Linear Algebra Notation and Definitions

This appendix provides notation, definitions and some well known results, mostly in linear algebra. We assume that all matrices and vectors are real unless we explicitly state otherwise.

A square matrix is symmetric if $A_{ij} = A_{ji}$.

1. Eigenvalues and Eigenvectors

A square matrix is *positive definite* if $x^T Ax > 0$ for all x and *positive semidefinite* if $x^T Ax \geq 0$ for all x.

The *eigendecomposition* of a square matrix is a factorization $A = V\Lambda V^{-1}$ where Λ is diagonal. The columns of V are called *eigenvectors* of A and the diagonal elements of Λ are called the *eigenvalues* of A. A real matrix may have complex eigenvalues. The expression $\Lambda(A)$ denotes the set of eigenvalues of A.

Not all matrices have an eigendecomposition, but every symmetric matrix has one. In particular, the eigenvalues of symmetric matrices are real and their eigenvectors are orthogonal to each other. Therefore, the eigendecomposition of symmetric matrices is of the form $A = V\Lambda V^T$. The eigenvalues of positive definite matrices are all positive, and the eigenvalues of positive semidefinite matrices are all non-negative.

For any matrix V, the product VV^T is symmetric and positive semidefinite.

2. The Singular Value Decomposition

Every matric $A \in \mathbb{C}^{m \times n}$, $m \geq n$ has a *singular value decomposition* (SVD) $A = U\Sigma V^*$ where $U \in \mathbb{C}^{m \times n}$ with orthonormal columns, $V \in \mathbb{C}^{n \times n}$ with orthonormal columns, and $\Sigma \in \mathbb{R}^{n \times n}$ is non-negative and diagonal. (This decomposition is sometimes called the *reduced* SVD, the full one being with a rectangular U and an m-by-n Σ.) The columns of U are called *left singular vectors*, the columns of V are called right singular vectors, and the diagonal elements of Σ are called *singular values*. The singular values are non-negative. We denote the set of singular values of A by $\Sigma(A)$.

3. Generalized Eigenvalues

Preconditioning involves two matrices, the coefficient matrix and the preconditioners. The convergence of iterative linears solvers for symmetric semidefinite problems depends on the generalized eigenvalues of the pair of matrices. A pair (S, T) of matrices is also called a pencil.

DEFINITION 3.1. Let S and T be n-by-n complex matrices. We say that a scalar λ is a *finite generalized eigenvalue* of the matrix pencil (pair) (S, T) if there is a vector $v \neq 0$ such that

$$Sv = \lambda T v$$

and $Tv \neq 0$. We say that ∞ is a *infinite generalized eigenvalue* of (S, T) if there exist a vector $v \neq 0$ such that $Tv = 0$ but $Sv \neq 0$. Note that ∞ is an eigenvalue of (S, T) if and only if 0 is an eigenvalue of (T, S). The finite and infinite eigenvalues of a pencil are *determined eigenvalues* (the eigenvector uniquely determines the eigenvalue). If both $Sv = Tv = 0$ for a vector $v \neq 0$, we say that v is an *indeterminate eigenvector*, because $Sv = \lambda T v$ for any scalar λ.

We order from smallest to largest. We will denote the kth eigenvalue of S by $\lambda_k(S)$, and the kth determined generalized eigenvalue of (S, T) by $\lambda_k(S, T)$. Therefore $\lambda_1(S) \leq \cdots \leq \lambda_l(S)$ and $\lambda_1(S, T) \leq \cdots \leq \lambda_d(S, T)$, where l is the number of eigenvalues S has, and d is the number of determined eigenvalues that (S, T) has.

DEFINITION 3.2. A pencil (S, T) is *Hermitian positive semidefinite* (H/PSD) if S is Hermitian, T is positive semidefinite, and $\text{null}(T) \subseteq \text{null}(S)$.

The generalized eigenvalue problem on H/PSD pencils is, mathematically, a generalization of the Hermitian eigenvalue problem. In fact, the generalized eigenvalues of an H/PSD can be shown to be the eigenvalues of an equivalent Hermitian matrix. The proof appears in the Appendix. Based on this observation it is easy to show that other eigenvalue properties of Hermitian matrices have an analogy for H/PSD pencils. For example, an H/PSD pencil, (S, T), has exactly $\text{rank}(T)$ determined eigenvalues (counting multiplicity), all of them finite and real.

A useful tool for analyzing the spectrum of an Hermitian matrix is the *Courant-Fischer Minimax Theorem* [**?**].

THEOREM 3.3. *(Courant-Fischer Minimax Theorem) Suppose that* $S \in \mathbb{C}^{n\times n}$ *is an Hermitian matrix, then*

$$\lambda_k(S) = \min_{\dim(U)=k} \max_{x\in U} \frac{x^*Sx}{x^*x}$$

and

$$\lambda_k(S) = \max_{\dim(V)=n-k+1} \min_{x\in V} \frac{x^*Sx}{x^*x}\,.$$

As discussed above, the generalized eigenvalue problem on H/PSD pencils is a generalization of the eigenvalue problem on Hermitian matrices. Therefore, there is a natural generalization of Theorem 3.3 to H/PSD pencils, which we refer to as the *Generalized Courant-Fischer Minimax Theorem.* We now state the theorem. For completeness the proof appears in the Appendix.

THEOREM 3.4. *(Generalized Courant-Fischer Minimax Theorem) Suppose that* $S \in \mathbb{C}^{n\times n}$ *is an Hermitian matrix and that* $T \in \mathbb{C}^{n\times n}$ *is an Hermitian positive semidefinite matrix such that* $\mathrm{null}(T) \subseteq \mathrm{null}(S)$. *For* $1 \le k \le \mathrm{rank}(T)$ *we have*

$$\lambda_k(S,T) = \min_{\substack{\dim(U)=k \\ U \perp \mathrm{null}(T)}} \max_{x\in S} \frac{x^*Sx}{x^*Tx}$$

and

$$\lambda_k(S,T) = \max_{\substack{\dim(V)=rank(T)-k+1 \\ V \perp \mathrm{null}(T)}} \min_{x\in S} \frac{x^*Sx}{x^*Tx}\,.$$