1 The Sidrauski model

There are many ways to bring money into the macroeconomic debate. Among the fundamental issues in economics the treatment of money is probably the LESS satisfactory and there is very little agreement about what is the right way to look at monetary issues.

The Sidrauski model assumes we derive direct utility from holding money. This utility is due to the liquidity services we get and not due to the value of money as an asset. In models like this there is no direct utility from holding total assets, only from money.

Assume the representative agent who lives for ever maximizes the following utility:

$$\max V_0 = \int_0^\infty u(c_t, m_t) e^{-\theta t} dt$$

where : $u_c, u_m > 0$, $u_{cc}, u_{mm} < 0$

s.t. : $C_t + \dot{K} + \dot{M}/P = wN + rK + G$

Assuming no population growth one writes:

$$\dot{K}/N = \dot{k}$$

$$\dot{M}/PN = \dot{m} + \pi m$$

Because:

$$\dot{m} = \frac{\dot{M}}{PN} = \frac{\dot{MPN} - \dot{PNM}}{P^2N^2} = \dot{M}/PN - \pi m$$

Using (1 - 4) one can rewrite the budget constraint in “per capita” terms:

$$c_t + \dot{k} + \dot{m} + \pi m = w + rk + g$$

Define TOTAL assets as $a = m + k$

$$\dot{a} = rk + w + g - c - \pi m$$

$$\dot{a} = [ra + w + g] - [c + (\pi + r) m]$$

$$\dot{a} = [\text{income}] - [\text{“consumption”}]$$
and assume NPG: \( \lim_{t \to \infty} a_t \exp \left\{ - \int_0^t r_v dv \right\} = 0 \) we can write down the current value Hamiltonian:

\[
H = u(c_t, m_t) + \lambda \left[ ra + w + g - c - (\pi + r) m \right]
\]

where \( a \) is the state variable and \( c, m \) are control variables.

\[
\begin{align*}
u_c(c_t, m_t) &= \lambda \\
u_m(c_t, m_t) &= \lambda (\pi + r) \\
\dot{\lambda} &= \theta \lambda - r \lambda
\end{align*}
\]

\( NPG: \ldots \)

To close the model we assume (competitive) equilibrium in the capital market and CRS in production + competitive labor market to get: \( r_t = f'(k_t) \), \( w_t = f(k_t) - f'(k_t) k_t \).

An IMPORTANT additional assumption is:

\[
g = \frac{G}{N} = (\dot{M}/P)N = (\dot{M}/M) (M/PN) \equiv \mu m.
\]

This assumption states that government transfers equal ALL the revenue from printing money.

In the steady state we want to assume that not only \( \dot{a} = 0 \) but also \( \dot{m} = \dot{\lambda} = 0 \). The last equality implies that in the steady state the value of assets does not change.

\[
\begin{align*}
\dot{m} &= 0 \implies \mu = \pi \\
\dot{\lambda} &= 0 \implies \theta = f'(k_{ss})
\end{align*}
\]

The first equality states that the rate of printing in the steady state equals the rate of inflation and the second condition implies that in the steady state money is “SUPER NEUTRAL” [the rate of money growth has no effect on real economic activity - the economy is at the “modified golden rule”] (This conditions are general and could be modified to account for economic growth etc.).
Using (5, 6) one gets the optimum quantity of money in the steady state. Consumption equals production (we assumed no depreciation), production is at the modified golden rule level and the demand for money is given by:

\[ u_m = (\mu + \theta) u_c \]

Since money has no effect on real variables in the steady state the best money rule (FRIEDMAN’S RULE) is to make the marginal utility from money = zero (satiation level). This happens if the government REDUCE money (have disinflation!!) at the rate equal to the time preference. We want to compensate people for the liquidity service they need. Since compensation in the steady state is free (printing/absorbing money is free) we should compensate them fully.

\[ FRIEDMAN'S \ RULE \implies -\mu = \theta \]
1.1 Non uniqueness in monetary models

Can the market determine the price level? In many general cases it cannot do so even if the model is well specified a la Sidrauski. This is an important issue since if there is more than one representative agent they have no mechanism to pick a price level everybody agree upon. The concept of equilibrium is ill in such cases.

Let us use a variant of the Sidrauski model. this time we will assume that money is needed for liquidity purposes in firms so the production function is a function of real balances. A very similar model could use separable instantaneous utility between money and consumption.

Assume individuals maximize:

$$\max V_0 = \int_0^\infty u(c_t) e^{-\theta t} dt$$  \hspace{1cm} (7)

s.t. $$\dot{M} = P[h(m_t) - c_t] + G$$  \hspace{1cm} (8)

where:h(m_t) is the production function and the budget constraint is written in nominal terms.

In real terms the dynamic budget constraint is (here $g = G/P$): $$\dot{M}/P = [h(m_t) - c_t] + g$$

Since $\dot{M}/P = \mu m$ and $\dot{M}/P = \dot{m} + \pi_t m$ then $\pi_t = \mu - \dot{m}/m$.

Using the above we can rewrite the real dynamic constraint as:

$$\dot{m}_t = h(m_t) - c_t - \pi_t m_t + g_t.$$  \hspace{1cm} (9)

The Current value Hamiltonian is:

$$H = u(c_t) + \lambda_t [h(m_t) - c_t - \pi_t \dot{m} + g_t] ,$$

and the FOC are:

$$u_t(c_t) = \lambda_t \implies \dot{\lambda}_t = u_{cc}(c_t) \dot{c}_t$$

$$\dot{\lambda}_t = \lambda_t \theta - \lambda_t h'(m_t) + \lambda_t \pi_t \implies \dot{\lambda}_t = -\lambda_t [h'(m_t) - \pi_t - \theta]$$  \hspace{1cm} (10)
Using (7 – 10) one can write the dynamic optimal behavior as:

\[
\dot{c} = -\frac{u_c}{u_{cc}} \left[ h'(m_t) + \frac{\dot{m}}{m} - \mu - \theta \right].
\]  

(11)

The full employment means output equal consumption (there is no investment here) and equilibrium in the money market implies demand=supply or \( c_t = h(m_t) \) and \( m_t = m_t^* \).

Taking derivative w.r.t. time of equilibrium in goods market yields:

\[
\dot{c} = h'(m_t) \dot{m}.
\]

(12)

Using (11 to 12) we can rewrite the dynamic optimal path in terms of real money balances:

\[
h'(m_t) \dot{m} = -\frac{u_c}{u_{cc}} \left[ h'(m_t) + \frac{\dot{m}}{m} - \mu - \theta \right]
\]

OR

\[
\left( \frac{\dot{m}}{m} \right) \frac{h'(m_t) m_t}{h(m_t)} + \frac{u_c(t)}{c \cdot u_{cc}(t)} = -\frac{u_c(t)}{c \cdot u_{cc}(t)} \left[ h'(m_t) - \mu - \theta \right]
\]

(13)

Define the elasticity of output (production) w.r.t money as \( \alpha (m) \):

\[
\alpha (m_t) \equiv \frac{h'(m_t) m_t}{h(m_t)}
\]

Define the elasticity of the MARGINAL utility w.r.t consumption as \( \beta (m) \) : (it can be written as a function of \( m \) since \( c = h(m) \))

\[
\beta (m) \equiv -\frac{u_c(t)}{c \cdot u_{cc}(t)}
\]

We can rewrite the dynamic equation as:

\[
\dot{m}/m = \frac{1}{\alpha (m) - \beta (m)} \left[ h'(m_t) - \mu - \theta \right]
\]

To examine this equation in a simple way let us assume simple functional forms:

\[
h(m) = m^\delta, \quad u(c_t) = \frac{c^{1-\gamma}}{1 - \gamma}
\]
so \( \alpha(m) = \delta \) and \( \beta(m) = \gamma \). In this case
\[
\dot{m}(t) = \frac{1}{\gamma} \left[ \delta (m_t)^{\delta-1} - \mu - \theta \right] m_t
\]
define: \( \Delta = \frac{1}{\delta - \frac{1}{\gamma}} \)
\[
\dot{m}(t) = \Delta \left[ \delta (m_t)^{\delta} - (\mu + \theta) m_t \right]
\]
In the steady state:
\[
m = \left( \frac{\mu + \theta}{\delta} \right)^{\frac{1}{\delta-1}}.
\]
Thus:
\[
\frac{\partial \dot{m}}{\partial m} \mid \text{steady state} = \Delta [(\delta - 1)(\mu + \theta)] = \Delta [\text{sign} (-)]
\]
Analyzing this equation in the \([\dot{m}, m]\) space depends crucially on the sign of \( \Delta \).

If \( \Delta > 0 \) we get non uniqueness. There is no way to determine the value of the price level in the steady state. Any price level people agree upon leads them to the steady state BUT how (why) can they all choose the same price level?

If \( \Delta < 0 \) we have uniqueness BUT in the general case the curve may bend back and we can have bubbles in the price level. Every price level that is higher than the unique steady state price level is an “acceptable” inflationary bubble.