EXCERCISE: DISTRIBUTED SUMMATION

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The relationship between matrices and graphs is deep and has many applications. In this exercise we will explore one interesting application that is not directly related to support preconditioning.

An undirected graph $G = (V, E)$ represents a communication network in which vertices have computational power and edges represent communication links between computers. The goal of the exercise is to design a synchronous distributed algorithm that will allow the vertices to sum the elements of a vector that is stored in the vertices. Initially every vertex $i \in V$ holds a real number x_i. The algorithm runs in synchronous steps that update another value associated every step, y_i. Initially, all the vertices set $y_i = x_i$. In every step, every vertex receives the y values of its neighbors in the graph and replaces y_i by $\sum_{(i,j)\in E} a_{ij} y_j$. We assume that $(i, i) \in E$ for all i. The goal is to find weights a_{ij} that will guarantree that all the y_i's converge to $\sum_{j\in V} x_j$. This amounts to distributed summation: initially every vertex holds a value, and at the end (after enough iterations) all of them hold an approximation of the sum of the values. Another goal is to estimate the convergence rate of the algorithm and to make it as high as possible.

(1) Let L be the unweighted Laplacian matrix of a graph G. Show that $\Lambda(L) \le 2d_{\max}$, where $d_{\max}$ is the maximal degree of a vertex in the graph (not including self loops).

(2) Clearly, a matrix $A = (a_{ij})$ represents one iteration of the distributed summation algorithm if and only if $A^k \longrightarrow \mathbf{1}$, the matrix of all ones. Give an expression for the eigendecompostion of the matrix of all ones.

(3) Given the result of the previous step, characterize the eigendecomposition of A itself.

(4) Show how to use L to construct A. Remember that the sparsity pattern of A and L should be the same. Hint: consider how shifting and scaling a matrix transforms its eigenvalues and eigenvectors.

(5) How does the spectrum of L affect the convergence of the summation algorithm?

(6) Can you come up with some link between the spectral properties of L that affect convergence and some combinatorial property of the graph G? You can try to gain some intuition using Matlab experiments. Is there an intuitive explanation for this phenomenon?