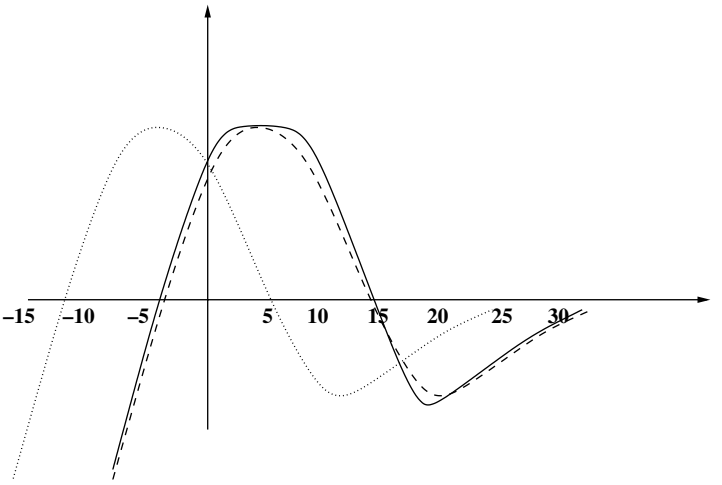

Fig. 13.5. Demonstration of an approximate equality given by (13.53): The sum of a function shifted by the 14 values c_n between 0 and 1 and multiplied by the coefficients α_n (c_n and α_n are given by (13.52) with $N = 13$, $\eta = 10$) yields approximately the same function shifted by the value 10. The *dotted line* shows $f(t)$; the *dashed line* shows$f(t - 10)$, the RHS of (13.53); and the *solid line* shows the sum, the LHS of (13.53)

the identity (13.45) for a particular weak measurement. We choose

$$c_n = n/N, \qquad \alpha_n = \frac{N!}{(N-n)!n!}\eta^n(1-\eta)^{N-n}\ , \tag{13.52}$$

where $n = 0, 1, ..., N$. Note, that the coefficients α_n are terms in the binomial expansion of $[\eta+(1-\eta)]^N$ and, in particular, $\sum_{n=0}^{N}\alpha_n = 1$. The corresponding "weak value" in this case is η and for a large class of functions (the functions with Fourier transform bounded by an exponential) we have an approximate equality

$$\sum_{n=0}^{N}\alpha_n f(t-c_n) \simeq f(t-\eta)\ . \tag{13.53}$$

The proof can be found in [87]. Here we only demonstrate it on a numerical example, Fig. 13.5. Even for a relatively small number of terms in the sum (14 in our example), the method works remarkably well. The shifts from 0 to 1 yield the shift by 10. The distortion of the shifted function is not very large. By increasing the number of terms in the sum, the distortion of the shifted function can be made arbitrarily small.

13.5.3 Classical Time Machines

A well-known example of a time machine is a rocket which takes a system to a fast journey. If the rocket is moving with velocity V and the duration of the journey (in the laboratory frame) is T, then we obtain the time shift (relative to the situation without the fast journey):

$$\delta t = T\left(1-\sqrt{1-\frac{V^2}{c^2}}\right)\ . \tag{13.54}$$

For typical laboratory velocities this effect is rather small, but it has been observed experimentally in precision measurements in satellites and, of course, the effect is observed on decaying particles in accelerators. In such a "time machine," however, the system necessarily experiences external force, and we consider this a conceptual disadvantage.

In our time machine we use, instead of the time dilation of special relativity, the gravitational time dilation. The relation between the proper time of the system placed in a gravitational potential ϕ and the time of the external observer ($\phi = 0$) is given by $d\tau = dt\sqrt{1+2\phi/c^2}$. We produce the gravitational potential by surrounding our system with a spherical shell of mass M and radius R. The gravitational potential inside the shell is $\phi = -GM/R$. Therefore, the time shift due to the massive shell surrounding our system, i.e., the difference between the time period T of the external observer at a large distance from the shell and the period of the time evolution of the system (the proper time), is

$$\delta t = T\left(1 - \sqrt{1 - \frac{2GM}{c^2 R}}\right) . \tag{13.55}$$

This effect, for any man-made massive shell, is too small to be observed by today's instruments. However, the conceptual advantage of this method is that we do not "touch" our system. Even the gravitational field due to the massive spherical shell vanishes inside the shell.

The classical time machine can only *slow down* the time evolution of the system. For any reasonable mass and radius of the shell, the change of the rate of the time flow is extremely small. In the next section we shall describe our quantum time machine which amplifies the effect of the classical gravitational time machine (for a spherical shell of the same mass), and makes it possible to speed up the time flow for an evolution of a system, as well as to change its direction.

13.5.4 Quantum Gravitational Time Machine

In our machine we use the gravitational time dilation and a quantum interference phenomenon which, due to the peculiar mathematical property discussed in Sect. 13.5.2, amplifies the time translation. We produce the superposition of states shifted in time by small values δt_n (due to spherical shells of different radii) given by the left-hand side of (13.51). Thus, we obtain a time shift by a possibly large, positive or negative, time interval Δt.

The wave function of a quantum system $\Psi(q,t)$, considered as a function of time, usually has a Fourier transform which decreases rapidly for large frequencies. Therefore, the sum of the wave functions shifted by small periods of time $\delta t_n = \delta t c_n$, and multiplied by the coefficients α_n, with c_n and α_n given by (13.52), is approximately equal to the wave function shifted by the large time $\Delta t = \delta t \eta$. Since the equality (13.53) is correct with the same coefficients for all functions with rapidly decreasing Fourier transforms, we obtain for each q, and therefore for the whole wave function,

$$\sum_{n=0}^{N} \alpha_n \Psi(q, t - \delta t_n) \simeq \Psi(q, t - \Delta t) . \tag{13.56}$$

Thus, a device which changes the state of the system from $\Psi(q,t)$ to the state given by the left-hand side of (13.56) generates the time shift of Δt. Let us now present a design for such a device and explain how it operates.

Our machine consists of the following parts: a massive spherical shell, a mechanical device—"the mover"—with a quantum operating system, and a measuring device which can prepare and verify states of this quantum operating system.

The massive shell of mass M surrounds our system and its radius R can have any of the values $R_0, R_1, ..., R_N$. Initially, $R = R_0$.

The mover changes the radius of the spherical shell at time $t = 0$, waits for an (external) time T, and then moves it back to its original state, i.e., to the radius R_0.

The quantum operating system (QOS) of the mover controls the radius to which the shell is moved for the period of time T. The Hamiltonian of the QOS has $N + 1$ nondegenerate eigenstates $|n\rangle$, $n = 0, 1, ..., N$. If the state of the QOS is $|n\rangle$, then the mover changes the radius of the shell to the value R_n.

The measuring device preselects and postselects the state of the QOS. It prepares the QOS before the time $t = 0$ in the initial state

$$|\Psi_{in}\rangle_{QOS} = \mathcal{N} \sum_{n=0}^{N} \alpha_n |n\rangle \ , \tag{13.57}$$

with the normalization factor

$$\mathcal{N} = \frac{1}{\sqrt{\sum_{n=0}^{N} |\alpha_n|^2}} \ . \tag{13.58}$$

After the mover completes its operation, i.e., after the time $t = T$, we perform another measurement on the QOS. One of the nondegenerate eigenstates of this measurement is the specific "final state"

$$|\Psi_f\rangle_{QOS} = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n\rangle \ . \tag{13.59}$$

Our machine works only if the postselection measurement yields the state (13.59). Unfortunately, this is a very rare event. We shall discuss the probability of obtaining the appropriate outcome in the next section.

Assume that the postselection measurement is successful, i.e., that we do obtain the final state (13.59). We will next show that in this case, assuming an appropriate choice of the radii R_n, our "time machine" shifts the wave function of the system by the time interval Δt. The time shift is defined relative to the situation in which the machine has not operated, i.e., the radius of the shell was not changed from the initial value R_0. In order to obtain the desired time shift $\Delta t = \delta t \eta$ we chose the radii R_n such that

$$\delta t_n \equiv \frac{n \delta t}{N} = T \left(\sqrt{1 - \frac{2GM}{c^2 R_0}} - \sqrt{1 - \frac{2GM}{c^2 R_n}} \right) \ . \tag{13.60}$$

The maximal time shift in the different terms of the superposition (the left-hand side of (13.51)) is $\delta t_N = \delta t$. The parameter η is the measure of the "quantum amplification" relative to the maximal (classical) time shift δt. If the radius R_0 of the shell is large enough that the time dilation due to the shell in its initial configuration can be neglected, (13.60) simplifies to

$$\delta t_n = T\left(1 - \sqrt{1 - \frac{2GM}{c^2 R_n}}\right) . \tag{13.61}$$

Let us assume then that we have arranged the radii according to (13.61) and we have prepared the quantum operating system of the mover in the state (13.57). Then, just prior to the operation of the time machine the overall state is the direct product of the corresponding states of the system, the shell, and the mover,

$$\mathcal{N}|\Psi(q,0)\rangle|R_0\rangle \sum_{n=0}^{N} \alpha_n n\rangle , \tag{13.62}$$

where $|R_0\rangle$ signifies that the shell, together with the mechanical part of the mover, is at the radius R_0. Although these are clearly macroscopic bodies. we assume that we can treat them quantum–mechanically. We also make an idealized assumption that these bodies do not interact with the environment, i.e., no element of the environment becomes correlated to the radius of the shell.

Once the mover has operated, changing the radius of the spherical shell, the overall state becomes

$$\mathcal{N}|\Psi(q,0)\rangle \sum_{n=0}^{N} \alpha_n |R_n\rangle|n\rangle . \tag{13.63}$$

For different radii R_n, we have different gravitational potentials inside the shell and, therefore, different relations between the flow of the proper time of the system and the flow of the external time. Thus, after the external time T has elapsed, just before the mover takes the radii R_n back to the value R_0, the overall state is

$$\mathcal{N} \sum_{n=0}^{N} \alpha_n |\Psi(q, T - \delta t_n)\rangle|R_n\rangle|n\rangle . \tag{13.64}$$

Note that now the system, the shell, and the QOS are correlated: the system is not in a pure quantum state. After the mover completes its operation, the overall state becomes

$$\mathcal{N} \sum_{n=0}^{N} \alpha_n |\Psi(q, T - \delta t_n)\rangle|R_0\rangle|n\rangle. \tag{13.65}$$

There is still a correlation between the system and the QOS.

The last stage is the postselection measurement performed on the QOS. It puts the QOS and, consequently, our quantum system, in a pure state. After the successful postselection measurement, the overall state is

$$\left(\sum_{n=0}^{N} \alpha_n |\Psi(q, T - \delta t_n)\rangle\right) |R_0\rangle \left(\frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n\rangle\right) . \tag{13.66}$$

We have shown that the wave function of the quantum system $\Psi(q,t)$ is changed by the operation of the time machine into $\sum_{n=0}^{N} \alpha_n |\Psi(q, T-\delta t_n)\rangle$. Up to the precision of the approximate equality (13.53) (which can be arbitrarily improved by increasing the number of terms N in the sum), this wave function is indeed $|\Psi(q, T-\Delta t)\rangle$! Note that for $\Delta t > T$, the state of the system at the moment the time machine has completed its operation is the state in which the system was before the beginning of the operation of the time machine.

13.5.5 The Probability of the Success of the Quantum Time Machine

The main conceptual weakness of our time machine is that usually it does not work. Successful postselection measurements corresponding to large time shifts are extremely rare. Let us estimate the probability of the successful postselection measurement in our example. The probability is given by the square of the norm of the vector obtained by projecting the state (13.66) on the subspace defined by state (13.59) of the QOS:

$$\text{Prob} = ||\frac{\mathcal{N}}{\sqrt{N+1}}(\sum_{n=0}^{N} \alpha_n |\Psi(q, T-\delta t_n)\rangle |R_0\rangle||^2 \ . \tag{13.67}$$

In order to obtain a time shift without significant distortion, the wave functions shifted by different times δt_n have to be such that the scalar products between them can be approximated by 1. Taking then the explicit form of α_n from (13.52), we evaluate the probability (13.67), obtaining

$$\text{Prob} \simeq \frac{\mathcal{N}^2}{N} \ . \tag{13.68}$$

The normalization factor $\mathcal{N}$ given by (13.58) decreases very rapidly for large N. Even if we use a more efficient choice of the initial and the final states of the QOS (see [3]) for the amplification, $\eta > 1$, the probability decreases with N as $1/(2\eta - 1)^N$.

The small probability of the successful operation of our time machine is, in fact, unavoidable. At the time just before the postselection measurement, the system is in a mixture of states correlated to the orthogonal states of the QOS (see (13.65)). The probability of finding the system at that time in the state $|\Psi(q, T-\Delta t)\rangle$, for Δt which differs significantly from the time periods δt_n, is usually extremely small. This is the probability to find the system, by a measurement performed "now," in the state in which it was supposed to be at some other time. For any real situation this probability is tiny but not equal precisely to zero, since all systems with bounded energies have wave functions with nonvanishing tails. The successful operation of our time machine is a particular way of "finding" the state of the quantum system shifted by the period of time $\Delta = \eta\delta$. Therefore, the probability for success

cannot be larger than the probability of finding the shifted wave function by direct measurement.

One can wonder what has been achieved by all this rather complicated procedure if we can obtain the wave function of the system shifted by the time period Δt simply by performing a quantum verification measurement at the time T of the state $|\Psi(q, T - \Delta t)\rangle$. There is a very small chance for the success of this verification measurement, but using our procedure the chance is even smaller. What our machine can do, and we are not aware of any other method which can achieve this, is to shift the wave function in time *without knowing* the wave function. If we obtain the desired result of the postselection measurement (the postselection measurement performed on *the measuring device*), we know that the wave function of the system, whatever it is, is shifted by the time Δt. Not only is the knowledge of the wave function of the system inessential for our method, but even the very nature of the physical system whose wave function is shifted by our time machine need not be known. The only requirement is that the energy distribution of the system decreases rapidly enough. If the expectation value of the energy can be estimated, then we can improve dramatically the probability of the success of our procedure. The level of difficulty of the time shift without distortion depends on the magnitude of the energy dispersion ΔE and not on the expectation value of energy $\langle E \rangle$. For quantitative analysis of this requirement see [87].

The operation of our time machine can be considered as *a superposition of time evolutions* [7] for different periods of time δt_n. This name is especially appropriate if the Hamiltonian of the system is bounded, since in this case the approximate equality (13.51) is correct for all states $|\Psi\rangle$.

13.5.6 Time Translation to the Past and to the Future

Let us spell out again what our machine does. Assume that the time evolution of the state of the system is given by $|\Psi(t)\rangle$. By this we mean that this is the evolution *before* the operation of the time machine and this is also the evolution later, provided we *do not* operate the time machine. The state $|\Psi(t)\rangle$ describes the actual past states of the system and the counterfactual future states of the system, i.e., the states which will be in the case we do not disturb the evolution of the system by the operation of our time machine. Define "now," $t = 0$, to be the time at which we begin the operation of the time machine. The time interval of the operation of the time machine is T. Moving the system to the *past* means moving it to the state in which the system actually was at some time $t < 0$. Moving the system to the future means moving it to the state in which it would have wound up after undisturbed evolution at some future time $t > T$. Evidently, the classical time machine does neither of these, since all it can achieve is that at time T the system is in the state corresponding to the time t, $0 < t < T$.

When we speed up or slow down the rate of the time evolution, the system passes through all states of its undisturbed evolution only once. More bizarre

is the situation when we reverse the direction of the time flow, thus ending up, after completing the operation of the time machine, in the state in which the system was before $t = 0$. In this case the system passes three times through some states during its evolution.

For our time machine to operate properly, it is essential that the system is isolated from the external world. In the case of the time translation to the state of the past, the system has to be isolated not only during the time of the operation of the time machine, but also during the whole period of intended time translation. If the system is to be moved to the state in which it was at the time t, $t < 0$, then it has to be isolated from the time t until the end of operation of the time machine. This seems to be a limitation of our time machine. It leads, however, to an interesting possibility. We can send a system to its *counterfactual past*, i.e., to the past in which it was supposed to be if it were isolated (or if it were in any environment chosen by us).

Consider an excited atom which we isolate in the vacuum at time $t = 0$ inside our time machine. And assume that our time machine made a successful time translation to a negative time t, such that $|t|$ is larger than the lifetime of the excited atomic state. Since the atom, now, is not in the environment it was in the past, we do not move the atom to its actual state in the past. Instead, we move the atom to the state of its counterfactual past. By this we mean the state of the isolated atom which, under its normal evolution in the vacuum during the time period $|t|$ winds up in the excited state. In fact, this is the state of the atom together with an incoming radiation field. The radiation field is exactly such that it will be absorbed by the atom. Although our procedure is very complicated and only very rarely successful, still, it is probably the easiest way to prepare the precise incoming electromagnetic wave which excites a single atom with probability one.

13.5.7 Experimental Realization of the Quantum Time-Translation Machine?!

Suter [88] has claimed to perform an experimental realization of the quantum time-translation machine using a classical Mach–Zehnder interferometer. The experimental setup of Suter, however, does not fall even close to the definition of the time machine. In his setup we know what is the system and what is its initial state. What he shows is that if we send a single mode of a radiation field through a birefringent retardation device which yields different retardations for two orthogonal polarizations, then placing the preselection polarization filter and the postselection polarization filter will lead to a much larger effect than can be achieved by preselection alone. Thus, it might seem like speeding up the time evolution, but this procedure fails all tests of universality. Different modes of radiation field speed up differently, an arbitrary wave packet is usually distorted, and for other systems (other particles) the device is not supposed to work at all.

Thus, the first basic requirement that the time machine has to work for various systems is not fulfilled from the beginning. And it cannot be easily modified since the "external" variable (which is supposed to be a part of the time machine) is the property of the system itself—the polarization of the radiation field. The next necessary requirement, that it works for a large class of the initial states of the system, cannot be fulfilled too. Indeed, he considers a superposition of only two time evolutions. This superposition can be identical to a longer evolution for a particular state, but not for a large class of states. As it has been shown [7, 87] a superposition of a large number of time evolutions is necessary for this purpose.

Suter, together with R. Ernst and M. Ernst, performed in the past another experiment which they called "An experimental realization of a quantum time-translation machine" [89]. In this experiment a very different system was used: the effect was demonstrated on the heteronuclear coupling between two nuclear spins. But the experimental setup was also applicable only to a specific system and only for a certain state. Therefore, the same criticism is applicable and, therefore, one should not call it an implementation of the time-translation machine.

Although the experiments of Suter are not implementations of the quantum time machine, still, they are interesting as *weak measurements*. The experiment of Suter with a birefringent retardation device can be considered as a weak measurement of a polarization operator. In fact, this is a variation of the experiments which were proposed [68] and performed [70] previously. The "weakness condition" of these two experiments follows from the localization of the beam (which was sent through a narrow slit). The "weak" regime of the experiment of Suter is achieved by taking the retardation small. The second experiment of Suter can be considered as the first weak measurement of a nuclear spin component.

13.6 Time Symmetry

13.6.1 Forward- and Backward-Evolving Quantum States

Before discussing the time symmetry of the pre- and postselected systems which are usually discussed in the framework of the two-state vector formalism, we will consider the question of differences between possibilities for manipulating forward-evolving quantum states (13.1) and backward-evolving state (13.6) which has been recently analyzed [90]. It is particularly important in the light of recent argument of Shimony [91] against equal status of forward- and backward-evolving quantum states.

A notable difference between forward- and backward-evolving states has to do with the *creation* of a particular quantum state at a particular time. In order to create the quantum state $|A = a\rangle$ evolving forward in time, we measure A before this time. We cannot be sure to obtain $A = a$, but if we

obtain a different result $A = a'$ we can always perform a unitary operation and thus create at time t the state $|A = a\rangle$. On the other hand, in order to create the backward-evolving quantum state $\langle A = a|$, we measure A after time t. If we do not obtain the outcome $A = a$, we cannot repair the situation, since the correcting transformation has to be performed at a time when we do not yet know which correction is required. Therefore, a backward-evolving quantum state at a particular time can be created only with some probability, while a forward-evolving quantum state can be created with certainty. (Only if the forward-evolving quantum state is identical to the backward-evolving state we want to create at time t, and only if we know that no one touches the system at time t, can the backward-evolving state be created with certainty, since then the outcome $A = a$ occurs with certainty. But this is not an interesting case.)

The formalism of quantum theory is time reversal invariant. It does not have an intrinsic arrow of time. The difference with regard to the creation of backward and forward evolving quantum state follows from the "memory's" arrow of time. We can base our decision of what to do at a particular time only on events in the past, since future events are unknown to us. The memory time arrow is responsible for the difference in our ability to manipulate forward- and backward-evolving quantum states. However, the difference is only in relation to creation of the quantum state. As we will see below there are no differences with measurements in the sense of "finding out" what is the state at a particular time.

The ideal (von Neumann) measurement procedure applies both to forward evolving quantum states and to backward-evolving quantum states. In both cases, the outcome of the measurement is known after the time of the measurement. All that is known about what can be measured in an ideal (nondemolition) measurement of a forward-evolving quantum state can be applied also to a backward-evolving quantum state. There are constraints on the measurability of nonlocal variables, i.e., variables of composite systems with parts separated in space. When we consider instantaneous nondemolition measurements (i.e., measurements in which, in a particular Lorentz frame during an arbitrarily short time, local records appear which, when taken together, specify the eigenvalue of the nonlocal variable), we have classes of measurable and unmeasurable variables. For example, the Bell operator variable is measurable, while some other variables [92], including certain variables with product state eigenstates [93, 94], cannot be measured.

The procedure for measuring nonlocal variables involves entangled ancillary particles and local measurements, and can get quite complicated. Fortunately, there is no need to go into detail in order to show the similarity of the results for forward- and backward-evolving quantum states. The operational meaning of the statement that a particular variable A is measurable is that in a sequence of three consecutive measurements of A—the first taking a long time and possibly including bringing separate parts of the system to the same location and then returning them, the second being short and non-

local, and the third, like the first, consisting of bringing together the parts of the system—all outcomes have to be the same. But this is a time symmetric statement; if it is true, it means that the variable A is measurable both for forward- and backward-evolving quantum states.

We need also to obtain the correct probabilities in the case that different variables are measured at different times. For a forward-evolving quantum state it follows directly from the linearity of quantum mechanics. For a backward-evolving quantum state, the simplest argument is the consistency between the probability of the final measurement, which is now $B = b$, given the result of the intermediate measurement $A = a$, and the result of the intermediate measurement given the result of the final measurement. We assume that the past is erased. The expression for the former is $|\langle A = a|B = b\rangle|^2$. For consistency, the expression for the latter must be the same, but this is what we need to prove.

In exactly the same way we can show that the same procedure for *teleportation* of a forward-evolving quantum state [95] yields also teleportation of a backward-evolving quantum state. As the forward-evolving quantum state is teleported to a space–time point in the future light cone, the backward-evolving quantum state is teleported to a point in the backward light cone. Indeed, the operational meaning of teleportation is that the outcome of a measurement in one place is invariably equal to the outcome of the same measurement in the other place. Thus, the procedure for teleportation of the forward-evolving state to a point in the future light cone invariably yields teleportation of the backward-evolving quantum state to the backward light cone.

The impossibility of teleportation of the backward-evolving quantum state outside the backward light cone follows from the fact that it will lead to teleportation of the forward-evolving quantum state outside the forward light cone, and this is impossible since it obviously breaks causality.

Another result which has been proved using causality argument is the *no cloning theorem* for backward-evolving quantum quantum states [90]. So, also in this respect there is no difference between forward- and backward-evolving quantum states.

The argument used above does not answer the question of whether it is possible to measure nonlocal variables in a *demolition measurement*. Demolition measurements destroy (for the future) the state and may be the quantum systems itself. Thus, obviously, a demolition measurement of a nonlocal variable of a quantum state evolving forward in time does not measure this variable for a quantum state evolving backward in time. Any nonlocal variable of a composite system can be measured with demolition for a quantum state evolving forward in time [96]. Recently, it has been shown [97] also that any nonlocal variable can be measured for a quantum state evolving backward in time. Moreover, the procedure is simpler and requires fewer entanglement resources.

The difference follows from the fact that we can change the direction of time evolution of a backward-evolving state along with complex conjugation of

the quantum wave (flipping a spin). Indeed, all we need is to prepare an EPR state of our system and an ancilla. Guarding the system and the ancilla ensures that the forward-evolving quantum state of the ancilla is the flipped state of the system. For a spin wave function we obtain $\alpha\langle\uparrow| + \beta\langle\downarrow| \rightarrow -\beta^*|\uparrow\rangle + \alpha^*|\downarrow\rangle$. For a continuous variable wave function $\Psi(q)$ we need the original EPR state $|q - \tilde{q} = 0, \quad p + \tilde{p} = 0\rangle$. Then, the backward-evolving quantum state of the particle will transform into a complex conjugate state of the ancilla $\Psi(q) \rightarrow \Psi^*(\tilde{q})$.

If the particle and the ancilla are located in different locations, then such an operation is a combination of time reversal and teleportation of a backward-evolving quantum state of a continuous variable [98].

We cannot flip and change the direction of time evolution of a quantum state evolving forward in time. To this end we would have to perform a Bell measurement on the system and the ancilla and to get a particular result (singlet). However, we cannot ensure this outcome, nor can we correct the situation otherwise. Moreover, it is easily proven that no other method will work either. If one could have a machine which turns the time direction (and flips) a forward-evolving quantum state, then one could prepare at will any state that evolves toward the past, thus signaling to the past and contradicting causality.

Let us consider now a pre- and postselected system. It is meaningless to ask whether we can perform a nondemolition measurement on a system described by a two-state vector. Indeed, the vector describing the system should not be changed *after* the measurement, but there is no such time: for a forward-evolving state, "after" means later, whereas for a backward-evolving state, "after" means before. It is meaningful to ask whether we can perform a *demolition* measurement on a system described by a two-state vector. The answer is positive [97], even for composite systems with separated parts.

Next, is it possible to teleport a two-state vector? Although we can teleport both forward- and backward-evolving quantum states, we cannot teleport the two-state vector. The reason is that the forward evolving state can be teleported only to the future light cone, while the backward-evolving state can be teleported only to the backward light cone. Thus, there is no space–time point to which both states can be teleported.

Finally, the answer to the question of whether it is possible to clone a two-state vector is negative, since neither forward-evolving nor backward-evolving quantum states can be cloned.

13.6.2 Time-Symmetric Aspects of Pre- and Postselected Systems

When a quantum system is described by the two-state vector (13.2) or the generalized two-state vector (13.7), the backward-evolving states enter on equal footing with the forward-evolving states. Note that the asymmetry in the procedure for obtaining the state (13.7) is not essential: we can start preparing $1/\sqrt{N}\sum_i \ |\Phi_i\rangle|i\rangle$ instead.

We will analyze now the symmetry under the interchange $\langle\Phi|\,|\Psi\rangle \leftrightarrow \langle\Psi|\,|\Phi\rangle$. This will be considered as a symmetry under reversal of the direction of the arrow of time. It is important to note that in general this interchange is not equivalent to the interchange of the measurements creating the two-state vector $A = a$ and $B = b$. An example showing the nonequivalence can be found in [99]. However, in order to simplify the discussion, we will assume that the free Hamiltonian is zero, and therefore $|\Psi\rangle = |A = a\rangle$ and $\langle\Phi| = \langle B = b|$. In this case, of course, the reversal of time arrow is identical to the interchange of the measurements at t_1 and t_2. If the free Hamiltonian is not zero, then an appropriate modification should be made [100].

The ABL rule for the probabilities of the outcomes of ideal measurements (13.9) is also explicitly time-symmetric: First, both $\langle\Phi|$ and $|\Psi\rangle$ enter the equation on equal footing. Second, the probability (13.9) is unchanged under the interchange $\langle\Phi|\,|\Psi\rangle \leftrightarrow \langle\Psi|\,|\Phi\rangle$.

The ABL rule for a quantum system described by a generalized two-state vector (13.7) is time-symmetric as well: $\langle\Phi_i|$ and $|\Psi_i\rangle$ enter the equation on equal footing. The manifestation of the symmetry of this formula under the reversal of the arrow of time includes complex conjugation of the coefficients. The probability (13.10) is unchanged under the interchange $\sum_i \alpha_i\langle\Phi_i|\,|\Psi_i\rangle \leftrightarrow \sum_i \alpha_i^*\langle\Psi_i|\,|\Phi_i\rangle$.

The outcomes of weak measurements, the weak values, are also symmetric under the interchange $\langle\Phi|\,|\Psi\rangle \leftrightarrow \langle\Psi|\,|\Phi\rangle$ provided we perform complex conjugation of the weak value together with the interchange. This is similar to complex conjugation of the Schrödinger wave function under the time reversal. Thus, also for weak measurements there is the time reversal symmetry: both $\langle\Phi|$ and $|\Psi\rangle$ enter the formula of the weak value on the same footing and there is symmetry under the interchange of the pre- and postselected states. The time symmetry holds for weak values of generalized two-state vectors (13.22): i.e., the interchange $\sum_i \alpha_i\langle\Phi_i|\,|\Psi_i\rangle \leftrightarrow \sum_i \alpha_i^*\langle\Psi_i|\,|\Phi_i\rangle$ leads to $C_w \leftrightarrow C_w^*$.

13.6.3 The Time Asymmetry

The symmetry is also suggested in using the language of "preselected" state and "postselected" state. In order to obtain the two-state vector (13.2) we need to preselect $A = a$ at t_1 and postselect $B = b$ at t_2. Both measurements might not yield the desired outcomes, so we need several systems out of which we pre- and postselect the one which is described by the two-state vector (13.2). However, the symmetry is not complete and the language might be somewhat misleading. It is true that we can only (post)select $B = b$ at t_2, but we can *prepare* instead of preselect $A = a$ at t_1. For preparation of $|a\rangle$ a single system is enough. If the measurement of A yields a different outcome a' we can perform a fast unitary operation which will change $|A = a'\rangle$ to $|A = a\rangle$ and then the time evolution to time t will bring the system to the state $|\Psi\rangle$. This procedure is impossible for creation of the backward-evolving state $\langle\Phi|$. Indeed, if the outcome of the measurement of B does not yield b,

we cannot read it and then make an appropriate unitary operation *before* t_2 in order to get the state $\langle \Phi |$ at time t. We need several systems to post-select the desired result (unless by chance the first system has the desired outcome).

Although the formalism includes situations with descriptions by solely forward-evolving quantum state and by solely backward-evolving quantum states, here also there is a conceptual difference. For obtaining backward-evolving state it was necessary to have a guarded ancilla in order to erase the quantum state evolving from the past. Of course, there is no need for this complication in obtaining forward-evolving quantum state. The difference is due to fixed "memory" arrow of time: we know the past and we do not know the future. This asymmetry is also connected to the concept of a measurement. It is asymmetric because, by definition, we do not know the measured value before the measurement and we do know it after the measurement.

13.6.4 If *Measurements* are Time-Asymmetric, How the Outcomes of Measurements are Time-Symmetric?

Taking this asymmetry of the concept of measurement into account, how one can understand the time symmetry of the formulae for the probability of an intermediate measurements (13.9), (13.10) and for the formulae of weak values (13.21), (13.22)?

This is because these formulae deal with the results of the measurements which, in contrast with the concept of measurement itself, are free from the time asymmetry of a measurement. The results of measurements represent the way the system affects other systems (in this case measuring devices) and these effects, obviously, do not exhibit the time asymmetry of our memory. The time asymmetry of measurement is due to the fact that the pointer variable of the measuring device is showing "zero" mark before the measurement and not after the measurement. But the result of the measurement is represented by the *shift* of the pointer position. (If originally the pointer showed "zero" it is also represented by the final position of the pointer.) This shift is independent of the initial position of the pointer and therefore it is not sensitive to the time asymmetry caused by asymmetrical fixing of the initial (and not final) position of the pointer. The relations described in the formulae of the two-state vector formalism are related to these shifts and, therefore, the time symmetry of the formulae follows from the underlying time symmetry of the quantum theory. The shifts of the pointer variable in weak measurements were considered as "weak-measurements elements of reality" [101] where "elements of reality were identified with "definite shifts." This approach was inspired by the EPR elements of reality which are definite outcomes of ideal measurements, i.e., definite shifts in ideal measurement procedures. The next section discusses a controversy related to ideal measurements.

13.6.5 Counterfactual Interpretation of the ABL Rule

Several authors criticized the TSVF because of the alleged conflict between counterfactual interpretations of the ABL rule and predictions of quantum theory [102, 20, 103, 104]. The form of all these inconsistency proofs is as follows: The probability of an outcome $C = c_n$ of a quantum measurement performed on a preselected system, given correctly by (13.12), is considered. In order to allow the analysis using the ABL formula, a measurement at a later time, t_2, with two possible outcomes, which we denote by "1_f" and "2_f," is introduced. The suggested application of the ABL rule is expressed in the formula for the probability of the result $C = c_n$

$$\begin{aligned} \mathrm{Prob}(C = c_n) = \mathrm{Prob}(1_f)\ \mathrm{Prob}(C = c_n\ ; 1_f) \\ +\mathrm{Prob}(2_f)\ \mathrm{Prob}(C = c_n\ ; 2_f)\ , \end{aligned} \tag{13.69}$$

where $\mathrm{Prob}(C = c_n\ ; 1_f)$ and $\mathrm{Prob}(C = c_n\ ; 2_f)$ are the conditional probabilities given by the ABL formula, (13.9), and $\mathrm{Prob}(1_f)$ and $\mathrm{Prob}(2_f)$ are the probabilities of the results of the final measurement. In the proofs, the authors show that (13.69) is not valid and conclude that the ABL formula is not applicable to this example and therefore it is not applicable in general.

One us (L.V.) has argued [105, 106, 21] that the error in calculating equality (13.69) does not arise from the conditional probabilities given by the ABL formula, but from the calculation of the probabilities $\mathrm{Prob}(1_f)$ and $\mathrm{Prob}(2_f)$ of the final measurement. In all three alleged proofs, the probabilities $\mathrm{Prob}(1_f)$ and $\mathrm{Prob}(2_f)$ were calculated on the assumption that no measurement took place at time t. Clearly, one cannot make this assumption here since then the discussion about the probability of the result of the measurement at time t is meaningless. Thus, it is not surprising that the value of the probability $\mathrm{Prob}(C = c_n)$ obtained in this way comes out different from the value predicted by the quantum theory. Straightforward calculations show that the formula (13.69) with the probabilities $\mathrm{Prob}(1_f)$ and $\mathrm{Prob}(2_f)$ calculated on the condition that the intermediate measurement has been performed leads to the result predicted by the standard formalism of quantum theory.

The analysis of counterfactual statements considers both *actual* and *counterfactual* worlds. The statement is considered to be true if it is true in counterfactual worlds "closest" to the actual world. In the context of the ABL formula, in the actual world the preselection and the postselection has been successfully performed, but the measurement of C has not (necessarily) been performed. On the other hand, in counterfactual worlds the measurement of C has been performed. The problem is to find counterfactual worlds "closest" to the actual world in which the measurement of C has been performed. The fallacy in all the inconsistency proofs is that their authors have considered counterfactual worlds in which C has not been measured.

Even if we disregard this fallacy there is still a difficulty in defining the "closest" worlds in the framework of the TSVF. In standard quantum theory

it is possible to use the most natural definition of the "closest" world. Since the future is considered to be irrelevant for measurements at present time t, only the period of time before t is considered. Then the definition is:

> (i) Closest counterfactual worlds are the worlds in which the system is described by the same quantum state as in the actual world.

In the framework of the TSVF, however, this definition is not acceptable. In the time-symmetric approach the period of time before and after t is considered. The measurement of C constrains the possible states immediately after t to the eigenstates of C. Therefore, if in the actual world the state immediately after t is not an eigenstate of C, no counterfactual world with the same state exists. Moreover, there is the same problem with the backward- evolving quantum state (the concept which does not exist in the standard approach) in the period of time before t. This difficulty can be solved by adopting the following definition of the closest world [106]:

> (ii) Closest counterfactual worlds are the worlds in which the results of all measurements performed on the system (except the measurement at time t) are the same as in the actual world.

For the preselected only situation, this definition is equivalent to (i), but it is also applicable to the symmetric pre- and postselected situation. The definition allows to construct *time-symmetric counterfactuals* in spite of common claims that such concept is inconsistent [107].

An important example of counterfactuals in quantum theory are "elements of reality" which are inspired by the EPR elements of reality. The modification of the definition of elements of reality applicable to the framework of the TSVF [34] is:

> (iii) If we can *infer* with certainty that the result of measuring at time t of an observable C is c, then, at time t, there exists an element of reality $C = c$.

The word "infer" is neutral relative to past and future. The inference about results at time t is based on the results of measurements on the system performed both before and after time t. Note that there are situations (e.g., the three-boxes example) in which we can "infer" some facts that cannot be obtained by neither "prediction" based on the past results nor "retrodiction" based on the future results separately.

The theorem (i) of Sect. 13.4.4 now can be formulated in a simple way: If $A = a$ is an element of reality then $A_w = a$ is the weak-measurement of reality. The theorem (ii) of Sect. 13.4.4 can be formulated as follows. If A is a dichotomic variable, a is an eigenvalue of A, and if $A_w = a$ is a weak-measurement element of reality, then $A = a$ is an element of reality.

The discussion about the meaning of time symmetric counterfactuals continues until today. Kastner changed her view on such counterfactuals from "inconsistent" to "trivial" [108]. See Vaidman's reply [109] and other very recent contributions on this issue [110, 111, 112].

13.7 Protective Measurements

Several years ago we proposed a concept of *protective measurements* [113, 114, 115] which provides an argument strengthening the consideration of a quantum state as a "reality" of some kind. We have shown that "protected" quantum states can be observed just on a single quantum system. On the other hand, if a single quantum state is "the reality" how "the two-state vector" can be "the reality"?

13.7.1 Protective Measurement of a Single Quantum State

In order to measure the quantum state of single system one has to measure expectation values of various observables. In general, the weak (expectation) value cannot be measured on a single system. However, it can be done if the quantum state is *protected* [113, 114]. The appropriate measurement interaction is again described the Hamiltonian (13.8), but instead of an impulsive interaction the adiabatic limit of slow and weak interaction is considered: $g(t) = 1/T$ for most of the interaction time T and $g(t)$ goes to zero gradually before and after the period T.

In this case the interaction Hamiltonian does not dominate the time evolution during the measurement, moreover, it can be considered as a perturbation. The free Hamiltonian H_0 dominates the evolution. In order to protect a quantum state this Hamiltonian must have the state to be a nondegenerate energy eigenstate. For $g(t)$ smooth enough we then obtain an adiabatic process in which the system cannot make a transition from one energy eigenstate to another, and, in the limit $T \to \infty$, the interaction Hamiltonian changes the energy eigenstate by an infinitesimal amount. If the initial state of the system is an eigenstate $|E_i\rangle$ of H_0 then for any given value of P, the energy of the eigenstate shifts by an infinitesimal amount given by the first-order perturbation theory: $\delta E = \langle E_i|H_{int}|E_i\rangle = \langle E_i|A|E_i\rangle P/T$. The corresponding time evolution $e^{-iP\langle E_i|A|E_i\rangle}$ shifts the pointer by the expectation value of A in the state $|E_i\rangle$. Thus, the probability distribution of the pointer variable, $e^{-(Q-a_i)^2/\Delta^2}$ remains unchanged in its shape, and is shifted by the expectation value $\langle A\rangle_i = \langle E_i|A|E_i\rangle$.

If the initial state of the system is a superposition of several nondegenerate energy eigenstates $|\Psi_1\rangle = \Sigma\alpha_i|E_i\rangle$, then a particular outcome $\langle A\rangle_i \equiv \langle E_i|A|E_i\rangle$ appears at random, with the probability $|\alpha_i|^2$ [61]. (Subsequent adiabatic measurements of the same observable A invariably yield the expectation value in the same eigenstate $|E_i\rangle$.)

13.7.2 Protective Measurement of a Two-State Vector

At first sight, it seems that protection of a two-state vector is impossible. Indeed, if we add a potential that makes one state a nondegenerate eigenstate,

then the other state, if it is different, cannot be an eigenstate too. (The states of the two-state vector cannot be orthogonal.) But, nevertheless, protection of the two-state vector is possible [116].

The procedure for protection of a two-state vector of a given system is accomplished by coupling the system to another pre- and postselected system. The protection procedure takes advantage of the fact that weak values might acquire complex values. Thus, the effective Hamiltonian of the protection might not be Hermitian. Non-Hermitian Hamiltonians act in different ways on quantum states evolving forward and backwards in time. This allows simultaneous protection of two different states (evolving in opposite time directions).

Let us consider an example [116] of a two-state vector of a spin-$\frac{1}{2}$ particle, $\langle\uparrow_y||\uparrow_x\rangle$. The protection procedure uses an external pre- and postselected system S of a large spin N that is coupled to our spin via the interaction

$$H_{prot} = -\lambda \mathbf{S} \cdot \sigma \ . \tag{13.70}$$

The external system is preselected in the state $|S_x{=}N\rangle$ and postselected in the state $\langle S_y{=}N|$, that is, it is described by the two-state vector $\langle S_y{=}N||S_x{=}N\rangle$. The coupling constant λ is chosen in such a way that the interaction with our spin-$\frac{1}{2}$ particle cannot change significantly the two-state vector of the protective system S, and the spin-$\frac{1}{2}$ particle "feels" the effective Hamiltonian in which S is replaced by its weak value,

$$\mathbf{S}_w = \frac{\langle S_y = N|(S_x, S_y, S_z)|S_x = N\rangle}{\langle S_y = N|S_x = N\rangle} = (N, N, iN) \ . \tag{13.71}$$

Thus, the effective protective Hamiltonian is

$$H_{eff} = -\lambda N(\sigma_x + \sigma_y + i\sigma_z) \ . \tag{13.72}$$

The state $|\uparrow_x\rangle$ is an eigenstates of this (non-Hermitian) Hamiltonian (with eigenvalue $-\lambda N$). For backward-evolving states the effective Hamiltonian is the hermitian conjugate of (13.72) and it has different (nondegenerate) eigenstate with this eigenvalue; the eigenstate is $\langle\uparrow_y|$.

In order to prove that the Hamiltonian (13.70) indeed provides the protection, we have to show that the two-state vector $\langle\uparrow_y||\uparrow_x\rangle$ will remain essentially unchanged during the measurement. We consider measurement which is performed during the period of time, between pre- and postselection which we choose to be equal one. The Hamiltonian

$$H = -\lambda \mathbf{S} \cdot \sigma + P\sigma_\xi \tag{13.73}$$

can be replaced by the effective Hamiltonian

$$H_{eff} = -\lambda N(\sigma_x + \sigma_y + i\sigma_z) + P\sigma_\xi \ . \tag{13.74}$$

Indeed, the system with the spin S can be considered as N spin-1/2 particles all preselected in $|\uparrow_x\rangle$ state and postselected in $|\uparrow_y\rangle$ state. The strength of the coupling to each spin-1/2 particle is $\lambda \ll 1$, therefore during the time of the measurement their states cannot be changed significantly. Thus, the forward-evolving state $|S_x{=}N\rangle$ and the backward-evolving state $\langle S_y{=}N|$ do not change significantly during the measuring process. The effective coupling to such system is the coupling to its weak values.

Good precision of the measurement of the spin component requires large uncertainty in P, but we can arrange the experiment in such a way that $P \ll \lambda N$. Then the second term in the Hamiltonian (13.74) will not change significantly the eigenvectors. The two-state vector $\langle\uparrow_y||\uparrow_x\rangle$ will remain essentially unchanged during the measurement, and therefore the measuring device on this single particle will yield $(\sigma_\xi)_w = \frac{\langle\uparrow_y|\sigma_\xi|\uparrow_x\rangle}{\langle\uparrow_y|\uparrow_x\rangle}$.

The Hamiltonian (13.73), with an external system described by the two-state vector $\langle S_y = N||S_x = N\rangle$, provides protection for the two-state vector $\langle\uparrow_y||\uparrow_x\rangle$. It is not difficult to demonstrate that any two-state vector obtained by pre- and postselection of the spin-$\frac{1}{2}$ particle can be protected by the Hamiltonian (13.73). A general form of the two-state vector is $\langle\uparrow_\beta||\uparrow_\alpha\rangle$ where $\hat{\alpha}$ and $\hat{\beta}$ denote some directions. It can be verified by a straightforward calculation that the two-state vector $\langle\uparrow_\beta||\uparrow_\alpha\rangle$ is protected when the two-state vector of the protective device is $\langle S_\beta = N||S_\alpha = N\rangle$.

At least formally we can generalize this method to make a protective measurement of an arbitrary two-state vector $\langle\Psi_2||\Psi_1\rangle$ of an arbitrary system. Let us decompose the post-selected state $|\Psi_2\rangle = a|\Psi_1\rangle + b|\Psi_\perp\rangle$. Now we can define "model spin" states: $|\Psi_1\rangle \equiv |\tilde{\uparrow}_z\rangle$ and $|\Psi_\perp\rangle \equiv |\tilde{\downarrow}_z\rangle$. On the basis of the two orthogonal states we can obtain all other "model spin" states. For example, $|\tilde{\uparrow}_x\rangle = 1/\sqrt{2}\ (|\tilde{\uparrow}_z\rangle + |\tilde{\downarrow}_z\rangle)$, and then we can define the "spin model" operator $\tilde{\sigma}$. Now, the protection Hamiltonian, in complete analogy with the spin-$\frac{1}{2}$ particle case is

$$H_{prot} = -\lambda \mathbf{S} \cdot \tilde{\sigma}\ . \tag{13.75}$$

In order to protect the state $\langle\Psi_2||\Psi_1\rangle$, the pre-selected state of the external system has to be $|S_z{=}N\rangle$ and the postselected state has to be $\langle S_\chi{=}N|$ where the direction $\hat{\chi}$ is defined by the "spin model" representation of the state $|\Psi_2\rangle$,

$$|\tilde{\uparrow}_\chi\rangle \equiv |\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle|\tilde{\uparrow}_z\rangle + \langle\Psi_\perp|\Psi_2\rangle|\tilde{\downarrow}_z\rangle\ . \tag{13.76}$$

However, this scheme usually leads to unphysical interaction and is good only as a gedanken experiment in the framework of nonrelativistic quantum theory where we assume that any Hermitian Hamiltonian is possible.

13.8 The TSVF and the Many-Worlds Interpretation of Quantum Theory

The TSVF fits very well into the many-worlds interpretation (MWI) [117], the preferred interpretation of quantum theory of one of us (L.V.) [118]. The

counterfactual worlds corresponding to different outcomes of quantum measurements have in the MWI an especially clear meaning: these are subjectively actual different worlds. In each world, the observers of the quantum measurement call their world the actual one, but, if they believe in the MWI they have no paradoxes about ontology of the other worlds. The apparent paradox that a weak value at a given time might change from an expectation value to a weak value corresponding to a particular postselection is solved in a natural way: in a world with pre-selection only (before the postselection) the weak value is the expectation value; then this world splits into several worlds according to results of the postselection measurement and in each of these worlds the weak value will be that corresponding to the particular postselection. The time-symmetric concepts of "elements of reality," "weak-measurements elements of reality" are consistent and meaningful in the context of a particular world. Otherwise, at time t, before the "future" measurements have been performed, the only meaningful concepts are the concepts of the standard, time-asymmetric approach.

One of us (Y.A.) is not ready to adopt the far reaching consequences of the MWI. He proposes another solution [119]. It takes the TSVF even more seriously than it was presented in this paper. Even at present, before the "future" measurements, the backward evolving quantum state (or its complex conjugate evolving forward in time) exists! It exists in the same way as the quantum state evolving from the past exists. This state corresponds to particular outcomes of all measurements in the future. An element of arbitrariness: "Why this particular outcome and not some other?" might discourage, but the alternative (without the many-worlds)—the collapse of the quantum wave—is clearly worse than that.

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References

1. Y. Aharonov, P.G. Bergmann, J.L. Lebowitz: Phys. Rev. B **134**, 1410 (1964)
2. Y. Aharonov, D. Albert, A. Casher, L. Vaidman: Phys. Lett. A **124**, 199 (1987)
3. Y. Aharonov, L. Vaidman: Phys. Rev. A **41**, 11 (1990)
4. Y. Aharonov, L. Vaidman: J. Phys. A **24**, 2315 (1991)
5. L. Vaidman, Y. Aharonov, D. Albert: Phys. Rev. Lett. **58**, 1385 (1987)

6. Y. Aharonov, D. Albert, L. Vaidman: Phys. Rev. Lett. **60**, 1351 (1988)
7. Y. Aharonov, J. Anandan, S. Popescu, L. Vaidman: Phys. Rev. Lett. **64**, 2965 (1990)
8. Y. Aharonov, S. Popescu, D. Rohrlich, L. Vaidman: Phys. Rev. A **48**, 4084 (1993)
9. M.B. Berry: J. Phys. A **27**, L391 (1994)
10. M.B. Berry, S. Popescu: J. Phys. A **39**, 6965 (2006)
11. S.M. Barnett, D.T. Pegg, J. Jeffers, O. Jedrkiewicz, R. Loudon: Phys. Rev. A **62**, 022313 (2000)
12. S.M. Barnett, D.T. Pegg, J. Jeffers, O. Jedrkiewicz: J. Phys. B **33**, 3047 (2000)
13. S.M. Barnett, D.T. Pegg, J. Jeffers: J. Mod. Opt. **47**, 1779 (2000)
14. D.T. Pegg, S.M. Barnett, J. Jeffers: Phys. Rev. A **66**, 022106 (2002)
15. D.T. Pegg, K.L. Pregnell: J. Mod. Opt. **51**, 1613 (2004)
16. R.B. Griffiths: J. Stat. Phys. **36**, 219 (1984)
17. M. Gell-Mann, J.B. Hartle: in *Physical Origins of Time Asymmetry*, ed. by J. Halliwell et al. (Cambridge University Press, Cambridge, 1994)
18. B. Reznik, Y. Aharonov: Phys. Rev. A **52**, 2538 (1995)
19. J. von Neumann: *Mathematical Foundations of Quantum Theory* (Princeton University Press, Princeton, NJ, 1983)
20. O. Cohen: Phys. Rev. A **51**, 4373 (1995)
21. L. Vaidman: Phys. Rev. A **57**, 2251 (1998)
22. Y. Aharonov, L. Vaidman: Phys. Rev. A **67**, 042107 (2003)
23. R.B. Griffiths: Phys. Rev. A **54**, 2759 (1996); **57**, 1604 (1998)
24. A. Kent: Phys. Rev. Lett. **78**, 2874 (1997)
25. R.B. Griffiths: *Consistent Quantum Theory* (Cambridge University Press, Cambridge, 2003)
26. R.E. Kastner: Found. Phys. **29**, 851 (1999)
27. L. Vaidman: Found. Phys. **29**, 865 (1999)
28. K.A. Kirkpatrick: J. Phys. A: Math. Theor. **36**, 4891 (2003)
29. T. Ravon, L. Vaidman: J. Phys. A: Math. Theor. **40**, 2873 (2007)
30. K.A. Kirkpatrick: J. Phys. A: Math. Theor. **40**, 2883 (2007)
31. C.R. Leavens, I. Puerto Giménez, D. Alonso, R. Sala Mayato: Phys. Lett. A **359**, 416 (2006)
32. O. Hosten ,M.T. Rakher, J.T. Barreiro, N.A. Peters, P.G. Kwiat: Nature **439**, 949 (2006)
33. L. Vaidman: The Impossibility of the Counterfactual Computation for all Possible Outcomes. Phys. Rev. Lett. **98**, 160403(2006). quant-ph/0610174
34. L. Vaidman: Phys. Rev. Lett. **70**, 3369 (1993)
35. L. Vaidman: ' "Elements of Reality" and the Failure of the Product Rule'. In *Symposium on the Foundations of Modern Physics*, ed. by P.J. Lahti, P. Bush, P. Mittelstaedt (World Scientific, Cologne, 1993) pp. 406–417
36. Y. Aharonov, D. Albert, L. Vaidman: Phys. Rev. D **34**, 1805 (1986)
37. L. Hardy: Phys. Rev. Lett. **68**, 2981 (1992)
38. S. Marcovitch, B. Reznik, L. Vaidman: Phys. Rev. A **75**, 022102 (2007)
39. S. Popescu, D. Rohrlich: Found. Phys. **24**, 379 (1994)
40. S. Benmenahem: Phys. Rev. A **39**, 1621 (1989)
41. D. Mermin: Phys. Rev. Lett. **74**, 831 (1995)
42. Y. Aharonov, B.G. Englert: Zeit. Natur. A **56** 16 (2001)
43. B.G. Englert, Y. Aharonov: Phys. Lett. A **284**, 1 (2001)

44. P.K. Aravind: Z. Natur. A **58**, 85 (2003)
45. P.K. Aravind: Z. Natur. A **58**, 682 (2003)
46. M. Horibe, A. Hayashi, T. Hashimoto: Phys. Rev. A **71**, 032337 (2005)
47. A. Hayashi, M. Horibe, T. Hashimoto: Phys. Rev. A **71**, 052331 (2005)
48. J.P. Paz, A.J. Roncaglia, M.A. Saraceno: Phys. Rev. A **72**, 012309 (2005)
49. G. Kimura, H. Tanaka, M. Ozawa: Phys. Rev. A **73**, 050301 (2006)
50. T. Durt: Int. J. Mod. Phys. B **20**, 1742 (2006)
51. O. Schulz, R. Steinhbl, M. Weber, B. Englert, C. Kurtsiefer, H. Weinfurter: Phys. Rev. Lett. **90**, 177901 (2003)
52. A. Botero: Ph.D. Thesis, The University of Texas at Austin (2003). quant-ph/0306082
53. Y. Aharonov, A. Botero: Phys. Rev. A **72**, 052111 (2005)
54. A. Tanaka: Phys. Lett. A **297**, 307 (2002)
55. H.M. Wiseman: Phys. Rev. A **65**, 032111 (2002)
56. A.D. Parks: J. Phys. A **36**, 7185 (2003)
57. L.M. Johansen: Phys. Lett. A **322** 298 (2004)
58. L.M. Johansen: Phys. Rev. Lett. **93**, 120402 (2004)
59. L.M. Johansen, A. Luis: Phys. Rev. A **70**, 052115 (2004)
60. O. Oreshkov, T. Brun: Phys. Rev. Lett. **95**, 110409 (2005)
61. W. Unruh: Ann. N.Y. Acad. Sci. **755**, 560 (1995)
62. K.J. Resch, J.S. Lundeen, A.M. Steinberg: Phys. Lett. A **324**, 125 (2004)
63. Y. Aharonov, A. Botero, S. Popescu, B. Reznik, J. Tollaksen: Phys. Lett. A **301**, 130 (2002)
64. K. Molmer: Phys. Lett. A **292**, 151 (2001)
65. S.E. Ahnert, M.C. Payne: Phys. Rev. A **70**, 042102 (2004)
66. J.S. Lundeen, K.J. Resch, A.M. Steinberg: Phys. Rev. A **72**, 016101 (2005)
67. S.E. Ahnert, M.C. Payne: Phys. Rev. A **72**, 016102 (2005)
68. J.M. Knight, L. Vaidman: Phys. Lett. A **143**, 357 (1990)
69. M. Duck, P.M. Stevenson, E.C.G. Sudarshan: Phys. Rev. D **40**, 2112 (1989)
70. N.W.M. Ritchie, J.G. Story, R.G. Hulet: Phys. Rev. Lett. **66**, 1107 (1991)
71. A.D. Parks, D.W. Cullin, D.C. Stoudt: Proc. R. Soc. Lond A **454**, 2997 (1998)
72. N. Brunner, A. Acin, D. Collins, N. Gisin, V. Scarani: Phys. Rev. Lett. **93**, 180402 (2003)
73. N. Brunner, V. Scarani, M. Wegmuller, M. Legre, N. Gisin: Phys. Rev. Lett. **93**, 203902 (2004)
74. D.R. Solli, C.F. McCormick, R.Y. Chiao, S. Popescu, J.M. Hickmann: Phys. Rev. Lett. **92**, 043601 (2004)
75. G.L. Pryde, J.L. O'Brien, A.G. White, T.C. Ralph, H.M. Wiseman: Phys. Rev. Lett. **94**, 220405 (2005)
76. Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, W.G. Unruh: Phys. Rev. A **57**, 4130 (1998)
77. J. Ruseckas, B. Kaulakys: Phys. Rev. A **66**, 052106 (2002)
78. S.E. Ahnert, M.C. Payne: Phys. Rev. A **69**, 042103 (2004)
79. Q. Wang, F.W. Sun, Y.S. Zhang, Jian-Li, Y.F. Huang, G.C. Guo: Phys. Rev. A **73**, 023814 (2006)
80. A.M. Steinberg: Phys. Rev. Lett. **74**, 2405 (1995)
81. R.Y. Chiao, A.M. Steinberg: Prog. Opt. **37**, 345 (1997)
82. Y. Aharonov, N. Erez, B. Reznik: Phys. Rev. A **65**, 052124 (2002)
83. D. Sokolovski, A.Z. Msezane, V.R. Shaginyan: Phys. Rev. A **71**, 064103 (2005)

84. D. Rohrlich, Y. Aharonov: Phys. Rev. A **66**, 042102 (2002)
85. K.J. Resch, A.M. Steinberg: Phys. Rev. Lett. **92**, 130402 (2004)
86. J.S. Lundeen, K.J. Resch: Phys. Lett. A **334**, 337 (2005)
87. L. Vaidman: Found. Phys. **21**, 947 (1991)
88. D. Suter: Phys. Rev. A **51**, 45 (1995)
89. D. Suter, M. Ernst, R.R. Ernst: Molec. Phys. **78**, 95 (1993)
90. L. Vaidman: J. Phys. A: Math. Theor. **40**, 3275 (2007)
91. A. Shimony: Found. Lett. **35**, 215 (2005)
92. S. Popescu, L. Vaidman: Phys. Rev. A **49**, 4331 (1994)
93. C.H. Bennett, D.P. DiVincenzo, C.A. Fuchs, T. Mor, E. Rains, P.W. Shor, J.A. Smolin, W.K. Wootters: Phys. Rev. A **59**, 1070 (1999)
94. B. Groisman, L. Vaidman: J. Phys. A: Math. Gen. **34**, 6881 (2001)
95. C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters: Phys. Rev. Lett. **70**, 1895 (1993)
96. L. Vaidman: Phys. Rev. Lett. **90**, 010402 (2003)
97. L. Vaidman, I. Nevo: Int. J. Mod. Phys. B **20**, 1528 (2006)
98. L. Vaidman: Phys. Rev. A **49**, 1473 (1994)
99. A. Shimony: Erken. **45**, 337 (1997)
100. L. Vaidman: Fortschr. Phys. **46**, 729 (1998)
101. L. Vaidman: Found. Phys. **26**, 895 (1996)
102. W.D. Sharp, N. Shanks: Phil. Sci. **60**, 488 (1993)
103. D.J. Miller: Phys. Lett. A **222**, 31 (1996)
104. R. E. Kastner: Stud. Hist. Phil. Mod. Phys. B **30**, 237 (1999)
105. L. Vaidman: Stud. Hist. Phil. Mod. Phys. **30**, 373 (1999)
106. L. Vaidman: Found. Phys. **29**, 755 (1999)
107. D. Lewis: Nous **13**, 455 (1979); reprinted in *Philosophical Papers Vol II* (Oxford University Press, Oxford 1986) p. 32
108. R.E. Kastner: Phil. Sci. **70**, 145 (2003)
109. L. Vaidman: preprint. http://philsci-archive.pitt.edu/archive/00001108/
110. R.E. Kastner: Stud. Hist. Philos. Mod. Phys. B **35**, 57 (2004)
111. D.J. Miller: Found. Phys. Lett. **19**, 321 (2006)
112. M.S. Leifer, R.W. Spekkens: Phys. Rev. Lett. **95**, 200405 (2005)
113. Y. Aharonov, L. Vaidman: Phys. Lett. A **178**, 38 (1993)
114. Y. Aharonov, J. Anandan, L. Vaidman: Phys. Rev. A **47**, 4616 (1993)
115. Y. Aharonov, J. Anandan, L. Vaidman: Found. Phys. **26**, 117 (1996)
116. Y. Aharonov, L. Vaidman: 'Protective Measurements of Two-State Vectors'. In *Potentiality, Entanglement and Passion-at-a-Distance*, ed. by R.S. Cohen et al. (Kluwer Academic Publishers, Dordrecht, 1997) pp. 1–8
117. H. Everett: Rev. Mod. Phys. **29**, 454 (1957)
118. L. Vaidman: 'Many-Worlds Interpretation of Quantum Mechanics', in *The Stanford Encyclopedia of Philosophy (Summer 2002 Edition)*, ed. by E. N. Zalta <http://plato.stanford.edu/archives/sum2002/entries/qm-anyworlds/>.
119. Y. Aharonov, E. Y. Gruss: Two-time interpretation of quantum mechanics (2005). quant-ph/0507269