
Consider a pure death process in which $i \to i-1$ at rate μ when $i \ge 1$. Find the transition probability $p_{i,j}(t)$.

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The process may be represented as $X_t = Y_{N(t)}$, where $N(t)$ is a Poisson process with rate μ, and the discrete-time Markov chain Y_n is deterministic: $p_Y(i, i-1) = 1$ for $i \ge 1$, and $p_Y(0,0) = 1$. Thus, for $i > j > 0$ we have

$$p_{i,j}(t) = \mathbb{P}\left(X(t) = j \,\middle|\, X(0) = i \right) = \mathbb{P}\left(Y_{N(t)} = j \,\middle|\, Y_0 = i \right) = \mathbb{P}\left(N(t) = i-j \right) = \frac{(\mu t)^{i-j}}{(i-j)!} \mathrm{e}^{-\mu t} \, .$$

Also,

$$p_{i,0}(t) = \mathbb{P}\left(X(t) = 0 \,\middle|\, X(0) = i \right) = \mathbb{P}\left(Y_{N(t)} = 0 \,\middle|\, Y_0 = i \right) = \mathbb{P}\left(N(t) \ge i \right) = \sum_{k=i}^{\infty} \frac{(\mu t)^k}{k!} \mathrm{e}^{-\mu t} \, .$$

Consider two machines that are maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i before it fails. The repair times for each unit are exponential with rate μ_i. They are repaired in the order in which they fail. (a) Let X_t be the number of working machines at time t. Is X_t a Markov chain? (b) Formulate a Markov chain model for this situation with state space $\{0, 1, 2, 12, 21\}$. (c) Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$. Find the stationary distribution.

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(a) X_t is probably not a Markov chain (unless $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$), since the future after $X_t = 1$ depends on the number of the working machine, and the past probably gives some information about this number.

(b) The queue to the repairman can be empty (state 0); it can contain a single machine 1 (state 1) or 2 (state 2); it can also contain machine 1 being repaired and machine 2 waiting (state 12), or other way round (state 21). We have transition rates

$$\begin{aligned} q(0,1) &= \lambda_1; & q(0,2) &= \lambda_2; \\ q(2,21) &= \lambda_1; & q(1,12) &= \lambda_2; \\ q(1,0) &= \mu_1; & q(2,0) &= \mu_2; \\ q(12,2) &= \mu_1; & q(21,1) &= \mu_2. \end{aligned}$$

In addition, $q(0,0) = -\sum_{x \ne 0} q(0,x) = -(\lambda_1 + \lambda_2)$, and similarly $q(1,1) = -(\lambda_2 + \mu_1)$, $q(2,2) = -(\lambda_1 + \mu_2)$, $q(12,12) = -\mu_1$, $q(21,21) = -\mu_2$.

(c) $\sum_x \pi(x) q(x,y) = 0$ for all y; that is,

$$\begin{gathered} \pi(1)\mu_1 + \pi(2)\mu_2 - \pi(0)(\lambda_1 + \lambda_2) = 0, \\ \pi(0)\lambda_1 + \pi(21)\mu_2 - \pi(1)(\lambda_2 + \mu_1) = 0, \\ \pi(0)\lambda_2 + \pi(12)\mu_1 - \pi(2)(\lambda_1 + \mu_2) = 0, \\ \pi(1)\lambda_2 - \pi(12)\mu_1 = 0, \\ \pi(2)\lambda_1 - \pi(21)\mu_2 = 0. \end{gathered}$$

For $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$ we get $3\pi(1) = 2\pi(12)$, $\pi(2) = 4\pi(21)$ and $2\pi(1)+4\pi(2) = 4\pi(0)$, $\pi(0) + 4\pi(21) = 5\pi(1)$, $3\pi(0) + 2\pi(12) = 5\pi(2)$; further, $\pi(1) + 2\pi(2) = 2\pi(0)$, $\pi(0) + \pi(2) = 5\pi(1)$, $3\pi(0) + 3\pi(1) = 5\pi(2)$. We get $\pi(1) = \frac{4}{11}\pi(0)$, $\pi(2) = \frac{9}{11}\pi(0)$, $\pi(12) = \frac{6}{11}\pi(0)$, $\pi(21) = \frac{9}{44}\pi(0)$; $1 = \pi(0) \cdot \bigl(1 + \frac{4}{11} + \frac{9}{11} + \frac{6}{11} + \frac{9}{44}\bigr)$; $\pi(0) = \frac{44}{129}$; $\pi(1) = \frac{16}{129}$, $\pi(2) = \frac{36}{129}$, $\pi(12) = \frac{24}{129}$, $\pi(21) = \frac{9}{129}$.
