7 Equivalence relations and measurability

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A theory of classification is interesting and deep. In addition, it helps to prove that all knots are a Borel set, not just an analytic set.

7a From classification problems to Borel equivalence relations

A natural starting point for a systematic theory of classification.

Here are examples of classification problems in different branches of mathematics. Do not worry if some examples are not quite clear to you.

7a1 Example. (Already mentioned in Sect. 5d.) Classification of tame knots. The set of tame knots was introduced and endowed with an equivalence relation "of the same type".

7a2 Example. The same but for all knots (tame, wild).

7a3 Example. (Lurked in the end of Sect. 5d.) The set of all compact subsets of $\mathbb{R}^3$ (say), up to homeomorphism (that is, with the equivalence relation "homeomorphic").

7a4 Example. The class of all countable compact metrizable spaces, up to homeomorphism.[1]

[1]Example 0.5 in: G. Hjorth, "Classification and orbit equivalence relations", AMS 2000.

7a5 Example. The set of all unitary operators in $\mathbb{C}^n$ (just matrices), with the equivalence relation "unitarily equivalent" (that is, "conjugate"): $U \sim V$ if and only if $V = W^{-1}UW$ for some unitary W.[1]

7a6 Example. The same, but in an infinite-dimensional separable Hilbert space.[2]

7a7 Example. The class of all separable Banach spaces, with the equivalence relation "linearly homeomorphic".

7a8 Example. The set of all von Neumann algebras of operators on a given infinite-dimensional separable Hilbert space, up to unitary equivalence.

7a9 Example. The class of all separable C^*-algebras, up to isomorphism.[3]

7a10 Example. The class of all finite graphs, up to isomorphism.

7a11 Example. The class of all countable graphs, up to isomorphism.

Each example specifies a class of objects endowed with an equivalence relation. In some examples (7a1, 7a2, 7a3, 7a5, 7a6, 7a8) the class is a set, in others (7a4, 7a7, 7a9, 7a10, 7a11) it is not, but it is possible to choose a set Z of these objects that intersects every equivalence class.[4] Moreover, the cardinality of continuum is enough for Z in all these examples. Sometimes (in 7a1, 7a10) a countable Z is enough, but this is not typical.

Thus we have a set Z of cardinality (at most) continuum, endowed with an equivalence relation "$\sim$", and the quotient set $Z/\sim$ (of equivalence classes[5]). According to the cardinality, there must exist a one-to-one map $Z/\sim \to \mathbb{R}$; the composition map $Z \to Z/\sim \to \mathbb{R}$ is a *complete invariant*. It means that two objects are equivalent if and only if the corresponding real numbers are equal. However, this is not a satisfactory solution of the classification problem, since this complete invariant is utterly nonconstructive. Existence of such a map $Z \to \mathbb{R}$, ensured by the choice axiom, gives us no new information about the given equivalence relation. Quite useless![6]

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[1] Hjorth, Example 0.2.

[2] Hjorth, Sect. 1.1.

[3] See also: I. Farah, A.S. Toms, A. Törnquist, "Turbulence, orbit equivalence, and the classification of nuclear C^*-algebras", J. für die reine und angew. Math. (online 2012).

[4] For example: in 7a4, countable compact subsets of $\mathbb{R}$ may be used (due to Mazurkiewicz-Sierpinski theorem). Or alternatively, compact metrics on $\{1, 2, \dots\}$.

[5] These are traditionally called classes, but they are sets, of course.

[6] "Therefore the notion of Borel reducibility provides a natural starting point for a systematic theory of classification which is both generally applicable, and manages to ban the trivialities provided by the Axiom of Choice." (Farah, Toms and Törnquist, p. 2.)

Here is a useful approach.[1] One invents a *parametrization* of the given classification problem, — a pair (X, f) of a Borel space X and a map f from X to the given class such that $f(X)$ intersects every equivalence class.[2] It does not matter whether f is one-to-one or not; the relation $f(x) \sim f(y)$ is relevant, the relation $f(x) = f(y)$ is not. One introduces an equivalence relation E_f on X:

$$x \; E_f \; y \quad \Longleftrightarrow \quad f(x) \sim f(y) \, .$$

Let (X, f) and (Y, g) be two parametrizations of the same classification problem. A *morphism* from (X, f) to (Y, g) is a map $\varphi : X \to Y$ such that $f(x) \sim g(\varphi(x))$ for all $x \in X$. Then

$$\forall x, y \in X \quad \big(x \; E_f \; y \quad \Longleftrightarrow \quad \varphi(x) \; E_g \; \varphi(y) \big)$$

(think, why). A morphism is usually highly non-unique; at least one morphism must exist (think, why). Let ψ be a morphism from (Y, g) to (X, f), then

$$\forall x \in X \quad x \; E_f \; \psi(\varphi(x)) \, ; \qquad \forall y \in Y \quad y \; E_g \; \varphi(\psi(y)) \, .$$

A clever choice of a parametrization respects the structure of the given classification problem. This is an informal idea, of course. Here are some indications of a "clever" parametrization.

First, for a "clever" (X, f), X should be "nice" (standard is very nice; analytic is less nice), and the set $E_f \subset X \times X$ should be "nice" (Borel measurable is very nice, analytic is less nice).

Second, for "clever" (X, f) and (Y, g), "nice" morphisms $\varphi : X \to Y$, $\psi : Y \to X$ should exist (Borel measurable is very nice).

7b Measurable parametrizations

A new kind of space, more general than measurable space, is appropriate for a set of equivalence classes.

Replacing Z with $Z/\!\sim$ we may assume that the given equivalence relation on Z is just the equality.

7b1 Definition. (a) A *measurable parametrization* of a set Z is a pair (X, f) of a measurable space X and a map f from X onto Z.

[1] Farah, Toms and Törnquist, Sect. 2.

[2] See Kechris, Sect. 12.E, for parametrizations of (a) Polish spaces, (b) Polish groups, (c) separable Banach spaces as in 7a7, (d) von Neumann algebras as in 7a8; all these are parametrized by standard Borel spaces.

(b) A measurable parametrization (X_1, f_1) of Z is *finer* than a measurable parametrization (X_2, f_2) of Z if $f_1 = f_2 \circ \varphi$ for some measurable map $\varphi : X_1 \to X_2$.

(c) Measurable parametrizations (X_1, f_1), (X_2, f_2) of Z are *equivalent* if (X_1, f_1) is finer than (X_2, f_2) and (X_2, f_2) is finer than (X_1, f_1).

(d) A *measurably parametrized space* is a set endowed with a measurable parametrization.[1]

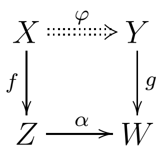
(e) A *measurably parametrizable space* is a set endowed with an equivalence class[2] of measurable parametrizations.

I often drop the word "measurably" before "parametrized" or "parametrizable".

Note some similarity between 7b1 and 3a1. In 3a1(c) equivalence classes may be avoided using topological spaces. I wonder, is there something like that for 7b1(e)?

7b2 Definition. Let (X, f) be a measurable parametrization of Z, and (Y, g) a measurable parametrization of W.

(a) A *morphism* from the measurably parametrized space (Z, X, f) to the measurably parametrized space (W, Y, g) is a map $\alpha : Z \to W$ such that $\alpha \circ f = g \circ \varphi$ for some measurable map $\varphi : X \to Y$.



(b) A morphism $\alpha : Z \to W$ is an *isomorphism* if α is invertible and $\alpha^{-1} : W \to Z$ is a morphism from (W, Y, g) to (Z, X, f).

(c) Two measurably parametrized spaces are *isomorphic* if there exists an isomorphism between them.

Thus, a parametrization (X_1, f_1) of Z is finer than (X_2, f_2) if and only if id_Z is a morphism from (Z, X_1, f_1) to (Z, X_2, f_2). The two parametrizations are equivalent if and only if id_Z is an isomorphism.

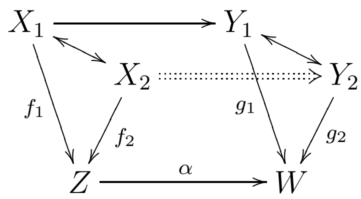
The composition of morphisms is a morphism (similarly to 1d4). "Isomorphic" is an equivalence relation (similarly to 1d6). The following lemma shows that morphisms (and isomorphisms) are well-defined also between parametrizable spaces.

[1] Items (d), (e) are not a standard terminology.

[2] Class indeed, not a set, which is problematic in ZFC. We restrict ourselves to such statements about measurably parametrizable spaces that can be evidently reformulated in terms of measurably parametrized spaces. (Beyond that, one can use ZGC, the Zermelo set theory with global choice known as Hilbert's global choice operator and used by Bourbaki.)

7b3 Lemma. Let (X_1, f_1) and (X_2, f_2) be equivalent measurable parametrizations of Z; (Y_1, g_1) and (Y_2, g_2) equivalent measurable parametrizations of W; and $\alpha : Z \to W$ a morphism from (Z, X_1, f_1) to (W, Y_1, g_1). Then α is also a morphism from (Z, X_2, f_2) to (W, Y_2, g_2).

Proof.



□

The same holds if (X_2, f_2) is finer than (X_1, f_1) and (Y_1, g_1) is finer than (Y_2, g_2).

Let (Z, P_1) and (Z, P_2) be parametrizable spaces (P_1, P_2 being equivalence classes of parametrizations). We say that P_1 is *finer* than P_2 if and only if id_Z is a morphism from (Z, P_1) to (Z, P_2). That is, (X_1, f_1) is finer than (X_2, f_2) for some (therefore all) $(X_1, f_1) \in P_1$, $(X_2, f_2) \in P_2$.

7b4 Core exercise. [1] P_2 is finer than P_1 if and only if there exist $((X, \mathcal{A}_1), f) \in P_1$ and $((X, \mathcal{A}_2), f) \in P_2$ such that $\mathcal{A}_1 \subset \mathcal{A}_2$.

Prove it.

Given a σ-algebra $\mathcal{A}$ on Z, we have a parametrization $\big((Z, \mathcal{A}), \mathrm{id}_Z\big)$ on Z; such a parametrization will be called *trivial.* A measurable space may be treated as a (trivial) parametrized space. Then, a morphism between measurable spaces is just a measurable map (and isomorphism is what it should be).

7b5 Core exercise. A parametrization (X, f) of Z is equivalent to some trivial parametrization if and only if there exist a σ-algebra $\mathcal{A}$ on Z and a map $\varphi : Z \to X$ such that $f \circ \varphi = \mathrm{id}_Z$ and f, φ are measurable (when Z is endowed by $\mathcal{A}$).

Prove it.

7c A parametrization not equivalent to trivial

A non-Borel analytic set may be treated as a nonstandard Borel space. However, being parametrized by a Borel set, it becomes something different.

[1] Using the axiom of choice.

Let $Z \subset \mathbb{R}$ be a set (endowed with its Borel σ-algebra), $\psi : Z \to \mathbb{R}$ a Borel function, and $B \subset \mathbb{R}^2$ a Borel set. Then $A = \{x \in Z : (x, \psi(x)) \in B\}$ is a Borel subset of Z (since $Z \ni x \mapsto (x, \psi(x)) \in \mathbb{R}^2$ is a Borel map), maybe not of $\mathbb{R}$. But if $B \subset Z \times \mathbb{R}$ then A is a Borel subset of $\mathbb{R}$ (no matter how bad is Z). Here is why. The function $(x, y) \mapsto y - \psi(x)$ is Borel measurable on the Borel set B, therefore $\{(x, y) \in B : y = \psi(x)\}$ is a Borel set in $\mathbb{R}^2$. Its projection A is a Borel set in $\mathbb{R}$ by 6d9! We conclude.

7c1 Lemma. If a Borel set $B \subset \mathbb{R}^2$ has a non-Borel projection $Z = \{x : \exists y \, (x, y) \in B\}$ then a Borel function $\psi : Z \to \mathbb{R}$ cannot satisfy the condition $\big(x, \psi(x)\big) \in B$ for all $x \in Z$.

In other words, such B does not admit a Borel uniformizing function.[1] Moreover, by the same argument, B does not admit an uniformizing function $\psi : Z \to \mathbb{R}$ such that the function $(x, y) \mapsto \psi(x)$ on B is Borel measurable.[2]

7c2 Core exercise. Let B and Z be as in 7c1. Then the parametrization (B, f) of Z, where $f : (x, y) \mapsto x$ is the projection, cannot be equivalent to a trivial parametrization.

Prove it.

Such (Z, B, f) is an example of a parametrizable space (Z, P_B) that is not a measurable space.

On the other hand, the subset Z of $\mathbb{R}$ has its Borel σ-algebra $\mathcal{B}(Z)$, and the measurable space $(Z, \mathcal{B}(Z))$ is itself a parametrizable space (Z, P_{trivial}). Clearly, id_Z is a morphism from (Z, P_B) to (Z, P_{trivial}). By 7c1, id_Z is not a morphism from (Z, P_{trivial}) to (Z, P_B). It means that P_B is strictly finer than P_{trivial}. In the spirit of 7b4 we have $((B, \mathcal{A}_2), f) \in P_B$ and $((B, \mathcal{A}_1), f) \in P_{\text{trivial}}$ where $\mathcal{A}_2 = \mathcal{B}(B)$ is the Borel σ-algebra of B, and $\mathcal{A}_1 = \sigma(f) = \{(A \times \mathbb{R}) \cap B : A \in \mathcal{B}(Z)\}$. The relation "$\mathcal{A}_1 \subset \mathcal{A}_2$" is equivalent to the relation "P_B is finer than P_{trivial}", of course; but the relation "P_B is strictly finer than P_{trivial}" is deeper than just "$\mathcal{A}_1 \subsetneq \mathcal{A}_2$". Indeed, if B is (say) a rectangle (and so, Z is an interval rather than a non-Borel set) then $\mathcal{A}_1 \subsetneq \mathcal{A}_2$ but P_B is equivalent to P_{trivial}.

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[1]Borel measurability of the projection is necessary but not sufficient for a Borel uniformizing function to exist; see D. Blackwell, "A Borel set not containing a graph", Ann. Math. Statist. **39**:4, 1345–1347 (1968); also Srivastava, Example 5.1.7. On the other hand, a universally measurable uniformizing function exists for every Borel (as well as analytic) set by the (Jankov-)von Neumann uniformization theorem, see Srivastava, Sect. 5.5 or Kechris, Sect. 18.A.

[2]In fact, Borel measurability of $(x, y) \mapsto \psi(x)$ on B is equivalent to Borel measurability of ψ (on Z), since a subset $A \subset Z$ is Borel measurable (in Z) if and only if $(A \times \mathbb{R}) \cap B$ is Borel measurable (which follows easily from 6a5). This is basically the Blackwell-Mackey theorem, see Srivastava, Th. 4.5.7 or Kechris, Exercise (14.16).

Why only $B \subset \mathbb{R}^2$? The same holds for $B \subset R_1 \times R_2$ whenever R_1, R_2 are standard Borel spaces.[1]

In particular we may consider the standard Borel space $\mathrm{Tr}(T)$ of all subtrees of the full infinitely splitting tree $T = \{1, 2, \dots\}^{<\infty}$ (recall Sect. 6c), and its subset $\mathrm{IF}(T)$ of all subtrees that have (at least one) infinite branch. The set $\mathrm{IF}(T)$ is analytic (by 6c12) and not Borel (which is partially proved in 6c14). The body $[T]$ is the standard Borel space of all infinite branches of T.

In the space $\mathrm{Tr}(T) \times [T]$ we consider the Borel set $B = \{(T_1, s) : s \in [T_1]\}$. Its first projection $Z = \{T_1 : [T_1] \ne \emptyset\} = \mathrm{IF}(T)$ is non-Borel. By 7c1 (generalized to standard Borel spaces), a Borel map $\psi : \mathrm{IF}(T) \to [T]$ cannot be uniformizing, that is, cannot satisfy the condition $\psi(T_1) \in [T_1]$ for all $T_1 \in \mathrm{IF}(T)$. An infinite branch cannot be chosen by a Borel function of the tree.

An evident uniformizing function, well-known as the leftmost branch, chooses the least k_1 such that the subtree over (k_1) has an infinite branch, the least k_2 such that the subtree over (k_1, k_2) has an infinite branch, and so on. This is not a Borel function (think, why).[2]

7c3 Core exercise. Let Z, (X, f), $\mathcal{A}$ and φ satisfy the conditions of 7b5. Then:

(a) A set $A \subset Z$ is $\mathcal{A}$-measurable if and only if $f^{-1}(A) \subset X$ is measurable. (Thus, $(Z, \mathcal{A})$ is a *quotient space* of X.)

(b) φ is an isomorphism from $(Z, \mathcal{A})$ to $\varphi(Z)$ (treated as a measurable subspace of X).

(c) $\varphi(Z) = \{x \in X : \varphi(f(x)) = x\}$.

(d) If X is countably separated then $\varphi(Z)$ is measurable.

Prove it.

7d Borel sets in the light of measurable parametrizations

All knots are not just an analytic set, they are a Borel set.

We still deal with T, $\mathrm{Tr}(T)$ and $\mathrm{IF}(T)$ as in 7c, but now we use topology.

It was noted in Sect. 6c (before 6c12) that the set $\mathrm{Tr}(T)$ is a closed subset of the space 2^T (homeomorphic to the Cantor set), thus, a compact metrizable space. The set of all (finite or infinite) branches is also a closed

[1]Then $y - \psi(x)$ cannot be used; instead of $y - \psi(x) = 0$, the relation $\big(\psi(x), y\big) \in D$ is used, the diagonal D being measurable.

[2]Still, it is universally measurable (think, why); see also Footnote 1 on page 102.

subset of 2^T. The set $[T]$ of all infinite branches is not closed (think, why) but still Polish;[1] a complete metric on $[T]$ introduced in Sect. 6c (before 6c2) is compatible (think, why).

All pairs "a subtree and its branch" are a closed set in $2^T \times 2^T$ (think, why). Thus, the set $B = \{(T_1, s) : s \in [T_1]\}$ is closed in $\mathrm{Tr}(T) \times [T]$. We see that $Z = \mathrm{IF}(T)$ is a projection of a closed set in the product of Polish spaces, therefore a continuous image of a Polish space. (And no wonder: every analytic set is.) The leftmost branch of T_1, being not a Borel function of T_1, is a Borel function of $[T_1]$ treated as an element of $\mathbf{F}([T])$ (think, why). It means that $[T_1]$ is not a Borel function of T_1. A wonder: the section $B_{T_1} = \{s : (T_1, s) \in B\}$ of the closed set B is not a Borel function of T_1.

This is the only reason for the absence of Borel uniformizing functions in such situations, according to the following result.

7d1 Proposition. For every Polish space X there exists a Borel map $d : \mathbf{F}(X) \setminus \{\emptyset\} \to X$ such that $d(F) \in F$ for all nonempty closed F.

Proof. We take a complete compatible metric ρ on X and a dense sequence $(x_n)_n$ in X. Given a nonempty closed F, we take the smallest n_1 such that $\mathrm{dist}(x_{n_1}, F) < 2^{-1}$. Then we take the smallest n_2 such that $\mathrm{dist}(x_{n_2}, F) < 2^{-2}$ and $\rho(x_{n_1}, x_{n_2}) < 2^{-1}$. And so on; $\mathrm{dist}(x_{n_k}, F) < 2^{-k}$ and $\rho(x_{n_{k-1}}, x_{n_k}) < 2^{-(k-1)}$. Finally, $d(F) = \lim_k x_{n_k}$. □

7d2 Core exercise. For every Polish space X,

(a) there exist Borel maps $d_n : \mathbf{F}(X) \setminus \{\emptyset\} \to X$ such that every nonempty closed F is the closure of $\{d_1(F), d_2(F), \dots\}$;

(b) every random closed set in X is the closure of some random countable set.

Prove it.

7d3 Proposition. If a parametrized space (Z, X, f) satisfies the conditions

(a) X is a Polish space with the Borel σ-algebra,

(b) for every $z \in Z$ the set $\{x \in X : f(x) = z\}$ is closed,

(c) for every open $U \subset X$ the set $f^{-1}(f(U))$ is Borel measurable,

then (X, f) is equivalent to the trivial parametrization of a standard Borel space.

Proof. We define a σ-algebra $\mathcal{A}$ on Z as in 7c3(a): $\mathcal{A} = \{A : f^{-1}(A) \text{ is measurable in } X\}$; note that $f : X \to Z$ is measurable.

[1] It is in fact a G_δ set dense in the set of all branches; every G_δ set in a Polish space is known to be Polish. In contrast, if T is finitely splitting then $[T]$ is closed in 2^T; see also 6c2.

For every open $U \subset X$ we have $\{z : f^{-1}(z) \cap U \ne \emptyset\} = f(U) \in \mathcal{A}$ by (c). Thus, the map $Z \ni z \mapsto f^{-1}(z) \in \mathbf{F}(X)$ is measurable. Using 7d1 we get a measurable map $\varphi : Z \to X$ such that $\varphi(z) \in f^{-1}(z)$ for all $z \in Z$, that is, $f \circ \varphi = \mathrm{id}_Z$. The conditions of 7b5 are satisfied by Z, X, f and $\mathcal{A}$. By 7c3(b), $(Z, \mathcal{A})$ is isomorphic to $\varphi(Z)$; by (a) and 7c3(d), $\varphi(Z)$ is a standard Borel space. $\square$

7d4 Proposition. All knots are a Borel subset of $\mathbf{F}(\mathbb{R}^3)$.

Proof (sketch). Denote by K the circle,[1] and by $\mathrm{Homeo}(K \to \mathbb{R}^3)$ the set of all homeomorphisms $\alpha : K \to \alpha(K) \subset \mathbb{R}^3$; it is a Borel subset of $C(K \to \mathbb{R}^3)$, recall 5d13. For $\alpha, \beta \in \mathrm{Homeo}(K \to \mathbb{R}^3)$ the relation $\alpha(K) = \beta(K)$ holds if and only if $\alpha = \beta \circ \gamma$ for some $\gamma \in \mathrm{Homeo}(K)$ (homeomorphism of K).

We check the conditions of 7d3 for $X = \mathrm{Homeo}(K \to \mathbb{R}^3)$, $Z \subset \mathbf{F}(\mathbb{R}^3)$ the set of all knots, and $f : \alpha \mapsto \alpha(K)$. Condition (a) holds since $\mathrm{Homeo}(K \to \mathbb{R}^3)$ is a G_δ set in the Polish space $C(K \to \mathbb{R}^3)$. Condition (b) holds clearly. Condition (c): let $U \subset \mathrm{Homeo}(K \to \mathbb{R}^3)$ be an open set, then $f^{-1}(f(U)) = \{\beta \circ \gamma : \beta \in U, \gamma \in \mathrm{Homeo}(K)\}$ is the union over γ of sets $\{\beta \circ \gamma : \beta \in U\}$; such a set is open, since the map $\beta \mapsto \beta \circ \gamma$ is a homeomorphism of the space $\mathrm{Homeo}(K \to \mathbb{R}^3)$.

Using 7d3, 7b5 and 7c3(a) we see that the quotient space $(Z, \mathcal{A})$ is a standard Borel space. Measurability of f ensures that $\mathcal{A}$ contains the Borel σ-algebra of Z. It remains to apply 6b3. $\square$

7e Micro-survey of advanced theory

A systematic theory of classification. (No proofs in this section.)

A *Borel equivalence relation* is, by definition, an equivalence relation E on a standard Borel space X such that E treated as a subset of $X \times X$ is Borel measurable. Every Borel equivalence relation E leads to a parametrized space $(X/E, X, f)$ where $f(x) = [x] \in X/E$ is the equivalence class of x. Thus, $E = E_f = \{(x, y) : f(x) = f(y)\}$.

Let us define a *Borel parametrizable space* as a parametrizable space that admits a parametrization (X, f) such that E_f is a Borel equivalence relation.

A *reduction* of a Borel equivalence relation E on X to a Borel equivalence relation F on Y is, by definition, a Borel map $\varphi : X \to Y$ such that $x_1 \, E \, x_2 \iff \varphi(x_1) \, F \, \varphi(x_2)$ for all $x_1, x_2 \in X$. Existence of a reduction is denoted $E \le_B F$ (or $F \ge_B E$) and called *Borel reducibility* of E to F.

Clearly, a reduction of E to F leads to a one-to-one morphism $X/E \to Y/F$. And conversely: let Z, W be Borel parametrizable spaces and (X, f),

[1]Or another compact metrizable space. . .

(Y, g) their parametrizations such that E_f, E_g are Borel equivalence relations; then every one-to-one morphism $Z \to W$ corresponds to some reduction of E_f to E_g.

We see that Borel reducibility of Borel equivalence relations is the same as existence of one-to-one morphism of Borel parametrizable spaces.

Here are several well-known examples of Borel equivalence relations, or equivalently, Borel parametrizable spaces.

7e1 Example. $\mathbb{R}/\mathbb{Q}$, the Vitali space. Here $X = \mathbb{R}$ (with the Borel σ-algebra, of course), and $x \sim y$ means that $x - y$ is rational.

7e2 Example. E_0, germs of binary sequences. Here $X = \{0, 1\}^\infty$, and $x \sim y$ means that $x_n = y_n$ for all n large enough.

7e3 Example. E_1, germs of real sequences. Here $X = \mathbb{R}^\infty$, and again, $x \sim y$ means that $x_n = y_n$ for all n large enough.

Below, $X \sim_B Y$ means $(X \le_B Y) \wedge (Y \le_B X)$.

7e4 Proposition. [1] $\mathbb{R}/\mathbb{Q} \sim_B E_0$.

In addition, every standard Borel space is itself a parametrizable space; the corresponding equivalence relation is just the equality. Thus, we add to the list $\mathbb{R}$ (the real line) and $\mathbb{N}$ (the natural numbers).

The following three "dichotomy theorems" hold for all Borel equivalence relations E. By 7e4, E_0 may be replaced with $\mathbb{R}/\mathbb{Q}$.

7e5 Theorem. [2] Either $E \le_B \mathbb{N}$ or $E \ge_B \mathbb{R}$.

Among Examples 7a1–7a11, only two (7a1 and 7a10) satisfy $E \le_B \mathbb{N}$, that is, have only countably many equivalence classes. In 7a4 there are exactly $\aleph_1$ equivalence classes! Well, I did not claim that 7a4 leads to a *Borel* parametrizable space…

7e6 Theorem. [3] Either $E \le_B \mathbb{R}$ or $E \ge_B E_0$.

When $E \le_B \mathbb{R}$, such E is called *smooth* (or tame, or concretely classifiable).[4] For a nontrivial but smooth Borel parametrizable space, recall Sect. 7c (a Borel parametrization of a non-Borel analytic set). Example 7a5 is smooth (think, why), but 7a6 is not, and moreover (see below)…

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[1] See Hjorth, Exercise 7.22.

[2] The Silver dichotomy. See Theorem 5.7.1 in: V. Kanovei, "Borel equivalence relations: structure and classification", AMS 2008.

[3] Harrington, Kechris and Louveau. Sometimes called the general Glimm-Effros dichotomy. See Kanovei, Th. 5.7.2.

[4] See Kechris, Exercise (18.20) and Kanovei, Sect. 7.2.

7e7 Theorem. [1] Let $E \le_B E_1$; then either $E \le_B E_0$ or $E \sim_B E_1$.

These statements are quite simple, but their proofs are quite complicated. Here is one more example and one more dichotomy.

7e8 Example. E_3. Here $X = \mathbb{R}^{\infty\times\infty}$, and $x \sim y$ means that $\forall m \, \exists N \, \forall n \ge N \;\; x_{m,n} = y_{m,n}$.

7e9 Theorem. [2] Let $E \le_B E_3$; then either $E \le_B E_0$ or $E \sim_B E_3$.

7e10 Proposition. [3] Neither $E_1 \le_B E_3$ nor $E_3 \le_B E_1$.

A lot of mutually incompatible Borel equivalence relations exist above E_0. In contrast, below E_0 they are linearly ordered.

Example 7a6 is not reducible to E_1, nor to E_3. This is a special case of a general result.[4] On one hand, E_1 and E_3 belong to the set of all Borel equivalence relations obtainable from $\mathbb{N}$ by (arbitrary combinations of) five special operations.[5] On the other hand, Example 7a6 is "turbulent".[6]

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[1] Kechris and Louveau. See Kanovei, Th. 5.7.3.

[2] Hjorth and Kechris. See Kanovei, Th. 5.7.6.

[3] See Kanovei, Lemma 13.9.5.

[4] See Kanovei, Th. 13.5.3.

[5] Union, disjoint union, product, Fubini product, and power; see Kanovei, Sect. 4.2.

[6] Kechris and Sofronidis; see Hjorth, Th. 3.27.

Hints to exercises

7b4: given $\varphi : X \to Y$, consider $(X, \sigma(\varphi))$.

7c2: otherwise, by 7b5, $\varphi \circ f : (x, y) \mapsto (x, \psi(x))$ is a Borel measurable map $B \to B$.

7c3: (a) $A = \varphi^{-1}(f^{-1}(A))$; (b) φ and $f|_{\varphi(Z)}$ are mutually inverse measurable maps; (d) use (c) and 6b7.

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