Spectral properties of acyclic matrices

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Abstract

A useful conduit between combinatorics and computational scientific computing is the association between graphs and matrices. Trees are fundamental combinatorial objects, and matrices whose associated graph is a tree often arise in applications and computational scientific computing. In this talk we discuss the spectral properties of such matrices.

1 Introduction

Let $A = [a_{ij}]$ be an n by n real matrix. We say that A is *combinatorially symmetric* if $a_{ji}a_{ij} > 0$ whenever $a_{ij} \neq 0$. The graph $G(A)$ of the combinatorially symmetric matrix A has vertices 1, 2, ..., n and an edge joining i and j if and only if $i \neq j$ and $a_{ij} \neq 0$. Note that $G(A)$ is independent of the diagonal entries A. An *acyclic matrix* is a combinatorially symmetric matrix whose graph is a tree.

Acyclic matrices naturally arise in many different applications, for example the classic oscillating spring systems, and chemistry, as well as the more recent small-world networks [HLS], and the classification of DNA strands[ZFSTGS]. Also, due to their rich, yet simple nature, acyclic matrices arise in graph-based pre-conditioning [BCHT, BGHNT, G]. Acyclic matrices generalize one of the most important classes of matrices–the tridiagonal matrices. After all, the irreducible, tridiagonal matrices are (up to permutational similarity) the combinatorial matrices whose graph is a tree.

Eigenvalues play a central role in each of these applications. For example, eigenvalues reflect certain combinatorial properties in networks and DNA, and are used to measure the quality of a pre-conditioner. Thus, it is of interest to study the spectrum of acyclic matrices.

This study was initiated by Parter [P] and Fiedler [F]. Both observed that each acylic matrix is diagonally similar to a symmetric matrix, and hence that the eigenvalues of an n by n acyclic matrix A are real, say $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, have equal algebraic and geometric multiplicity, and the eigenvalues of principal submatrices interlace those of A. Fielder generalized the Perron-Frobenius theorem by proving that the sign-patterns of eigenvectors of A corresponding to λ_k are largely determined by k and the sign-pattern of A. In addition, he showed that if λ is an eigenvalue with multiplicity greater than 1, then there is j such that every eigenvector of A corresponding to λ has jth coordinate equal to zero. Such an index is a λ-*Fiedler vertex* of $G(A)$. Parter showed the surprising result that if λ is an eigenvalue with multiplicity greater than 1, then there exists an index j such that the multiplicity of λ as a eigenvalue of the principal submatrix $A(j)$ is *more* than the multiplicity of λ as a eigenvalue of A. Such a index is a λ-*Parter vertex* of $G(A)$.

Parter and Fiedler's work has been revived by the recent interest in the inverse-eigenvalue problem for acyclic graphs:

> Determine which n-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, are eigenvalues of an n by n acyclic matrix.

Partial results can be found in [BF, JDS, KS].

In this talk we will provide new proofs of the existence of Fiedler and Parter vertices, and then use these techniques to obtain new results about eigen**vectors** of acyclic matrices, and new insights to the inverse eigenvalue problem. The proofs are geometric rather than algebraic, that is, they are based on properties of eigenvectors rather than characteristic polyomials, and they result in effective algorithms for finding Fiedler and Parter vertices.

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