3 Topology as a powerful helper

Inventive use of basic topological ideas (such as convergence, compactness, continuity) leads to deeper understanding of Borel sets. At last we prove that the probability of being nonempty is well-defined for every random Borel set.

3a Metric spaces, Polish spaces

The basic topological ideas we need.

3a1 Definition. (a) A *metric space* is a pair (X, ρ) of a set X and a *metric* ρ on X, that is, a function $\rho : X \times X \to [0, \infty)$ such that $\rho(x, y) = 0 \iff x = y$, $\rho(x, y) = \rho(y, x)$, $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

(b) Let ρ_1, ρ_2 be two metrics on X; ρ_2 is *stronger* than ρ_1 if $\rho_2(x_n, x) \to 0 \implies \rho_1(x_n, x) \to 0$ for all $x, x_1, x_2, \dots \in X$;[1] further, ρ_1, ρ_2 are *equivalent,* if $\rho_1(x_n, x) \to 0 \iff \rho_2(x_n, x) \to 0$ for all $x, x_1, x_2, \dots \in X$.

(c) A *metrizable space*[2] is a pair (X, R) where X is a set and R is an equivalence class of metrics on X (*metrizable topology*; metrics of R are called *compatible*).

3a2 Core exercise. If ρ is a metric on X then $\rho_1 : (x, y) \mapsto \min(1, \rho(x, y))$ is a metric equivalent to ρ.

Prove it.

[1] However, a Cauchy sequence in (X, ρ_2) need not be Cauchy in (X, ρ_1).

[2] Equivalently, and usually, a metrizable space is defined as a special case of a topological space; but here we do not need the notion of general (not just metrizable) topological space.

The relation $x_n \to x$ ("convergence") is well-defined (as $\rho(x_n, x) \to 0$, of course) both in a metric space and in a metrizable space. Closed sets, open sets and continuous maps are defined as usual. If ρ_2 is stronger than ρ_1 then every set closed in (X, ρ_1) is closed in (X, ρ_2); the same holds for open sets. The identity map $x \mapsto x$ is continuous from (X, ρ_2) to (X, ρ_1) if and only if ρ_2 is stronger than ρ_1.

A *subspace* of a metric space (X, ρ) is a metric space of the form $(Y, \rho|_{Y\times Y})$ where $Y \subset X$. Clearly, a subspace of a metrizable space is a well-defined metrizable space. For $y, y_n \in Y$ the conditions "$y_n \to y$ in Y" and "$y_n \to y$ in X" are equivalent.

The product of two metrizable spaces[1] (X_1, ρ_1) and (X_2, ρ_2) is the metrizable space $(X_1 \times X_2, \rho)$ where $\rho\big((x_1, x_2), (y_1, y_2)\big) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2)$ or, equivalently, $\max\big(\rho_1(x_1, y_1),\ \rho_2(x_2, y_2)\big)$. For $x = (x_1, x_2)$, $x_n = (x_{n,1}, x_{n,2}) \in X_1 \times X_2$ the relation $x_n \to x$ holds if and only if $x_{n,1} \to x_1$ and $x_{n,2} \to x_2$.[2]

3a3 Example. (a) $\mathbb{R}$ is a metric space with its usual metric $(x, y) \mapsto |x-y|$, and a metrizable space with its usual convergence;

(b) $\mathbb{R}^d$ is a metrizable space (being $\mathbb{R}\times\dots\times\mathbb{R}$), with its usual (coordinatewise) convergence;

(c) every subset of $\mathbb{R}^d$ is a metrizable space;

(d) the measure algebra $\mathcal{A}/\overset{\mu}{\sim}$ of a probability space is a metric space with the metric $(A, B) \mapsto \mu(A \triangle B)$ (recall 2a6); its convergence is the "topological convergence", one of the two modes of convergence treated in 2d13–2d14.

A Cauchy sequence $(x_n)_n$ in a metric space (X, ρ) is defined as usual: $\sup_k \rho(x_n, x_{n+k}) \to 0$ as $n \to \infty$.

3a4 Core exercise. Prove that the metric ρ_1 introduced in 3a2 has the same Cauchy sequences as ρ.

3a5 Definition. (a) A metrizable space (as well as its metrizable topology) is *compact* if every sequence has a convergent subsequence.

(b) A metrizable space (as well as its metrizable topology) is *separable* if some sequence is dense.

(c) A metric space (as well as its metric) is *complete* if every Cauchy sequence is convergent.

(d) A metrizable space (as well as its metrizable topology) is *Polish* if it is separable, and in some compatible metric it is complete.

[1] Again, sometimes we work with equivalence classes implicitly, via their elements ("representatives").

[2] Sorry, x_1 and x_2 are ambiguous...

However, "Polish metric space" is ambiguous; for some authors it is "separable complete metric space", while others mean completeness in some equivalent metric.

3a6 Core exercise. A metrizable space is separable if and only if there exists a *countable base,* that is, a sequence $(U_n)_n$ of open sets such that $U = \cup_{n:U_n\subset U} U_n$ for every open set U.

Prove it.

3a7 Core exercise. A subspace of a separable space is separable.

Prove it.

3a8 Core exercise. (a) A compact space is separable.

(b) A compact space is complete in every compatible metric.

Prove it.

Thus, a compact space is Polish.

The space $[0, 1]$ is compact; $(0, 1)$ is not compact, and not complete, but still Polish (being homeomorphic to $\mathbb{R}$).

3a9 Core exercise. (a) The product of two compact spaces is compact.

(b) The product of two Polish spaces is Polish.

Prove it.

3a10 Core exercise. In the measure algebra $\mathcal{A}/\overset{\mu}{\sim}$ (recall 3a3(d)),

(a) $\operatorname{dist}(A_1, A_1 \cup A_2 \cup \dots) \le \operatorname{dist}(A_1, A_2) + \operatorname{dist}(A_2, A_3) + \dots;$

(b) $\operatorname{dist}(A_1, A_1 \cap A_2 \cap \dots) \le \operatorname{dist}(A_1, A_2) + \operatorname{dist}(A_2, A_3) + \dots;$

(c) if $\sum_n \operatorname{dist}(A_n, A_{n+1}) < \infty$ then $\liminf_n A_n \overset{\mu}{\sim} \limsup_n A_n$, that is, $\operatorname{dist}\big(\cup_n \cap_k A_{n+k}, \cap_n \cup_k A_{n+k}\big) = 0;$

(d) $\mathcal{A}/\overset{\mu}{\sim}$ is a complete metric space;

(e) $\mathcal{A}/\overset{\mu}{\sim}$ is (topologically) closed in $2^X/\overset{\mu}{\sim}$.

Prove it.

3a11 Core exercise. The measure algebra over $(0, 1)$ with Lebesgue measure is not compact.

Prove it.

3b Why do we need Polish spaces

A wonder: "large" spaces will be instrumental in understanding Borel sets in "small" spaces.

3b1 Definition. Let $(X, \mathcal{A})$ be a measurable space. A subset of X is *universally measurable* if it is μ-measurable for every probability measure μ on $(X, \mathcal{A})$.

Note that the universally measurable sets are a σ-algebra $\cap_\mu \mathcal{A}_\mu \supset \mathcal{A}$.

Warning. Every measurable set is universally measurable, but an universally measurable set is generally not measurable! This terminological anomaly appears because the word "measurable" is used differently in two contexts, of measurable spaces and of measure spaces.

3b2 Core exercise. Prove that the following two claims are equivalent:

(a) all standard Borel spaces satisfy 2d20(a,b);

(b) for every Borel set $S \subset \{0,1\}^\infty \times \{0,1\}^\infty$, its first projection $\{x : \exists y\,(x,y) \in S\}$ is universally measurable in $\{0,1\}^\infty$;

(c) the same for $\mathbb{R} \times \mathbb{R}$ instead of $\{0,1\}^\infty \times \{0,1\}^\infty$.

Dramatic is the history of 3b2(c)! Lebesgue claimed in 1905 a lemma: the projection of a Borel set is a Borel set, with such proof: the projection of a rectangle is an interval; the projection of the union (of arbitrary sequence of sets) is the union of projections; and the projection of the intersection is the intersection of projections. In 1917, Souslin[1] found an error in Lebesgue's proof: the projection of the intersection need not be the intersection of projections. Moreover, Souslin constructed a counterexample to Lebesgue's lemma, thus starting the theory of the so-called analytic sets. Lusin[2] proved in a note of 1917, just after Souslin's, that the projection of a Borel set is Lebesgue measurable. In 1930, in the preface to a book by Lusin, Lebesgue expressed his joy that he was inspired to commit such a fruitful error…

We want to prove universal measurability of the projection of a Borel set. Could topology help? The projection of a compact set is compact; but a Borel set need not be compact, of course. Here is a daring idea: the topology being auxiliary, we may replace it. Let us adapt the topology to the given Borel set!

The new topology should be stronger than the original one, then the projection map will remain continuous. And the given Borel set should turn into something more special; maybe closed? or even compact?

First of all, "compact" is too good, since then its projection must be compact; but clearly, its projection can be (at least) an arbitrary Borel set.

[1]Mikhail Ya. Souslin (=Suslin), 1894–1919, Russia; a research student of Lusin.

[2]Nikolai N. Lusin (=Luzin), 1883–1950, Russia.

Maybe, a σ-compact set, that is, a countable union of compact sets?[1] No, this is still too good, since the projection need not be an F_σ set.

Maybe, closed? No; every set is closed in the discrete topology ($\rho(x,y) = 1$ whenever $x \ne y$). However, the discrete topology is not separable. Maybe a closed set in a separable space? No; every subset Z of (say) $[0,1]$ is closed (and open) in the separable metric

$$\rho(x,y) = \begin{cases} |x-y| & \text{for } x,y \in Z \text{ or } x,y \notin Z, \\ 1 & \text{for } x \in Z,\ y \notin Z \text{ or } x \notin Z,\ y \in Z. \end{cases}$$

The right choice appears to be, a Polish topology. And then, the set may be closed, or open, or even both closed and open.

3c Adapting Polish topology

Being a helper, topology will be adapted to a given Borel set.

The Cantor set $\{0,1\}^\infty$ is our first example of a Polish (and even compact) space with an infinite algebra of clopen (that is, closed and open) sets. Stronger topologies on $\{0,1\}^\infty$ will give more examples, not compact (and even not σ-compact) but still Polish.

For every metrizable space (X,R) all clopen sets are an algebra (think, why); we denote it by $\operatorname{Clopen}(X,R)$. If R_2 is stronger than R_1 then $\operatorname{Clopen}(X,R_1) \subset \operatorname{Clopen}(X,R_2)$.

3c1 Lemma. Let (X,R_0) be a Polish space, $\mathcal{E} = \operatorname{Clopen}(X,R_0)$, and $G \in \mathcal{E}_\sigma$. Then there exists a Polish topology R on X, stronger than R_0, such that $G \in \operatorname{Clopen}(X,R)$.

For the proof we take $E_1, E_2, \dots \in \mathcal{E}$ such that $E_n \uparrow G$, define $N : X \to \{0,1,2,\dots\}$ by $N(x) = \min\{n : x \in E_n\}$ for $x \in G$ and $N(x) = 0$ for $x \in X \setminus G$, choose a complete metric $\rho_0 \in R_0$ such that $\forall x,y\ \rho_0(x,y) \le 1$, and define

$$\rho(x,y) = \begin{cases} \rho_0(x,y) & \text{if } N(x) = N(y), \\ 1 & \text{otherwise.} \end{cases}$$

Exercise 3c2 completes the proof.

[1] Note that $\mathbb{R}^d$ is σ-compact. Do not confuse "σ-compact" with "locally compact". A dense countable subset of $\mathbb{R}^d$ is σ-compact but not locally compact, nor Polish (use Baire category). And here is an example of a σ-compact, but not locally compact, Polish space: $\{\lambda e_k : \lambda \in \mathbb{R}, k = 1,2,\dots\}$ where e_k are orthonormal vectors of a Hilbert space.

3c2 Core exercise. (a) ρ is a metric on X, and $\rho_0(\cdot,\cdot) \le \rho(\cdot,\cdot) \le 1$;

(b) $\rho(x_n,x) \to 0$ if and only if $\rho_0(x_n,x) \to 0$ and $N(x_n) \to N(x)$;

(c) (X,ρ) is separable;

(d) a sequence $(x_n)_n$ is Cauchy in (X,ρ) if and only if it is Cauchy in (X,ρ_0) and $N(x_n)$ converges;

(e) (X,ρ) is complete;

(f) G is clopen in (X,ρ).

Prove it.

By 3c1 and 1b7, an open (or closed) subset of the Cantor set can be made clopen. Can we do the same for G_δ (or F_σ) sets by repeating the process? Yes, but not just now; before serving a G_δ set we need to serve countably many open sets with a single Polish topology.

3c3 Lemma. Let R_1, R_2 be metrizable topologies on X. Then there exists a metrizable topology R on X such that $x_n \to x$ in (X,R) if and only if $x_n \to x$ both in (X,R_1) and in (X,R_2).

3c4 Core exercise. Let $\rho_1 \in R_1$ and $\rho_2 \in R_2$; define ρ by $\rho(x,y) = \max\big(\rho_1(x,y), \rho_2(x,y)\big)$, and R by $\rho \in R$; prove that R satisfies 3c3.

Clearly, R is the weakest among all metrizable topologies that are stronger than R_1, and R_2; we denote it

$$R = R_1 \vee R_2\,.$$

3c5 Core exercise. Let $R = R_1 \vee R_2$. If R_1, R_2 are separable then R is separable.

Prove it.

3c6 Lemma. Let $R = R_1 \vee R_2$ and in addition, R_1 and R_2 be stronger than some metrizable topology R_0. If R_1, R_2 are Polish then R is Polish.

Proof. Separability is ensured by 3c5. We choose complete $\rho_1 \in R_1$, $\rho_2 \in R_2$, define ρ by $\rho(x,y) = \max\big(\rho_1(x,y), \rho_2(x,y)\big)$ and prove its completeness.

Let $(x_n)_n$ be Cauchy in (X,ρ). Then it is Cauchy, and therefore convergent, both in (X,ρ_1) and in (X,ρ_2). We have $\rho_1(x_n,x) \to 0$ and $\rho_2(x_n,y) \to 0$. It remains to prove that $x = y$, which follows from the fact that $x_n \to x$ in (X,R_0) and $x_n \to y$ in (X,R_0). ☐

3c7 Lemma. Let $R_1, R_2, \dots$ be metrizable topologies on X. Then there exists a metrizable topology R on X such that $x_n \to x$ in (X,R) if and only if $x_n \to x$ in (X,R_i) for all i.

Proof. We choose $\rho_i \in R_i$ such that $\rho_i(\cdot,\cdot) \le 1$, define ρ by

$$\rho(x,y) = \sup_i \frac{1}{i}\rho_i(x,y)$$

(clearly ρ is a metric stronger than each ρ_i) and R by $\rho \in R$.[1] If $\rho_i(x_n,x) \to 0$ for all i,[2] then

$$\limsup_n \rho(x_n,x) \le \limsup_n \max\bigl(\rho_1(x_n,x),\dots,\rho_i(x_n,x);\tfrac{1}{i}\bigr) \le \frac{1}{i}$$

for all i, therefore $\rho(x_n,x) \to 0$. □

Once again, R is the weakest among all metrizable topologies that are stronger than each R_i; we denote it

$$R = R_1 \vee R_2 \vee \dots$$

3c8 Lemma. Let $R = R_1 \vee R_2 \vee \dots$. If all R_i are separable then R is separable.

Proof. By 3c5, $R_1 \vee \dots \vee R_i$ is separable for each i. A sequence $(x_n)_n$ dense in $(X, R_1 \vee \dots \vee R_i)$ satisfies $\inf_n \rho(x,x_n) \le 1/i$ for all $x \in X$ (here ρ is constructed as in the proof of 3c7). The union of dense (at most) countable sets for $i = 1,2,\dots$ is dense in (X,R). □

3c9 Core exercise. Let $R = R_1 \vee R_2 \vee \dots$ and in addition, each R_i be stronger than some metrizable topology R_0. If each R_i is Polish then R is Polish.

Prove it.

3c10 Core exercise. Let (X,R_0) be a Polish space, $\mathcal{E} = \operatorname{Clopen}(X,R_0)$, and $G_1, G_2, \dots \in \mathcal{E}_\sigma$. Then there exists a Polish topology R on X, stronger than R_0, such that $G_1, G_2, \dots \in \operatorname{Clopen}(X,R)$.

Prove it.

Combining 3c10 and 3c1 we can get $A \in \operatorname{Clopen}(X,R)$ for $A \in \mathcal{E}_{\sigma\delta}$, and much more!

3c11 Proposition. Let (X,R_0) be a Polish space, $\mathcal{E} = \operatorname{Clopen}(X,R_0)$, and $A \in \sigma(\mathcal{E})$. Then there exists a Polish topology R on X, stronger than R_0, such that $A \in \operatorname{Clopen}(X,R)$.

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[1] Strangely enough, R depends symmetrically on R_i, which cannot be said about ρ and ρ_i. But we really need some coefficient tending to 0; otherwise we get a metric much stronger than needed.

[2] Not uniformly in i, in general.

Proof. We consider the set

$$\mathcal{A} = \bigcup_R \operatorname{Clopen}(X, R)$$

of all such $A \subset X$. Clearly, $\mathcal{E} \subset \mathcal{A}$. It is sufficient to prove that $\mathcal{A}$ is a σ-algebra.

Clearly, $\sim\mathcal{A} \subset \mathcal{A}$. We'll prove that $\mathcal{A}_\sigma \subset \mathcal{A}$. Let $A \in \mathcal{A}_\sigma$, that is, $A = A_1 \cup A_2 \cup \dots$, $A_n \in \mathcal{A}$. We take Polish topologies R_n on X, stronger than R_0, such that $A_n \in \operatorname{Clopen}(X, R_n)$. By 3c9, $R_\infty = R_1 \vee R_2 \vee \dots$ is a Polish topology. The algebra $\mathcal{E}_\infty = \operatorname{Clopen}(X, R_\infty)$ contains all A_n, thus, $A \in (\mathcal{E}_\infty)_\sigma$. By 3c1, $A \in \operatorname{Clopen}(X, R)$ for some Polish topology R stronger than R_∞. □

3c12 Theorem. For every Borel subset B of the Cantor set X there exists a Polish topology R on X, stronger than the usual topology on X, such that B is clopen in (X, R).

Proof. For the algebra $\mathcal{E} = \operatorname{Clopen}(X, R_0)$ (where R_0 is the usual topology), $\sigma(\mathcal{E})$ is the whole Borel σ-algebra on X; use Prop. 3c11. □

All that is pretty useless for a connected X, but the following is useful.

3c13 Extra exercise. Let (X, R_0) be a Polish space, and $G \subset X$ an open set. Then there exists a Polish topology R on X, stronger than R_0, such that $G \in \operatorname{Clopen}(X, R)$.

Prove it.

3c14 Definition. The *Borel σ-algebra* on a metrizable space is the σ-algebra generated by all open sets. A *Borel set* is a set that belongs to the Borel σ-algebra.

(Compare it with 1c1.)

3c15 Extra exercise. For every Borel subset B of a Polish space (X, R_0) there exists a Polish topology R on X, stronger than the usual topology on X, such that B is clopen in (X, R).

Prove it.

3d Measurability implies continuity

No, this title is not a mistake! True, a measurable function need not be continuous in a given topology; but we adapt topology to the function.

3d1 Proposition. Let $\varphi : \{0,1\}^\infty \to Y$ be a Borel map from the Cantor set to a separable metrizable space Y. Then there exists a Polish topology R on $\{0,1\}^\infty$, stronger than its usual topology, such that φ is continuous on $(\{0,1\}^\infty, R)$.

3d2 Lemma. For arbitrary Borel subsets $B_1, B_2, \dots$ of the Cantor set X there exists a Polish topology R on X, stronger than the usual topology on X, such that $B_1, B_2, \dots \in \text{Clopen}(X, R)$.

Proof. Combine 3c12 and 3c9. □

Proof of Prop. 3d1. We choose a countable base $(U_n)_n$ of Y and note that the sets $\varphi^{-1}(U_n)$ are Borel measurable. Lemma 3d2 gives us R such that all these sets are clopen, therefore open, in (X, R). □

Note that 3d1 turns into 3c12 for a two-element space Y, and into a special case of 3d2 for a countable Y with the discrete topology.

3e Analytic sets

The next class of sets after the Borel σ-algebra.

3e1 Definition. A subset of a Polish space is *analytic* if it is the image of some Polish space under some continuous map.

3e2 Proposition. Every Borel subset of the Cantor set is analytic.

Proof. Follows from Theorem 3c12, since a closed subset of a Polish space is itself a Polish space. □

The same holds for the space $\{0,1\}^\infty \times \{0,1\}^\infty$, since it is homeomorphic to the Cantor set.

Clearly, a continuous image of an analytic set is analytic. By 3e2, a continuous image of a Borel subset of $\{0,1\}^\infty$ (or $\{0,1\}^\infty \times \{0,1\}^\infty$) is analytic. In particular, we get the following.

3e3 Proposition. For every Borel set $S \subset \{0,1\}^\infty \times \{0,1\}^\infty$, its first projection $\{x : \exists y\, (x, y) \in S\}$ is analytic.

Thus, in order to prove 3b2(b) it is sufficient to prove that every analytic subset of the Cantor set is universally measurable.

3f Analytic sets are universally measurable

The space is "large", but its relevant part is "small".

3f1 Core exercise. Let (X, R) be a metrizable space and $\mathcal{B}$ its Borel σ-algebra.

(a) the measurable space $(X, \mathcal{B})$ is separated;

(b) if (X, R) is separable then $(X, \mathcal{B})$ is countably generated, and therefore, a Borel space.

Prove it.[1]

Let μ be a probability measure on a separable metrizable space X (endowed with its Borel σ-algebra, of course). Recall the σ-algebra Sandwich$(\mathcal{K})$ (see 2a15–2a16) where $\mathcal{K}$ is the set of all compact subsets of X. Does this σ-algebra contain open sets? In $\mathbb{R}^d$ every open set is σ-compact (that is, belongs to $\mathcal{K}_\sigma$), but generally it is not.[2] And nevertheless…

3f2 Proposition. If X is Polish then Sandwich$(\mathcal{K})$ contains all open sets, and therefore, all Borel sets.

3f3 Core exercise. If X is σ-compact then

(a) every closed set is σ-compact,

(b) every open set is σ-compact.

Prove it.

Thus, 3f2 is reduced to 3f4 below.

3f4 Proposition. For every probability measure on a Polish space there exists a σ-compact set of full measure.

A wonder: every probability measure on a "large" space sits on a "small" (but maybe dense) set.

Proposition 3f4 fails for non-finite measures.

3f5 Core exercise. If a separable metric space X is not σ-compact then there exists a measure μ on X such that $\mu(K) = 0$ for all compact $K \subset X$, but $\mu(X) = \infty$.

Prove it.

Proposition 3f4 fails also for non-Polish spaces.

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[1] In fact, if (X, R) is Polish then $(X, \mathcal{B})$ is a standard Borel space.

[2] In a separable Hilbert space, a nonempty open set cannot be σ-compact by the Baire category theorem.

3f6 Extra exercise. Let a metrizable space X be homeomorphic to a set $Y \subset \mathbb{R}$ that is not Lebesgue measurable. Then

(a) there exists a probability measure on X with no σ-compact set of full measure; and moreover,

(b) there exists a probability measure on X such that every compact set is of measure zero.

Prove it.

Given a set A in a metric space X and a number $r \ge 0$, we denote by A_{+r} the closed r-neighborhood of A:

$$A_{+r} = \Big\{ x \in X : \inf_{a \in A} \rho(x,a) \le r \Big\} .$$

3f7 Core exercise. Let X be a separable metric space, and μ a probability measure on X. Then for every $r, \varepsilon > 0$ there exists a finite set $J \subset X$ such that $\mu\big(J_{+r}\big) \ge 1 - \varepsilon$.

Prove it.

3f8 Core exercise. Let X be a complete separable metric space, $J_n \subset X$ finite sets, and $r_n > 0$, $r_n \to 0$. Then the set

$$\bigcap_n (J_n)_{+r_n}$$

is compact.

Prove it.

Proof of Prop. 3f4. Given $\varepsilon > 0$, we choose $\varepsilon_n > 0$ such that $\sum_n \varepsilon_n \le \varepsilon$, and $r_n > 0$ such that $r_n \to 0$. First, we take a finite set $J_1 \subset X$ such that the closed set $F_1 = (J_1)_{+r_1}$ satisfies $\mu(F_1) \ge 1 - \varepsilon_1$. Second, we take a finite set $J_2 \subset F_1$ such that the closed set $F_2 = F_1 \cap (J_2)_{+r_2}$ satisfies $\mu(F_2) \ge \mu(F_1) - \varepsilon_2 \ge 1 - \varepsilon_1 - \varepsilon_2$. And so on. The set $K = \cap_n F_n$ is compact, and $\mu(K) = \lim_n \mu(F_n) \ge 1 - \sum_n \varepsilon_n \ge 1 - \varepsilon$. $\square$

Thus, 3f2 is proved. It follows that (in a Polish space)[1]

(3f9) $\sup\{\mu(K) : K \subset A,\, K \text{ is compact}\} =$
$\sup\{\mu(F) : F \subset A,\, F \text{ is closed}\} = \mu(A) = \inf\{\mu(G) : G \supset A,\, G \text{ is open}\}$

for every Borel set $A \subset X$.

In a separable metrizable (rather than Polish) space, (3f9) fails by 3f6.

Well, this was a rather easy task. Now we enter a similar but somewhat more complicated way in order to prove that analytic sets are universally measurable.

[1] "Inner regularity" and "outer regularity".

3f10 Definition. An *outer measure*[1] on a set X is a map $\mu^* : 2^X \to [0, +\infty]$ such that

(a) $\mu^*(\emptyset) = 0$,

(b) $\mu^*(A) \le \mu^*(B)$ whenever $A \subset B \subset X$ ("monotonicity");

(c) $\mu^*(A_1 \cup A_2 \cup \dots) \le \mu^*(A_1) + \mu^*(A_2) + \dots$ for all $A_1, A_2, \dots \subset X$ ("countable subadditivity").

An outer measure μ^* is *upward σ-continuous,* if

(d) $A_n \uparrow A$ implies $\mu^*(A_n) \uparrow \mu^*(A)$ for all $A, A_1, A_2, \dots \subset X$.

We really need only $\mu^* : 2^X \to [0, 1]$.

Every probability measure μ leads to the corresponding outer measure μ^*, see 2a3(a) and 2a5. But in general an outer measure does not correspond to any measure (even if X is finite).[2]

3f11 Core exercise. (a) If an outer measure corresponds to some probability measure then it is upward σ-continuous. Prove it.

(b) In general, an outer measure need not be upward σ-continuous. Give a counterexample.

3f12 Core exercise. Let ν^* be an outer measure on a set Y, and $\varphi : X \to Y$. Then the formula

$$\mu^*(A) = \nu^*\big(\varphi(A)\big)$$

defines an outer measure μ^* on X. If ν^* is upward σ-continuous then μ^* is upward σ-continuous.

Prove it.

Even if ν^* is a measure, μ^* is generally not. (Try a projection. . .)

We turn to Polish spaces X, Y, a continuous map $\varphi : X \to Y$ and a probability measure ν on Y. We want to prove that the analytic set $\varphi(X) \subset Y$ is ν-measurable, that is, $\nu_*\big(\varphi(X)\big) = \nu^*\big(\varphi(X)\big)$. We consider the outer measure μ^* on X corresponding to ν^* as in 3f12. Upward σ-continuity is ensured for ν^* by 3f11(a) and for μ^* by 3f12.

3f13 Core exercise. $\nu_*\big(\varphi(X)\big) \ge \sup\{\mu^*(K) : K \subset X,\ K \text{ is compact}\}$.

Prove it.

In order to prove that $\nu_*\big(\varphi(X)\big) = \nu^*\big(\varphi(X)\big)$ it is sufficient to prove that

$$\sup\{\mu^*(K) : K \subset X,\ K \text{ is compact}\} = \mu^*(X)\,, \tag{3f14}$$

[1] Called also "abstract outer measure" and "exterior measure".

[2] Try a two-element $X = \{a, b\}$; if μ^* corresponds to some μ then either $\mu^*(\{a\}) + \mu^*(\{b\}) = \mu^*(X)$ or $\mu^*(\{a\}) = \mu^*(\{b\}) = \mu^*(X)$.

which looks similarly to 3f4. However, upward σ-continuity fails to ensure (3f14).[1]

3f15 Definition. An outer measure μ^* on a metrizable space X is *outer regular* if $\mu^*(A) = \inf\{\mu^*(G) : G \supset A,\ G \text{ is open}\}$ for all $A \subset X$.

3f16 Core exercise. Let X, Y be metrizable spaces, ν^* an outer measure on Y, $\varphi : X \to Y$ a continuous map, and μ^* the corresponding outer measure on X (as in 3f12). If ν^* is outer regular then μ^* is outer regular.

Prove it.

3f17 Core exercise. If μ is a probability measure on a Polish space then μ^* is outer regular.

Prove it.

3f18 Extra exercise. Does 3f17 hold on a separable metrizable (rather than Polish) space?

In our situation, outer regularity is ensured for ν^* by 3f17 and for μ^* by 3f16. And now we may forget about Y.

3f19 Proposition. Let μ^* be an outer regular, upward σ-continuous outer measure on a Polish space X, $\mu^*(X) < \infty$. Then (3f14) holds.

3f20 Core exercise. Let X be a separable metric space, and μ^* an upward σ-continuous outer measure on X, $\mu^*(X) < \infty$. Then for every $r, \varepsilon > 0$ there exists a finite set $J \subset X$ such that $\mu^*\big(J_{+r}\big) \ge \mu^*(X) - \varepsilon$.

Prove it.

The proof of Prop. 3f19 is similar to the proof of Prop. 3f4. Assume $\mu^*(X) = 1$. Given $\varepsilon > 0$, we take ε_n, r_n and construct finite J_n, closed F_n and compact K as before; still, $\mu^*(F_n) \ge 1-\varepsilon$ for all n. However, the relation $\mu^*(K) = \lim_n \mu^*(F_n)$ is problematic.

Using the outer regularity we take an open set $G \supset K$ such that $\mu^*(G) \le \mu^*(K)+\varepsilon$. Now, if there exists n such that $F_n \subset G$ then $\mu^*(K)+\varepsilon \ge \mu^*(G) \ge \mu^*(F_n) \ge 1 - \varepsilon$, that is, $\mu^*(K) \ge 1 - 2\varepsilon$. However, existence of such n is problematic.[2]

We modify the construction, ensuring $J_1 \subset J_2 \subset \dots$ (this is evidently possible). Now $J_n \subset F_m$ for all m, n (think, why), therefore $J_n \subset K$ for all n. Thus, $F_n \subset K_{+r_n}$. Exercise 3f21 completes the proof of Prop. 3f19.

[1]A counterexample: an outer measure that vanishes on all σ-compact sets and their subsets, and is equal to 1 on all other sets.

[2]Generally, if $F_n \downarrow K \subset G$ where G is open, K is compact and F_n are closed, the relation $F_n \subset G$ may fail for all n. Example: $K = [0,1]$, $F_n = [0,1] \cup [n,\infty)$, $G = (-1,2)$.

3f21 Core exercise. If K is compact and $G \supset K$ is open then $K_{+r} \subset G$ for some $r > 0$.

Prove it.

We conclude (see 3f19 and 3f13).

3f22 Theorem. Analytic sets in Polish spaces are universally measurable.

And finally (see 3e3, 3b2 and 2d20)...

3f23 Theorem. For every probability space $(X, \mathcal{A}, \mu)$, every standard Borel space $(Y, \mathcal{B})$ and every random measurable subset S of Y, defined on $(X, \mathcal{A}, \mu)$, the set $\{x \in X : S(x) \neq \emptyset\}$ is μ-measurable.

3f24 Core exercise. In the conditions of Theorem 3f23, the set

$$\{x \in X : S(x) \cap B \neq \emptyset\}$$

is μ-measurable for every $B \in \mathcal{B}$.

Prove it.

Hints to exercises

3a2: $\min(1, a + b) \le \min(1, a) + \min(1, b)$.

3a6: "only if": use $\frac{1}{m}$-neighborhood of x_n.

3a7: use 3a6; or alternatively, take $y_n \in Y$ such that $\rho(y_n, x_n) \le 2 \inf_{y \in Y} \rho(y, x_n)$.

3a8: (a) if a space is not separable then $\inf\{\rho(x_m, x_n) : m \ne n\} > 0$ for some $(x_n)_n$.

3a9: (a) subsequence of subsequence.

3a10: (a) $A_1 \cup A_2 \cup \dots = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \dots$;
(b) use (a);
(c) apply (a), (b) to the tail;
(d) use (c);
(e) use (d).

3a11: use binary digits.

3b2: (a,b) recall 2d20(b) and defs (2b9, 2d3+2d18);
(c) use 2b8.

3c2: (c) use 3a7.

3c5: if $(U_n)_n$ is a countable base of R_1 and $(V_n)_n$ — of R_2 then $(U_m \cap V_n)_{m,n}$ is a countable base of R.
Or alternatively, take a sequence $(x_n)_n$ dense in (X, R_1) and a sequence $(y_n)_n$ dense in (X, R_2); then, for each pair (m, n), choose $z_{m,n} \in X$ such that $\max\big(\rho(z_{m,n}, x_m), \rho(z_{m,n}, y_n)\big) \le 2 \sup_{z \in X} \max\big(\rho(z, x_m), \rho(z, y_n)\big)$.

3c9: similar to 3c6.

3c10: use 3c1 and 3c9.

3f3: (b) use (a).

3f5: try μ with only two values, 0 and ∞.

3f7: the union of finite sets J_n can be dense.

3f8: given $(x_n)_n$, first, choose a subsequence situated inside $\{a_1\}_{+r_1}$ for some $a_1 \in J_1$.

3f11: (a) $\mu^*(A_n) = \mu(B_n)$, $B_n \supset A_n$; $A_n \uparrow A$; then $\mu(B_n \setminus B_{n+1}) = 0$.
(b) let $\mu^*(A)$ depend only on the cardinality of A.

3f12: $\varphi(A_1 \cup A_2 \cup \dots) = \varphi(A_1) \cup \varphi(A_2) \cup \dots$

3f13: $\varphi(K)$ is measurable.

3f16: $\varphi^{-1}(G)$ is open.

3f17: use (3f9).

3f20: similar to 3f7.

3f21: otherwise $x_n \in K$, $y_n \notin G$, $\rho(x_n, y_n) \to 0$; choose a convergent sequence.

3f24: use 3f23.

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