Some Highlights of Combinatorial Matrix Theory

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Combinatorial Matrix Theory (CMT) is the name generally ascribed to the very successful partnership between Matrix Theory (MT) and Combinatorics & Graph Theory (CGT). It concerns the use of combinatorial and graph-theoretic ideas and reasoning in the study of linear algebraic properties of matrices, and the application of linear algebraic properties of matrices to combinatorial and graph-theoretic questions. CMT also deals with the study of intrinsic properties of matrices, usually with some assumed characteristics, viewed as arrays of numbers (data structures) and not as algebraic objects.

The key to the partnership of MT and CGT is the adjacency matrix of a graph. A graph with n vertices has an adjacency matrix A of order n which is a symmetric (0,1)-matrix. A bipartite graph whose vertices are bipartitioned into sets of size m and n has a bi-adjacency matrix which is a (0,1)-matrix of size m by n. From graphs and bipartite graphs we can proceed to multigraphs and bipartite multigraphs, and then to weighted graphs and weighed bipartite graphs, leading to arbitrary symmetric matrices and arbitrary rectangular matrices and matrices whose entries are 0's and indeterminates. These paths are reversible and lead back from matrices to graphs.

The classical Perron-Frobenius theory concerning spectral properties of nonnegative matrices is one of the early success stories of exploiting the combinatorial structure of matrices. The theory of strongly regular graphs is one of the early success stories of the application of elementary, but powerful, linear algebraic techniques in the study of graphs. Combinatorial tiling enumeration problems often rely on linear algebra. Graph coloring, sometimes aided by linear algebra, is used to model matrix partitioning problems motivated by computational questions, some of which seek to exploit sparsity structure. Graph coloring is also used to model frequency assignment problems in wireless communications.

Two of the most studied combinatorial classes of matrices are the class $\mathcal{A}(R, S)$ of (0,1)-matrices with prescribed row and column sums (marginals) and the class (really, a polytope) Ω_n of doubly stochastic matrices of order n. Doubly stochastic matrices can be regarded as continuous analogues of permutation matrices, and they find application in Markov chain theory. Classes $\mathcal{A}(R, S)$ have a rich structure and arise in such seemingly disparate fields as wildlife ecology and imaging.

In this expository talk we shall discuss these ideas and some particularly nice instances of them. General references are given below.

References

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