

# On the Factor Width of Symmetric Matrices

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## Abstract

We define a matrix concept called *factor width*. This gives a hierarchy of matrix classes for symmetric positive semidefinite matrices. We prove that the set of symmetric matrices with factor width at most two is exactly the class of (possibly singular) symmetric H-matrices with positive diagonals,  $H^+$ . Finally, we pose several open questions regarding factor width.

## 1 Introduction

Symmetric positive definite and semidefinite (SPD and SPSD, respectively) matrices arise frequently in applications and have been studied by many authors. It is well known that a Cholesky decomposition  $A = LL^T$ , where  $L$  is lower triangular, exists for any SPD matrix  $A$ . In this extended abstract we characterize SPSD matrices in terms of rectangular factorizations of the type  $A = VV^T$ , where  $V$  may have more columns than rows and generally has no triangular structure.

## 2 The Factor Width of a Symmetric Matrix

**Definition 1** *The factor width of a real symmetric matrix  $A$  is the smallest integer  $k$  such that there exists a (rectangular) matrix  $V$  where  $A = VV^T$  and each column of  $V$  contains at most  $k$  non-zeros.*

For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

has factor width two because  $A = VV^T$ , where  $V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$ .

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It is easy to see that a matrix has factor width one if and only if it is diagonal and non-negative. Furthermore, any symmetric, positive semidefinite matrix of order  $n$  has factor width at most  $n$ .

**Lemma 2** *For any positive  $k \leq n$ , there exist matrices of order  $n$  with factor width  $k$ .*

We defer all proofs to the full paper.

In conclusion, the concept of factor width defines a family of matrix classes. Let  $FW(k)$  denote the set of matrices with factor width  $k$  or less. It is easy to see that  $FW(k)$  is a convex cone for any  $k$  and  $FW(n)$  is precisely the cone of symmetric positive semidefinite (SPSD) matrices of order  $n$ .

### 3 Factor-Width-2 Matrices are H-Matrices

We briefly state our main results about factor-width two matrices.

**Lemma 3** *If  $A$  is symmetric, diagonally dominant then  $A$  has factor width at most two.*

**Definition 4** *A matrix  $A$  is an  $H^+$ -matrix if  $A$  is an  $H$ -matrix (i.e., its comparison matrix is an  $M$ -matrix) and  $A$  has non-negative diagonal entries.*

We allow both  $M$ - and  $H$ -matrices to be singular.

**Theorem 5** *A matrix has factor width two if and only if it is a symmetric  $H^+$ -matrix.*

### 4 Open problems

We have defined factor width and characterized the matrix classes  $FW(1)$ ,  $FW(2)$ , and  $FW(n)$ . Does  $FW(k)$  correspond to any known matrix class for other values of  $k$ ? In particular, what is  $FW(3)$ ? It is easy to show that matrices from the finite element method have low factor width, so these matrix classes may be of practical interest.

A second open question is how to compute the factor width of a given matrix. Is there any efficient (polynomial-time) algorithm for the general case? From our characterizations, it follows that  $FW(k)$  matrices can be recognized in linear time for  $k = 1$  and in polynomial time for  $k = 2$ , but what about  $k = 3$ ?

A third question is how many columns are required in the factor  $V$  for a given  $A$  and integer  $k$  such that  $A = VV^T$  is a  $FW(k)$  decomposition. This can be viewed as the  $FW(k)$  rank.

In scientific computing one often has to solve systems  $Ax = b$ , where  $A$  is SPD. It is easy to show that matrices from the finite element method have low factor width, so inverting (solving) low factor-width matrices is of great practical interest.