Exam of 01.07.2002 — Solutions

1

X, Y are independent, distributed uniformly $\mathrm{U}(0,1)$. Their joint distribution is uniform on the square $(0,1) \times (0,1)$.

1a

$\mathbb{P}\left(A \right) = \frac{1}{2}$, since the corresponding region (a triangle) is of area $\frac{1}{2}$.

$\mathbb{P}\left(A \mid X = \frac{1}{6} \right) = \mathbb{P}\left(X < Y \mid X = \frac{1}{6} \right) = \mathbb{P}\left(\frac{1}{6} < Y \mid X = \frac{1}{6} \right) = \mathbb{P}\left(\frac{1}{6} < Y \right) = \frac{5}{6}$. Similarly, $\mathbb{P}\left(A \mid X = x \right) = \mathbb{P}\left(x < Y \right) = 1 - x$ for $x \in (0,1)$, thus $\mathbb{P}\left(A \mid X \right) = 1 - X$.

$\mathbb{E}\mathbb{P}\left(A \mid X \right) = \mathbb{E}(1-X) = 1 - \frac{0+1}{2} = \frac{1}{2} = \mathbb{P}\left(A \right)$.

1b

$\mathbb{P}\left(B \right) = \frac{5}{9}$, since the corresponding domain (the square minus two triangles) is of area $1 - 2 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9}$.

$\mathbb{P}\left(A \cap B \right) = \frac{1}{2}\mathbb{P}\left(B \right) = \frac{5}{18}$, according to the area of the corresponding domain.

Yes, A and B are independent, since $\mathbb{P}\left(A \cap B \right) = \frac{5}{18} = \frac{1}{2} \cdot \frac{5}{9} = \mathbb{P}\left(A \right)\mathbb{P}\left(B \right)$.

1c

$\mathbb{P}\left(B \mid X = \frac{1}{6} \right) = \mathbb{P}\left(|X - Y| < \frac{1}{3} \mid X = \frac{1}{6} \right) = \mathbb{P}\left(|\frac{1}{6} - Y| < \frac{1}{3} \mid X = \frac{1}{6} \right) = \mathbb{P}\left(|\frac{1}{6} - Y| < \frac{1}{3} \right) = \mathbb{P}\left(-\frac{1}{6} < Y < \frac{1}{2} \right) = \mathbb{P}\left(0 < Y < \frac{1}{2} \right) = \frac{1}{2}$;

$\mathbb{P}\left(A \cap B \mid X = \frac{1}{6} \right) = \mathbb{P}\left(\frac{1}{6} < Y < \frac{1}{2} \right) = \frac{1}{3}$;

no, $\mathbb{P}\left(A \cap B \mid X = \frac{1}{6} \right) = \frac{1}{3}$ differs from $\mathbb{P}\left(A \mid X = \frac{1}{6} \right) \mathbb{P}\left(B \mid X = \frac{1}{6} \right) = \frac{5}{6} \cdot \frac{1}{2}$.

1d

For $x \in (\frac{2}{3}, 1)$, conditionally on $X = x$, events A and B become $Y \in (x, 1)$ and $Y \in (x - \frac{1}{3}, 1)$. Here, A implies B, therefore A and B are positively correlated.

For $x \in (0, \frac{1}{3})$ we get $A : Y \in (x, 1)$ and $B : Y \in (0, \frac{1}{3} + x)$. Here $\overline{A}$ implies B; negative correlation.

For $x \in [\frac{1}{3}, \frac{2}{3}]$ we get $A : Y \in (x, 1)$ and $B : Y \in (x - \frac{1}{3}, x + \frac{1}{3})$. The probability of A is $1 - x$. Given B, the conditional probability of A equals to $\frac{1}{2}$. (Of course, everything is conditioned by $X = x$.) For $x \in [\frac{1}{3}, \frac{1}{2}]$ the conditional probability is smaller, which means negative correlation. For $x \in [\frac{1}{2}, \frac{2}{3}]$ the correlation is positive. So,

$x < \frac{1}{2}$: negative correlation, $\mathbb{P}\left(A \cap B \mid X = x \right) < \mathbb{P}\left(A \mid X = x \right) \mathbb{P}\left(B \mid X = x \right)$;

$x = \frac{1}{2}$: no correlation;

$x > \frac{1}{2}$: positive correlation, $\mathbb{P}\left(A \cap B \mid X = x \right) > \mathbb{P}\left(A \mid X = x \right) \mathbb{P}\left(B \mid X = x \right)$.

1e

$$F_{Y|A}(y) = \mathbb{P}\left(Y \le y \mid A \right) = \frac{\mathbb{P}\left(X < Y \le y \right)}{\mathbb{P}\left(A \right)} = \frac{y^2/2}{1/2} = y^2 \quad \text{for } 0 < y < 1$$

(the probability is calculated as the area of the triangle), therefore $f_{Y|A}(y) = 2y$ for $y \in (0,1)$, otherwise 0.

Similarly, $1 - F_{X|A}(x) = \mathbb{P}\left(X > x \mid A \right) = (1-x)^2$, thus $f_{X|A}(x) = 2(1-x)$ for $x \in (0,1)$, otherwise 0.

1f

$\mathbb{E}\left(Y \mid A \right) = \int y f_{Y|A}(y)\, dy = \int_0^1 y \cdot 2y\, dy = \frac{2}{3}$;
$\mathbb{E}\left(X \mid A \right) = \int_0^1 x \cdot 2(1-x)\, dx = \frac{1}{3}$.

1g

The unconditional distribution of (X,Y) is uniform on the square, therefore the conditional distribution is uniform on the triangle, which means

$$f_{X,Y|A}(x,y) = \begin{cases} 1/\mathbb{P}\left(A \right) = 2 & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The equality $f_{X,Y|A}(x,y) = f_{X|A}(x) f_{Y|A}(y)$ does not hold. Indeed, if $0 < y < x < 1$ then the left-hand side vanishes but the right-hand side does not.

2

2a

The function $f(x) = \frac{|x-t|-|x-s|}{t-s}$ is equal to $+1$ for $x \le s$ and to -1 for $x \ge t$; in the middle, for $x \in [s,t]$, we have $-1 \le f(x) \le 1$.

On the other hand, $2F_X(s) - 1 = \mathbb{P}\left(X \le s \right) - \mathbb{P}\left(X > s \right) = \mathbb{E}\, g(X)$, where $g(x) = +1$ for $x \le s$ and -1 for $x > s$. We observe that $g(X) \le f(X)$ always, therefore $\mathbb{E}\, g(X) \le \mathbb{E}\, f(X)$, which means

$$2F_X(s) - 1 \le \frac{U(t) - U(s)}{t - s}\,.$$

The other inequality is similar: $f(x) \le h(x)$, where $h(x) = +1$ for $x < t$ and -1 for $x \ge t$.

2b

For each $\varepsilon > 0$,

$$2F_X(t) - 1 \le \frac{U(t+\varepsilon) - U(t)}{\varepsilon} \le 2F_X\big((t+\varepsilon)-\big) - 1 \le 2F_X(t+\varepsilon) - 1\,.$$

F_X is continuous at a given point t, therefore for $\varepsilon \to 0+$ we have $2F_X(t+\varepsilon)-1 \to 2F_X(t)-1$; by the sandwich argument,

$$\frac{U(t+\Delta t)-U(t)}{\Delta t} \to 2F_X(t)-1 \quad \text{for } \Delta t \to 0+\ .$$

On the other hand,

$$2F_X(t-\varepsilon)-1 \le \frac{U(t)-U(t-\varepsilon)}{\varepsilon} \le 2F_X(t-)-1\,;$$

for $\varepsilon \to 0+$ we have $2F_X(t-\varepsilon)-1 \to 2F_X(t-)-1 = 2F_X(t)-1$; by the sandwich argument, $\frac{U(t)-U(t-\varepsilon)}{\varepsilon} \to 2F_X(t)-1$, that is,

$$\frac{U(t+\Delta t)-U(t)}{\Delta t} \to 2F_X(t)-1 \quad \text{for } \Delta t \to 0-\ .$$

2c

X has a density f_X, therefore F_X is continuous everywhere, and 2b gives us $U'(t) = 2F_X(t)-1$ for all t. Assume now that f_X is continuous at a given point t. It follows that F_X is differentiable at t, and $F'_X(t) = f_X(t)$. Therefore U' is differentiable at t, and $U''(t) = 2F'_X(t) = 2f_X(t)$.

3

3a

We have to prove that the set $\{\omega : Y(\omega) \le y\}$ is an event, for every $y \in \mathbb{R}$. It follows from the equality

$$\{\omega : Y(\omega) \le y\} = \begin{cases} \emptyset & \text{for } -\infty < y < 0, \\ \{\omega : X(\omega) \le y\} \cup (\Omega \setminus A) & \text{for } 0 \le y < \infty. \end{cases}$$

3b

$\mathbb{P}\big(Y \le X\big) = 1$, therefore $Y^* \le X^*$.

3c

First, $\mathbb{P}\big(Y \ge 0\big) = 1$, therefore $Y^* \ge 0$. Second, $\mathbb{P}\big(Y = 0\big) \ge \mathbb{P}\big(\Omega \setminus A\big) = 1 - \mathbb{P}\big(A\big)$, therefore $Y^* = 0$ on an interval of length $\ge 1 - \mathbb{P}\big(A\big)$. The interval begins at 0, since Y^* is never negative.

3d

$$\mathbb{E}Y = \int_0^1 Y^*(p)\,dp = \int_0^{1-\mathbb{P}(A)} \underbrace{Y^*(p)}_{=0}\,dp + \int_{1-\mathbb{P}(A)}^1 \underbrace{Y^*(p)}_{\le X^*(p)}\,dp\,.$$

3e ....

$X + 100 \ge 0$; (3d) gives

$$\mathbb{E}\big((X+100)\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 (X+100)^*(p)\,dp\,.$$

Therefore

$$\mathbb{E}\big(X\cdot\mathbf{1}_A + 100\cdot\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 (X^*(p)+100)\,dp\,;$$

$$\mathbb{E}\big(X\cdot\mathbf{1}_A\big) + 100\,\mathbb{P}\left(A\right) \le \int_{1-\mathbb{P}(A)}^1 X^*(p)\,dp + 100\,\mathbb{P}\left(A\right);$$

$$\mathbb{E}\big(X\cdot\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 X^*(p)\,dp\,.$$

3f ....

For any $M \in (0,\infty)$ consider the random variable $X_M = \max(X, -M)$. Similarly to (3e),

$$\mathbb{E}\big(X_M\cdot\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 X_M^*(p)\,dp\,.$$

However, $X_M^*(p) = \max(X^*(p), -M)$ (monotone transformation); also, $X_M \ge X$ always; thus,

$$\mathbb{E}\big(X\cdot\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 \max(X^*(p), -M)\,dp\,.$$

It holds for every $M \in (0,\infty)$, and (assuming $\mathbb{P}\left(A\right) \ne 1$) we may choose M such that $X^*\big(1-\mathbb{P}\left(A\right)\big) > -M$, getting

$$\mathbb{E}\big(X\cdot\mathbf{1}_A\big) \le \int_{1-\mathbb{P}(A)}^1 X^*(p)\,dp\,.$$