Moderate deviations for random fields and random complex zeroes

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What is the most remarkable stationary random point process on the plane? The Poisson process is *hors concours;* being invariant (in distribution) under all measure preserving transformations of the plane, it is unrelated to the geometry of the plane. Chaotic analytic zero points (CAZP), actively investigated the last 10 years, are a point process on the complex plane $\mathbb{C}$ invariant under isometries (shifts, rotations, reflections) of $\mathbb{C}$. It consists of zeroes of the random entire function $\psi : \mathbb{C} \to \mathbb{C}$ defined by

$$\psi(z) = \sum_{k=0}^{\infty} \frac{\zeta_k z^k}{\sqrt{k!}} \,,$$

where $\zeta_0, \zeta_1, \dots$ are independent standard complex Gaussian random variables. (The process ψ is not stationary, but its zeroes are.)

On small distances the zeroes repel each other, like particles of the one-component plasma (OCP; not to be confused with the two-component Coulomb gas). On large distances the three models are related as follows:

Poisson — OCP — CAZP

in the sense explained below, see (4).

We consider so-called smooth linear statistics $Z(h)$ and $Z_0(h)$, — random variables indexed by compactly supported C^2-smooth test functions $h : \mathbb{C} \to \mathbb{R}$, — defined by

$$Z(h) = \sum_{z:\psi(z)=0} h(z) \,; \quad Z_0(h) = Z(h) - \mathbb{E}\, Z(h) = Z(h) - \frac{1}{\pi} \int h \,\mathrm{d}m$$

(m being Lebesgue measure on $\mathbb{C}$). We introduce rescaled test functions h_r,

$$h_r(z) = h\Big(\frac{1}{r} z\Big) \quad \text{for } r \in (0,\infty),\ z \in \mathbb{C} \,,$$

and examine the asymptotic behavior of $Z_0(h_r)$ as $r \to \infty$.

Asymptotic normality [6]: there exists an absolute constant $\sigma \in (0,\infty)$ such that for every test function h and every $c \in \mathbb{R}$,

$$\mathbb{P}\bigg(\frac{r}{\sigma \|\Delta h\|} Z_0(h_r) > c \bigg) \to \frac{1}{\sqrt{2\pi}} \int_c^\infty \mathrm{e}^{-x^2/2} \,\mathrm{d}x \quad \text{as } r \to \infty \,; \tag{1}$$

here Δ is the Laplace operator, and $\| \cdot \|$ is the norm in $L_2(\mathbb{C})$.

Note that $Z_0(h_r)$ is multiplied by r, not divided by r. This is not a mistake! Large-scale fluctuations in CAZP are much smaller than in the Poisson process.

Moderate deviations [7]:

$$\ln \mathbb{P}\bigg(\frac{r}{\sigma \|\Delta h\|} Z_0(h_r) > c \bigg) \sim \ln \bigg(\frac{1}{\sqrt{2\pi}} \int_c^\infty \mathrm{e}^{-x^2/2} \,\mathrm{d}x \bigg) \tag{2}$$

as $r \to \infty$, $\frac{c \log^2 r}{r} \to 0$.

That is, for every ε there exist R and δ such that the ratio of the left-hand side to the right-hand side is ε-close to 1 for all $r \ge R$ and all c such that $|c|\frac{\log^2 r}{r} \le \delta$.

The asymptotic normality is a special case, $c = \text{const}$. In the case $c \to \infty$ the right-hand side may be replaced with $(-c^2/2)$. (For the other tail, note that $Z_0(-h) = -Z_0(h)$.)

It is natural to expect that (2) holds whenever $\frac{c}{r} \to 0$; higher c could satisfy a large deviations principle, with a non-quadratic rate function.

Question. Does (2) hold when $\frac{r}{\log^2 r} \ll c \ll r$?

LINEAR RESPONSE [7]:

$$\frac{1}{r^2\varepsilon^2} \ln \mathbb{E} \exp\bigl(\varepsilon r^2 Z_0(h_r)\bigr) \to \frac{\sigma^2}{2}\|\Delta h\|^2 \tag{3}$$

as $r \to \infty$, $\varepsilon \log^2 r \to 0$.

By the Gärtner (-Ellis) theorem, (3) implies (2).

I like to call (3) 'linear response principle' for the following reason. In the spirit of equilibrium statistical physics (by analogy with Gibbs measures) we may treat the test function h as a (given, nonrandom) external field that multiplies all probabilities by $\frac{\exp Z_0(h)}{\mathbb{E}\exp Z_0(h)}$ (as if $Z_0(h)$ was subtracted from the Hamiltonian; the inverse temperature $\beta = 1$ is assumed). The response of the observable $Z_0(h)$ to this external field is the new average of $Z_0(h)$ (according to the new probabilities), equal to

$$\mathbb{E}\left(Z_0(h) \cdot \frac{\exp Z_0(h)}{\mathbb{E}\exp Z_0(h)} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=1} \ln \mathbb{E}\exp \lambda Z_0(h)\,.$$

Taking into account that the function $\lambda \mapsto \ln \mathbb{E}\exp\lambda Z_0(h)$ is convex, we see that the response is approximately linear if and only if $\ln \mathbb{E}\exp Z_0(h)$ is approximately quadratic (in h, when h is small in an appropriate sense).

Physical intuition suggests that the linear response principle should hold under quite general conditions. Accordingly, the moderate deviations principle (MDP) for random fields should hold under quite general conditions. In contrast, available general results on MDP are scanty, need restrictive assumptions and considerable effort [1], [2], [7]. What could it mean? Maybe my physical intuition is naive; in this case important counterexamples should be found.

Question. Does the moderate deviations principle (and moreover, the linear response principle) for stationary random fields hold under general conditions, such as exponential decay of correlations and finite exponential moments?

It follows from (3) that

$$\mathbb{E}\left(Z_0(g_r) \cdot \frac{\exp Z_0(h_r)}{\mathbb{E}\exp Z_0(h_r)} \right) \sim \sigma^2 \int g_r \Delta^2 h_r \,\mathrm{d}m \quad \text{as } r \to \infty$$

for all test functions g, h. (Here $\Delta^2 h_r = \Delta(\Delta h_r)$.) This is the response of the observable $Z_0(g_r)$ to the external field h_r. In more physical language, the response of the macroscopic observable $\sum_{z:\psi(z)=0} g(z)$ to the macroscopic external field h

is $\sigma^2 \int g\Delta^2 h \, \mathrm{d}m$. It means that $\sigma^2\Delta^2 h$ is the response (to h) of the macroscopic density of the chaotic analytic zero points.

For the Poisson process the response is proportional to h. For OCP it is (believed to be) proportional to Δh, which is electrostatic screening of external charges [4]. We summarize:

$$\begin{array}{ccccc} \text{Model} & : & \text{Poisson} & \text{OCP} & \text{CAZP} \\ \text{Response to } h \text{ is proportional to} & : & h & \Delta h & \Delta^2 h \end{array} \tag{4}$$

See also three toy models discussed in [6] (the end of the introduction).

I restrict myself to smooth (C^2) test functions. The number of random points in the disk of large radius r is a different story. It satisfies the Jancovici-Lebowitz-Manificat law [5], like OCP [3]. Its moderate deviations are a boundary effect. In contrast, (2) is a bulk effect. What happens to *large* deviations in the bulk? I do not know.

References

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