6 Good sets and their equivalence classes

6a Lebesgue measurable sets

Lebesgue measurable subsets of $[0,1]$ are a σ-algebra[1] [2] $\mathcal{A}_m$ that contains the σ-ideal of null sets. It is generated by open sets (or just open intervals) and null sets. It contains the Borel σ-algebra, generated by open sets (or just open intervals) only. All subsets of the Cantor set are Lebesgue measurable, but only some of them are Borel measurable.

The measure of symmetric difference, $(A,B) \mapsto m(A \triangle B)$ for $A, B \in \mathcal{A}_m$, is a pseudometric on $\mathcal{A}_m$. The corresponding metric space $\mathcal{A}_m / \overset{m}{\sim}$ consists of equivalence classes $[A] = \{B \in \mathcal{A}_m : m(A \triangle B) = 0\}$ with the metric $\rho([A],[B]) = m(A \triangle B)$.

Operations of complement, difference and finite or countable intersection and union are well-defined for equivalence classes. If $[A_n] = [B_n]$ for all n then $[\cup_n A_n] = [\cup_n B_n]$, etc. Also the partial order is well-defined for equivalence classes: $[A] \le [B] \iff [A \setminus B] = [\emptyset]$.

In practice one often deals with equivalence classes only indirectly, via their (arbitrary) representatives.

The metric space $\mathcal{A}_m / \overset{m}{\sim}$ is separable and complete. It may be treated as a closed subset of the (separable and complete) Hilbert space $L_2([0,1])$ (with an equivalent metric). Anyway, I recall you the proof. Separability: rational intervals and their finite unions are dense in $\mathcal{A}_m / \overset{m}{\sim}$, since their closure leads to a σ-algebra containing all intervals and all null sets. Completeness: every Cauchy sequence contains a subsequence $(A_n)_n$ such that $\sum_n \rho(A_n, A_{n+1}) < \infty$. By the first Borel-Cantelli lemma, $\limsup_n (A_n \triangle A_{n+1})$ is a null set. That is, $1\!\!1_{A_n}(x)$ converge for almost all $x \in [0,1]$ to $1\!\!1_A$ for some A. We have $\liminf_n A_n \subset A \subset \limsup_n A_n$ (up to a null set), therefore $A \in \mathcal{A}_m$.

[1] That is, closed under finite and countable unions (and intersections) and complement.
[2] Here m stands for Lebesgue measure.

Now, $\int |1\!\!1_{A_n} - 1\!\!1_A| \, \mathrm{d}m \to 0$ by the bounded convergence theorem; that is, $m(A_n \triangle A) \to 0$.

Every countable subset of $\mathcal{A}_m$ has a supremum in $\mathcal{A}_m$; that is, for $A_1, A_2, \dots \in \mathcal{A}_m$ there exists (necessarily unique) the least $A \in \mathcal{A}_m$ such that $\forall n \;\; A_n \subset A$; namely, $A = \cup_n A_n$. For uncountable subset of $\mathcal{A}_m$ the supremum need not exist; $\cup_{i \in I} A_i$ may be nonmeasurable (even if each A_i is a singleton) and then the supremum does not exist (think, why).

In contrast, *every* subset of $\mathcal{A}_m / \overset{m}{\sim}$ has a supremum! (You see, for the singletons $[A_i] = [\emptyset]$.) The reason is simple: the supremum of a dense countable subset is also the supremum of the whole set (think, why).

Thus, $\mathcal{A}_m / \overset{m}{\sim}$ is a complete Boolean algebra; it is well-known as the measure algebra (and can be defined axiomatically). It is (or was?) somewhat controversial, whether or not probability theory can use the measure algebra *instead* of probability space.[1]

Some equivalence classes of $\mathcal{A}_m / \overset{m}{\sim}$ contain both open sets and closed sets; for example, intervals (and their finite unions). These closed sets are not unique: one may (at least) add a point. Similarly, from an open set one may delete a point.

Some equivalence classes contain open sets but no closed sets; for example, a dense open set not of full measure. The complement contains closed sets but no open sets.

Some equivalence classes satisfy $0 < m\big(\big(A \cap (s,t)\big) < t - s$ for all $(s,t) \subset [0,1]$; these contain neither open sets nor closed sets.

Lebesgue measure is regular:

$$m(A) = \sup_{\text{compact } K \subset A} m(K) = \inf_{\text{open } G \supset A} m(G) \,. \tag{6a1}$$

It follows readily that each equivalence class contains both F_σ sets and G_δ sets.

Similarly to the relation $U \Vdash A$ we may introduce a relation $U \Vdash_m A$ (or equivalently, $U \Vdash_m [A]$), "A is of full measure on U", by

$$U \Vdash_m A \quad \Longleftrightarrow \quad m(U \setminus A) = 0 \,.$$

The union of all such U is the greatest such U (due to 5d5), that is, the greatest U such that $U \subset A$ up to a null set ("interior" of $[A]$). If $[A]$ contains open sets, we get the greatest one. But generally this is not the case, and it may happen that the "interior" is empty both for $[A]$ and for the complement. And even if $[A]$ contains open sets, the greatest one need

[1] See: "Measure algebra" in Encyclopedia of mathematics wiki, and references therefrom.

not be regular as defined in Sect. 6b. (If $[A]$ contains a dense open set not of full measure then it cannot contain a regular open set, since all sets of $[A]$ are dense).

6a2 Theorem. (Lebesgue's density theorem) If $A \subset \mathbb{R}$ is Lebesgue measurable then

$$\frac{1}{2\delta} m\big(A \cap [x-\delta, x+\delta]\big) \to 1 \quad \text{as } \delta \to 0$$

for almost all $x \in A$.

(For a null set A this holds vacuously.) The same applies to $\mathbb{R} \setminus A$, giving

$$\frac{1}{2\delta} m\big(A \cap [x-\delta, x+\delta]\big) \to 0 \quad \text{as } \delta \to 0$$

for almost all $x \notin A$.

Thus, the set

$$\varphi(A) = \Big\{ x \in \mathbb{R} : \frac{1}{2\delta} m\big(A \cap [x-\delta, x+\delta]\big) \to 1 \Big\}$$

belongs to $[A]$. We have a selector, $[A] \mapsto \varphi(A) \in [A]$.

6a3 Exercise. If $A \subset \mathbb{R}$ is Lebesgue measurable then the set $\varphi(A)$ is Borel measurable.

Prove it.

Note that $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$, but inclusions $\varphi(A \cup B) \supset \varphi(A) \cup \varphi(B)$ and $\varphi(\mathbb{R} \setminus A) \subset \mathbb{R} \setminus \varphi(A)$ are generally strict.

Proof of Th. 6a2. [1] We have to prove that $A \setminus \varphi(A)$ is a null set. We note that $A \setminus \varphi(A) = \cup_\varepsilon A_\varepsilon$ where

$$A_\varepsilon = \Big\{ x \in A : \liminf_{\delta \to 0} \frac{1}{2\delta} m\big(A \cap [x-\delta, x+\delta]\big) < 1 - \varepsilon \Big\} ;$$

it is sufficient to prove that each A_ε is a null set. A set $B = A_\varepsilon$ satisfies $B_\varepsilon = B$ (think, why). Now we may forget A, assume that a number $\varepsilon > 0$ and a set B of positive measure satisfy $B_\varepsilon = B$, and seek a contradiction. In addition we assume that B is bounded (otherwise consider $B \cap (-n, n)$) and, for simpler notation, that $m(B) = 1$ (otherwise rescale).

Using (6a1) we take open $G \supset B$ such that $(1-\varepsilon) m(G) < 1$. Let us define a "good" interval as a closed interval I such that $I \subset G$, $m(I) > 0$ and

$$m(B \cap I) \le (1-\varepsilon) m(I) \, .$$

[1] Following Oxtoby, Sect. 3.

The equality $B_\varepsilon = B$ ensures, for every $x \in B$, existence of arbitrarily short good intervals centered at x. On the other hand, B cannot be covered by disjoint good intervals $I_1, I_2, \dots$, not even up to a null set, since

$$m\Big(B \cap \bigcup_n I_n\Big) = \sum_n m(B \cap I_n) \le (1-\varepsilon) \sum_n m(I_n) =$$
$$= (1-\varepsilon) m\Big(\bigcup_n I_n\Big) \le (1-\varepsilon) m(G) < 1 = m(B)\,.$$

Seeking a contradiction we try to construct such I_n by a (sub)greedy algorithm as follows.

On each step n we choose a good I_n disjoint to $I_1 \cup \dots \cup I_{n-1}$ and suboptimal in the sense that

$$m(I_n) \ge \frac{1}{2} \sup \Big\{ m(I) : I \text{ is good, disjoint to } I_1 \cup \dots \cup I_{n-1} \Big\}\,.$$

The process never stops, but misses a set

$$C = B \setminus \bigcup_n I_n\,, \quad m(C) \ge 1 - (1-\varepsilon) m(G) > 0\,.$$

We introduce (not just good) intervals J_n by

$$J_n = [t_n - 3r_n, t_n + 3r_n] \quad \text{if} \quad I_n = [t_n - r_n, t_n + r_n]\,.$$

We take N such that

$$\sum_k m(I_{N+k}) < \frac{1}{3} m(C)\,,$$

then $\sum_k m(J_{N+k}) < m(C)$; we choose

$$\tilde{x} \in C \setminus \bigcup_k J_{N+k} = B \setminus (I_1 \cup \dots \cup I_N \cup J_{N+1} \cup J_{N+2} + \dots)$$

and a good interval $\tilde{I}$ centered at $\tilde{x}$ and disjoint to the closed set $I_1 \cup \dots \cup I_N$. We wonder, why does the greedy algorithm prefer the short intervals I_{N+k} to $\tilde{I}$. The only possible reason is that $\tilde{I}$ intersects some I_{N+k}. We consider the least such k;

$$\tilde{I} \cap (I_1 \cup \dots \cup I_{N+k-1}) = \emptyset\,, \quad \tilde{I} \cap I_{N+k} \ne \emptyset\,.$$

Then, why prefer I_{N+k} to $\tilde{I}$? Only because $m(I_{N+k}) \ge \frac{1}{2} m(\tilde{I})$. And here is a contradiction:

$$\tilde{I} \text{ is centered at } \tilde{x}\,; \quad \tilde{x} \notin J_{N+k}\,;$$
$$m(I_{N+k}) \ge \frac{1}{2} m(\tilde{I})\,; \quad \tilde{I} \cap I_{N+k} \ne \emptyset\,.$$

Namely, denoting by y the center of I_{N+k} (and J_{N+k}) we have $\frac{3}{2}m(I_{N+k}) < |x-y| \le \frac{1}{2}m(\tilde{I}) + \frac{1}{2}m(I_{N+k}) \le \frac{3}{2}m(I_{N+k})$. $\square$

6b Regular open, regular closed

6b1 Definition. In a metrizable space,

(a) an open set G is *regular* if $\mathrm{Int}(\mathrm{Cl}(G)) \subset G$ (and therefore $\mathrm{Int}(\mathrm{Cl}(G)) = G$).[1]

(b) a closed set F is *regular* if $\mathrm{Cl}(\mathrm{Int}(F)) \supset F$ (and therefore $\mathrm{Cl}(\mathrm{Int}(F)) = F$).

For example, the closed set $\{0\} \subset \mathbb{R}$ is not regular, and the open set $\mathbb{R} \setminus \{0\}$ is not regular; but $(0,\infty)$ and $[0,\infty)$ are regular. The union of two regular open sets need not be regular, but their intersection is regular (since $\mathrm{Int}(\mathrm{Cl}(G_1 \cap G_2)) \subset \mathrm{Int}(\mathrm{Cl}(G_1)) \cap \mathrm{Int}(\mathrm{Cl}(G_2))$ trivially).

Clearly, G is regular open if and only if $X \setminus G$ is regular closed.

We may also consider pairs (G,F) of an open set G and closed F such that $F = \mathrm{Cl}(G)$ and $G = \mathrm{Int}(F)$. Calling them "regular pairs"[2] we observe canonical bijections between regular pairs, regular open sets, and regular closed sets.

An open G is regular if and only if every neighborhood of every boundary point contains points of all the three kinds: interior, boundary *and exterior.* The same holds for closed sets.

(6b2) If G is regular open, U is open, and $U \setminus G \ne \emptyset$ then $U \setminus \mathrm{Cl}(G) \ne \emptyset$.

(6b3) If G_1, G_2 are regular open and $G_1 \ne G_2$ then there exists a point interior to one of G_1, G_2 and exterior to the other.

6b4 Lemma. The closure of an arbitrary open set is a regular closed set.

Thus, the interior of an arbitrary closed set is a regular open set.

Proof of 6b4. Let G be open and $F = \mathrm{Cl}(G)$, then G is a dense subset of (F and) $\mathrm{Int}(F)$, therefore $\mathrm{Cl}(\mathrm{Int}(F)) = \mathrm{Cl}(G) = F$. $\square$

Another proof of 6b4. Consider the "exterior" map $\psi : G \mapsto X \setminus \mathrm{Cl}(G) = \mathrm{Int}(X \setminus G)$ (of the set of all open sets to itself). Clearly,

$$G_1 \subset G_2 \implies \psi(G_1) \supset \psi(G_2)\,. \tag{6b5}$$

[1] $G \in \mathrm{RO}(X)$ according to Kechris (Sect. 8.G).

[2] Not a standard terminology.

Also, $\psi(\psi(G)) = \operatorname{Int}\big(X \setminus (X \setminus \operatorname{Cl}(G))\big) = \operatorname{Int}(\operatorname{Cl}(G))$, which implies, first,

$$\psi(\psi(G)) \supset G\,, \tag{6b6}$$

and second, that G is regular if and only if $\psi(\psi(G)) \subset G$.

Regularity of $\operatorname{Cl}(G)$ is equivalent to regularity of $X \setminus \operatorname{Cl}(G) = \varphi(G)$, thus, to $\psi(\psi(\psi(G))) \subset \psi(G)$. The latter follows immediately from (6b5), (6b6). $\square$

6b7 Exercise. (a) For an open G the set $\operatorname{Int}(\operatorname{Cl}(G))$ is the least regular open set containing G;

(b) for a closed F the set $\operatorname{Cl}(\operatorname{Int}(F))$ is the greatest regular closed set contained in F.

Prove it.

The union of regular open sets need not be regular. Nevertheless, by 6b7, in the partially ordered set of all regular open sets *every* subset has the supremum,

$$\sup_{i\in I} G_i = \operatorname{Int}\left(\operatorname{Cl}\left(\bigcup_{i\in I} G_i\right)\right). \tag{6b8}$$

Similarly, in the partially ordered set of all regular closed sets *every* subset has the infimum,

$$\inf_{i\in I} F_i = \operatorname{Cl}\left(\operatorname{Int}\left(\bigcap_{i\in I} F_i\right)\right). \tag{6b9}$$

These two partially ordered sets are isomorphic (both being isomorphic to the partially ordered set of regular pairs). Thus, each of these partially ordered sets (or rather, *this* partially ordered set) is complete: every subset has the infimum and the supremum.

By the way, regular pairs are closer to the ancient idea of geometric body than sets. Speaking about a ball, ancient geometers did not mean the open ball, nor the closed ball!

> A set $H \subset \mathbb{R}^n$ is called *regular-open*, if it equals the interior of its closure. H is said to be a *geometric body,* if it is bounded, regular-open and Jordan measurable.
>
> Miklós Laczkovich[1]

[1] "Paradoxical decompositions: a survey of recent results", in: First European Congress of Mathematics (Paris 1992), Vol. 2, Part 2, 159–184. (See page 180.)

6c Sets with Baire property

Let X be a metrizable space. The σ-ideal of meager sets leads to an equivalence relation[1] on subsets of X, and the corresponding equivalence classes: $[A] = [B]$ if $A \triangle B$ is meager.

Once again, operations of complement, difference and finite or countable intersection and union are well-defined for equivalence classes. If $[A_n] = [B_n]$ for all n then $[\cup_n A_n] = [\cup_n B_n]$, etc. Also the partial order is well-defined for equivalence classes: $[A] \le [B] \iff [A \setminus B] = [\emptyset]$. (These facts hold for arbitrary σ-ideals.)

6c1 Exercise. The boundary of an open set is nowhere dense, but not always null.

Prove it.

Thus, every open set is equivalent to its closure. Also, every closed set is equivalent to its interior. In contrast to Sect. 6a, now an equivalence class contains open sets if and only if it contains closed sets. Moreover, in this case it contains regular open sets, as well as regular closed sets.

In a completely metrizable space, by 6b3 and Baire category theorem 3a11, regular open sets are never equivalent; the same holds for regular closed sets.

6c2 Definition. A set A in a metrizable space X has the *Baire property* if $[A]$ contains open (equivalently, closed) sets. Symbolically: $A \in \mathrm{BP}(X)$.

6c3 Proposition. $\mathrm{BP}(X)$ is a σ-algebra.

In Sect. 6a the situation is different: sets equivalent to open sets are a system closed under countable unions[2] and finite intersections but not countable intersections (since an equivalence class with no open sets still contains G_δ sets.)

The σ-algebra $\mathrm{BP}(X)$ is generated by open sets and meager sets. It contains the Borel σ-algebra, generated by open sets only. All subsets of the Cantor set have the Baire property, but only some of them are Borel measurable.

Proof of 6c3. Clearly, $\emptyset \in \mathrm{BP}(X)$, and $A \in \mathrm{BP}(X)$ implies $X \setminus A \in \mathrm{BP}(X)$. Let $A = \cup_n A_n$ and each $A_n \in \mathrm{BP}(X)$, that is, $[A_n] = [U_n]$ for some open U_n. Then $[A] = [U]$ where $U = \cup_n U_n$ is open. $\square$

[1] Denoted by "$=^*$" by Kechris (Sect. 8.F).

[2] This fails for uncountable unions (think, why).

Regretfully, $\mathrm{BP}(X)$ should not be called "Baire σ-algebra", and sets with the Baire property should not be called "Baire measurable", since these terms mean something different.

From now on (till the end of Sect. 6c) X is a completely metrizable space. The corresponding set $\mathrm{BP}(X)/\!\sim$ of equivalence classes,[1] as a partially ordered set, is canonically isomorphic to the partially ordered set of regular open sets (or pairs), since each equivalence class $[A]$ contains exactly one regular open set, denote it $U(A)$. Thus, $\mathrm{BP}(X)/\!\sim$ is a complete Boolean algebra. Also, we have a selector, $[A] \mapsto U(A) \in [A]$.

Compare this situation with Lebesgue's density theorem 6a2. There, informally, a set appears to be of full measure in an "infinitesimal neighborhood" of x, for almost every $x \in A$ (but generally not of full measure in any "true" neighborhood). Here, in contrast, a set A is comeager in some (true, not "infinitesimal") neighborhood of x, for most of $x \in A$ (and the proof here is much simpler than that in Sect. 6a).

Also, compare the selector $[A] \mapsto \varphi(A) \in [A]$ of Sect. 6a with the selector $[A] \mapsto U(A) \in [A]$ here. Once again, the selector respects the intersection, but not union, nor complement. That is, $U(A \cap B) = U(A) \cap U(B)$ (since the latter is regular), while the inclusions $U(A \cup B) \supset U(A) \cup U(B)$ and $U(X \setminus A) \subset X \setminus U(A)$ are generally strict. Also, $\varphi(A)$ is Borel measurable (in fact, $F_{\sigma\delta}$), while $U(A)$ is regular open.

The relation $U \Vdash A$ (equivalently, $U \Vdash [A]$) is easy to describe in terms of the regular open set $U(A) \in [A]$. First, $U \Vdash A \iff U \Vdash U(A)$. Second, this evidently holds if $U \subset U(A)$. Otherwise it cannot hold (by (6b2) and the Baire category theorem). We see that

$$U \Vdash A \iff U \subset U(A)\,. \tag{6c4}$$

6c5 Exercise. The following two conditions are equivalent for an arbitrary $A \subset X$:

(a) A has the Baire property;

(b) there exist disjoint open sets $U, V \subset X$ such that $U \cup V$ is dense and $U \Vdash A$, $V \Vdash X \setminus A$.

Prove it.

6c6 Exercise. The Banach-Mazur game of a set with the Baire property is determined.

Prove it.

Thus, a Vitali set does not have the Baire property.

[1] $\mathrm{BP}(X)/\,\mathrm{MGR}(X)$, according to Kechris (Sect. 8.G).

We know that $[0,1] = A \uplus B$ for some A, B such that A is of full measure but meager, while B is comeager but null. An arbitrary subset of A has the Baire property but need not be Lebesgue measurable. An arbitrary subset of B is Lebesgue measurable but need not have the Baire property. In particular, let V be a Vitali set, then $V \cap A$ is meager but not Lebesgue measurable, while $V \cap B$ is null but does not have the Baire property.

6d No isomorphism

We have two complete partially ordered sets (in fact, Boolean algebras): $\mathcal{A}_m/\overset{m}{\sim}$ and $\mathrm{BP}/\!\sim$ (I mean, $\mathrm{BP}([0,1])/\!\sim$; let both be taken over $[0,1]$). They differ in their relations to open sets on $[0,1]$ etc., but we still did not observe any intrinsic distinction. Are they isomorphic as partially ordered sets? In other words: whether there exists an order preserving bijection $\varphi : \mathrm{BP}/\!\sim \to \mathcal{A}_m/\overset{m}{\sim}$, or not? "Order preserving" means

$$a \le b \quad \Longleftrightarrow \quad \varphi(a) \le \varphi(b) \quad \text{for all } a, b \in \mathrm{BP}/\!\sim .$$

We know that $\mathrm{BP}/\!\sim$ is isomorphic to RO (regular open sets). Also, every homeomorphism $[0,1] \to [0,1]$ leads readily to an automorphism (that is, isomorphism to itself) of $\mathrm{BP}/\!\sim$ (in contrast to $\mathcal{A}_m/\overset{m}{\sim}$).

6d1 Exercise. An isomorphism (that is, order preserving bijection) $\varphi : \mathrm{BP}/\!\sim \to \mathcal{A}_m/\overset{m}{\sim}$ preserves operations of complement, difference and finite or countable intersection and union, as well as suprema and infima of arbitrary sets.

Prove it.

6d2 Exercise. Let $\varphi : \mathrm{BP}/\!\sim \to \mathcal{A}_m/\overset{m}{\sim}$ be an isomorphism. Then the function

$$[0,1] \ni t \mapsto m\big(\varphi([0,t])\big) \in [0,1]$$

is a homeomorphism of $[0,1]$.

Prove it.

Thus, having some φ we can construct another φ satisfying

$$m\big(\varphi([0,t])\big) = t \quad \text{for all } t \in [0,1] ;$$

this is assumed from now on.

6d3 Exercise. Let an open $G \subset [0,1]$ be a finite union of open intervals; then

$$m\big(\varphi([G])\big) = m(G) .$$

Prove it.

We take open intervals $I_1, I_2, \dots \subset [0,1]$ such that $G = I_1 \cup I_2 \cup \dots$ is dense in $[0,1]$ but $m(G) < 1$, and consider $G_n = I_1 \cup \dots \cup I_n$.

6d4 Exercise. Prove that $\sup_n [G_n] = \big[[0,1]\big]$ (the equivalence class of the whole $[0,1]$).

On the other hand, $m\big(\varphi([G_n])\big) = m(G_n)$, therefore

$$m\Big(\sup_n \varphi([G_n])\Big) = \sup_n m\big(\varphi([G_n])\big) = \sup_n m(G_n) = m(G) < 1\,.$$

And we got a contradiction:

$$m\Big(\varphi\Big(\sup_n [G_n]\Big)\Big) = 1\,, \quad m\Big(\sup_n \varphi([G_n])\Big) < 1\,.$$

We summarize.

6d5 Proposition. The partially ordered sets $\mathcal{A}_m/\overset{m}{\sim}$ and $\mathrm{BP}/\sim$ are not isomorphic.

Hints to exercises

6a3: the function $x \mapsto m(A \cap [x-\delta, x+\delta])$ is Lipschitz continuous.

6d1: start with suprema and infima.

6d2: use (6b8), (6b9).

6d3: use 6d1.

6d4: use (6b8).

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