CHAPTER 6

Embeddings and Combinatorial Support Bounds

To bound $\sigma(A, B)$ using the Symmetric-Product-Support Lemma, we need to factor A and B into $A = UU^T$ and $B = VV^T$, and we need to find a W such that $U = VW$. We have seen that if A and B are diagonally dominant, then there is an almost trivial way to factor A and B such that U and V are about as sparse as A and B. But how do we find a W such that $U = VW$? In this chapter, we show that when A and B are weighted (but not signed) Laplacians, we can construct such W using an embedding of the edges of G_A into paths in G_B. Furthermore, when W is constructed from an embedding, the bounds on $\|W\|_2$ can be interpreted as combinatorial bounds on the quality of the embedding.

1. Defining W using Path Embeddings

We start with the construction of a matrix W such that $U = VW$ The following lemma is the key to the construction of W.

LEMMA 1.1. *Let $(i_1, i_2, \ldots, i_\ell)$ be a sequence of integers between* 1 *and n, such that $i_j \neq i_{j+1}$ for $j = 1, \ldots \ell - 1$. Then*

$$\langle i_1, -j_\ell \rangle = \sum_{j=1}^{\ell-1} \langle i_j, -j_{j+1} \rangle \ ,$$

where all the edge vectors are length n.

PROOF. By induction. The base of the induction clearly holds: For $\ell = 2$, $\langle i_1, -j_\ell \rangle = \sum_{j=1}^{1} \langle i_j, -j_{j+1} \rangle = \langle i_1, -j_2 \rangle$. Suppose that the claim is true for $\ell - 1 \geq 2$.

$$\begin{aligned}
\sum_{j=1}^{\ell-1} \langle i_j, -j_{j+1} \rangle &= \sum_{j=1}^{\ell-2} \langle i_j, -j_{j+1} \rangle + \langle i_{\ell-1}, -j_\ell \rangle \\
&= \langle i_1, -j_{\ell-1} \rangle + \langle i_{\ell-1}, -j_\ell \rangle \\
&= \begin{bmatrix} \vdots \\ +1 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ +1 \\ \vdots \\ -1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ +1 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ \vdots \end{bmatrix} \\
&= \langle i_1, -j_\ell \rangle \ .
\end{aligned}$$

□

To see why this lemma is important, consider the role of a column of W. Suppose that the columns of U and V are all positive edge vectors. Denote column c of U by

$$U_{:\,,c} = \langle \min(i_1, i_\ell), -\max(i_1, i_\ell) \rangle = (-1)^{i_1 > i_\ell} \langle i_1, -i_\ell \rangle \ ,$$

where the $(-1)^{i_1 > i_\ell}$ evaluates to -1 if $i_1 > i_\ell$ and to 1 otherwise. This column corresponds to the edge (i_1, i_ℓ) in G_{UU^T}. Now let $(i_1, i_2, \ldots, i_\ell)$ be a simple path in G_{VV^T} (a simple path is a sequence of vertices $(i_1, i_2, \ldots, i_\ell)$ such that (i_j, i_{j+1}) is an edge in the graph for $1 \leq j < \ell$ and such that any vertex appears at most once on the path). If $U = VW$, then

$$U_{:\,,c} = VW_{:\,,c} = \sum_{r=1}^{k} V_{:\,,r} W_{r,c} \ .$$

Let $r_1, r_2, \ldots, r_{\ell-1}$ be the columns of V that corresponds to the edges of the path $(i_1, i_2, \ldots, i_\ell)$, in order. That is, $V_{:\,,r_1} = \langle \min(i_1, i_2), -\max(i_1, i_2) \rangle$, $V_{:\,,r_2} = \langle \min(i_2, i_3), -\max(i_2, i_3) \rangle$, and so on. By the lemma,

$$\begin{aligned}
U_{:\,,c} &= (-1)^{i_1 > i_\ell} \langle i_1, -i_\ell \rangle \\
&= (-1)^{i_1 > i_\ell} \sum_{j=1}^{\ell-1} \langle i_j, -i_{j+1} \rangle \\
&= (-1)^{i_1 > i_\ell} \sum_{j=1}^{\ell-1} (-1)^{i_j > i_{j+1}} V_{:\,,r_j} \ .
\end{aligned}$$

It follows that if we define $W_{:\,,c}$ to be

$$W_{r,c} = \begin{cases} (-1)^{i_1 > i_\ell}(-1)^{i_j > i_{j+1}} & r = r_j \text{ for some } 1 \le j < \ell \\ 0 & \text{otherwise,} \end{cases}$$

then we have $U_{:\,,c} = VW_{:\,,c}$. We can constuct all the columns of W in this way, so that W satisfies $U = VW$.

A path of edge vectors that ends in a vertex vector supports the vertex vector associated with the first vertex of the path.

LEMMA 1.2. *Let $(i_1, i_2, \ldots, i_\ell)$ be a sequence of integers between* 1 *and n, such that $i_j \neq i_{j+1}$ for $j = 1, \ldots \ell - 1$. Then*

$$\langle i_1 \rangle = \langle i_\ell \rangle + \sum_{j=1}^{\ell-1} \langle i_j, -i_{j+1} \rangle \ ,$$

where all the edge and vertex vectors are length n.

PROOF.

$$-\langle i_\ell \rangle + \sum_{j=1}^{\ell-1} \langle i_j, -i_{j+1} \rangle = \langle i_\ell \rangle + \langle i_1, -i_\ell \rangle = \langle i_1 \rangle \ .$$

□

Let's try to generalize these ideas. There are several issues that we must address if we want to construct W for arbitrary pairs of diagonally-dominant matrices: vertex vectors, negative edge vectors, and scaled edge and vertex vectors. It turns out that handling matrices with positive offdiagonals, which give rise to negative edge vectors, is considerably more complex than handling only positive edge vectors and vertex vectors. Therefore, we leave this issue for a later chapter. Handling vertex vectors and scaled vertex and positive edge vectors, however, is easy. The next theorem shows how path embeddings can be used to construct W for any pair of weighted (unsigned) Laplacians.

THEOREM 1.3. *Let A and B be weighted (unsigned) Laplacians and let U and V be their canonical incidence factors. Let π be a path embedding of the edges and strictly-dominant vertices of G_A into G_B, such that for an edge (i_1, i_ℓ) in G_A, $i_1 < i_\ell$, we have*

$$\pi(i_1, i_\ell) = (i_1, i_2, \ldots, i_\ell)$$

for some simple path $(i_1, i_2, \ldots, i_\ell)$ in G_B, and such that for a strictly-dominant i_1 in G_A,

$$\pi(i_1) = (i_1, i_2, \ldots, i_\ell)$$

for some simple path $(i_1, i_2, \ldots, i_\ell)$ in G_B that ends in a strictly-dominant vertex i_ℓ in G_B. Denote by $c_V(i_j, i_{j+1})$ the index of the column of V that is a scaling of $\langle i_j, -i_{j+1}\rangle$. That is,

$$V_{:\,,c_V(i_j,i_{j+1})} = \sqrt{-B_{i_j,i_{j+1}}}\,\langle \min(i_j, i_{j+1}), -\max(i_j, i_{j+1})\rangle\ .$$

Similarly, denote by $c_V(i_j)$ the index of the column of V that is a scaling of $\langle i_j\rangle$,

$$V_{:\,,c_V(i_j)} = \sqrt{B_{i_j,i_j} - \sum_{\substack{i_k=1\\ i_k\neq i_j}}^{n} \left|B_{i_k,i_j}\right|}\,\langle i_j\rangle\ ,$$

and similarly for U.

We define a matrix W as follows. For a column index $c_U(i_1, i_\ell)$ with $i_1 < i_\ell$ we define

$$W_{r,c_U(i_1,i_\ell)} = \begin{cases} (-1)^{i_j > i_{j+1}}\sqrt{A_{i_1,i_\ell}/B_{i_j,i_{j+1}}} & \text{if } r = c_V(i_j, i_{j+1}) \text{ for} \\ & \text{some edge } (i_j, i_{j+1}) \text{ in } \pi(i_1, i_\ell) \\ 0 & \text{otherwise.} \end{cases}$$

For a column index $c_U(i_1)$, we define

$$W_{r,c_U(i_1)} = \begin{cases} \sqrt{\frac{A_{i_1,i_1} - \sum_{j\neq i_1}\left|A_{i_1,j}\right|}{B_{i_\ell,i_\ell} - \sum_{j\neq i_\ell}\left|B_{i_\ell,j}\right|}} & \text{if } r = c_V(i_\ell) \\ (-1)^{i_j > i_{j+1}}\sqrt{\frac{A_{i_1,i_1} - \sum_{k\neq i_1}\left|A_{i_1,k}\right|}{\left|B_{i_j,i_{j+1}}\right|}} & \text{if } r = c_V(i_j, i_{j+1}) \text{ for} \\ & \text{some edge } (i_j, i_{j+1}) \text{ in } \pi(i_1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $U = VW$.

PROOF. For scaled edge-vector columns in U we have

$$
\begin{aligned}
VW_{:\,,c_U(i_1,i_\ell)} &= \sum_r V_{:\,,r} W_{r,c_U(i_1,i_\ell)} \\
&= \sum_{\substack{r=c_V(i_j,i_{j+1}) \\ \text{for some edge} \\ (i_j,i_{j+1}) \text{ in } \pi(i_1,i_\ell)}} V_{:\,,r} W_{r,c_U(i_1,i_\ell)} \\
&= \sum_{j=1}^{\ell-1} \sqrt{\left|B_{i_j,i_{j+1}}\right|} \left\langle \min(i_j,i_{j+1}), -\max(i_j,i_{j+1}) \right\rangle (-1)^{i_j > i_{j+1}} \sqrt{\frac{A_{i_1,i_\ell}}{B_{i_j,i_{j+1}}}} \\
&= \sqrt{|A_{i_1,i_\ell}|} \sum_{j=1}^{\ell-1} (-1)^{i_j > i_{j+1}} \left\langle \min(i_j,i_{j+1}), -\max(i_j,i_{j+1}) \right\rangle \\
&= \sqrt{|A_{i_1,i_\ell}|} \sum_{j=1}^{\ell-1} \langle i_j, -i_{j+1} \rangle \\
&= \sqrt{|A_{i_1,i_\ell}|} \langle i_1, -i_\ell \rangle \\
&= U_{:\,,c_U(i_1,i_\ell)} \,.
\end{aligned}
$$

For scaled vertex-vector columns in U we have

$$\begin{aligned}
VW_{:\,,c_U(i_1)} &= \sum_r V_{:\,,r} W_{r,c_U(i_1)} \\
&= V_{:\,,c_V(i_\ell)} W_{c_V(i_\ell),c_U(i_1,i_\ell)} + \sum_{\substack{r=c_V(i_j,i_{j+1}) \\ \text{for some edge} \\ (i_j,i_{j+1}) \text{ in } \pi(i_1)}} V_{:\,,r} W_{r,c_U(i_1)} \\
&= \sqrt{B_{i_\ell,i_\ell} - \sum_{j \neq i_\ell} |B_{i_k,i_\ell}|}\,\langle i_\ell \rangle \sqrt{\frac{A_{i_1,i_1} - \sum_{j\neq i_1} |A_{i_1,j}|}{B_{i_\ell,i_\ell} - \sum_{j \neq i_\ell} |B_{i_\ell,j}|}} \\
&\quad + \sum_{j=1}^{\ell-1} \sqrt{\left|B_{i_j,i_{j+1}}\right|}\,\langle \min(i_j,i_{j+1}), -\max(i_j,i_{j+1}) \rangle \\
&\qquad \cdot (-1)^{i_j > i_{j+1}} \sqrt{\frac{A_{i_1,i_1} - \sum_{j\neq i_1} |A_{i_1,j}|}{\left|B_{i_j,i_{j+1}}\right|}} \\
&= \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|}\,\langle i_\ell \rangle \\
&\quad + \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|} \\
&\qquad \cdot \sum_{j=1}^{\ell-1} (-1)^{i_j > i_{j+1}} \langle \min(i_j,i_{j+1}), -\max(i_j,i_{j+1}) \rangle \\
&= \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|}\,\langle i_\ell \rangle + \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|} \sum_{j=1}^{\ell-1} \langle i_j, -i_{j+1} \rangle \\
&= \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|}\,(\langle i_\ell \rangle + \langle i_1, -i_\ell \rangle) \\
&= \sqrt{A_{i_1,i_1} - \sum_{j \neq i_1} |A_{i_1,j}|}\,\langle i_1 \rangle \\
&= U_{:\,,c_U(i_1)}\,.
\end{aligned}$$

□

This theorem plays a fundamental role in many applications of support theory, and it merits additional discussion. For a path embedding that satisfies the hypothesis to exist, G_B must satisfy certain connectivity constraints. Are they essential for the existence of a matrix W

such that $U = VW$? The next lemma shows that these constraints are essential.

LEMMA 1.4. *Let $A = UU^T$ and $B = VV^T$ be weighted (but not signed) Laplacians with arbitrary symmetric-product factorizations. The following conditions are necessary for the equation $U = VW$ to hold for some matrix W (by Theorem 1.3, these conditions are also sufficient).*

(1) *For each edge (i, j) in G_A, either i and j are in the same connected component in G_B, or the two components of G_B that contain i and j both include a strictly-dominant vertex.*
(2) *For each strictly-dominant vertex i in G_A, the component of G_B that contains i includes a strictly-dominant vertex.*

PROOF. Suppose for contradiction that one of the conditions is not satisfied, but that there is a W that satisfies $U = VW$. Without loss of generality, we assume that the vertices are ordered such that vertices that belong to a connected component in G_B are consecutive. Under that assumption,

$$V = \begin{bmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V_k \end{bmatrix} ,$$

and

$$B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{bmatrix} = \begin{bmatrix} V_1V_1^T & & & \\ & V_2V_2^T & & \\ & & \ddots & \\ & & & V_kV_k^T \end{bmatrix} .$$

The blocks of V are possibly rectangular, whereas the nonzero blocks of B are all diagonal and square.

We now prove the necessity of the first condition. Suppose for some edge (i, j) in G_A, i and j belong to different connected components of G_B (without loss of generality, to the first two components), and that one of the components (w.l.o.g. the first) does not have a strictly-dominant vertex. Because this component does not have a strictly-dominant vertex, the row sums in $V_1V_1^T$ are exactly zero. Therefore, $V_1V_1^T\vec{1} = \vec{0}$, so V_1 must be rank deficient.

Since (i, j) is in G_A, the vector $\langle i, -j\rangle$ is in the column space of the canonical incidence factor of A, and therefore in the column space of any U such that $A = UU^T$. If $U = VW$, then the vector $\langle i, -j\rangle$ must also be in the column space of V, so for some x

$$\langle i, -j \rangle = Vx = \begin{bmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} V_1 x_1 \\ V_2 x_2 \\ \vdots \\ V_k x_k \end{bmatrix} .$$

Therefore, $V_1 x_1$ is a vertex vector. By Theorem 1.3, if V_1 spans a vertex vector, it spans all the vertex vectors associated with the vertices of the connected component. This implies that V_1 is full rank, a contradiction.

The necessity of the second condition follows from a similar argument. Suppose that vertex i is strictly dominant in G_A and that it belongs to a connected component in G_B (w.l.o.g. the first) that does not have a vertex that is strictly dominant in G_B. This implies that for some y

$$\langle i \rangle = Vy = \begin{bmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V_k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} V_1 y_1 \\ V_2 y_2 \\ \vdots \\ V_k y_k \end{bmatrix} .$$

Again $V_1 y_1$ is a vertex vector, so V_1 must be full rank, but it cannot be full rank because $V_1 V_1^T$ has zero row sums. □

On the other hand, even if U and V are the canonical incidence factors of A and B, not every W such that $U = VW$ corresponds to a path embedding. In particular, a column of W can correspond to a linear combination of multiple paths. Also, even if W does correspond to a path embedding, the paths are not necessarily simple. A linear combination of scaled positive edge vectors that correspond to a simple cycle can be identically zero, so the coefficients of such linear combinations can be added to W without affecting the product VW. However, it seems that adding cycles to a path embedding cannot reduce the 2-norm of W, so cycles are unlikely to improve support bounds.

2. Combinatorial Support Bounds

To bound $\sigma(A, B)$ using the Symmetric-Produce-Support Lemma, we need to factor A into $A = UU^T$, B into $B = VV^T$, find a matrix W such that $U = VW$, and to bound the 2-norm of W from above. We have seen how to factor A and B (if they are weigthed Laplacians) and how to construct an appropriate W combinatorially, using graph embeddings. We now show how to use combinatorial metrics of the path embeddings to bound $\|W\|_2$.

Bounding the 2-norm directly is hard, because the 2-norm is not related in a simple way to the entries of W. But the 2-norm can be bounded using simpler norms, such as the Frobenius norm, the infinity norm, and the 1-norm. We will show that these simpler norms have natural and useful combinatorial interpretations when W represents a path embedding. The bounds on the 2-norm that we use are

$$\begin{aligned} \|W\|_2^2 &\leq \|W\|_F^2 = \sum_{r=1}^{k}\sum_{c=1}^{m} W_{r,c}^2 \\ \|W\|_2^2 &\leq \|W\|_1\|W\|_\infty = \left(\max_{c=1}^{m}\sum_{r=1}^{k}|W_{r,c}|\right)\left(\max_{r=1}^{k}\sum_{c=1}^{m}|W_{r,c}|\right) \end{aligned}$$

To keep the notation and the definition simple, we now assume that A and B are weigted Laplacians with zero row sums. We will show later how to deal with positive row sums. We also assume that W corresponds to a path embedding π. The following definitions provide a combinatorial interpretation of these bounds.

Definition 2.1. The *weighted dilation* of an edge of G_A in an path embedding π of G_A into G_B is

$$\text{dilation}_\pi(i_1, i_2) = \sum_{\substack{(j_1,,j_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \sqrt{\frac{A_{i_1,i_2}}{B_{j_1,j_2}}}\ .$$

The *weigted congestion* of an edge of G_B is

$$\text{congestion}_\pi(j_1, j_2) = \sum_{\substack{(i_1,,i_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \sqrt{\frac{A_{i_1,i_2}}{B_{j_1,j_2}}}\ .$$

The weighted stretch of an edge of G_A is

$$\text{stretch}_\pi(i_1, i_2) = \sum_{\substack{(j_1,,j_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \frac{A_{i_1,i_2}}{B_{j_1,j_2}}\ .$$

The weighted crowding of an edge in G_B is

$$\text{crowding}_\pi(j_1, j_2) = \sum_{\substack{(i_1,,i_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \frac{A_{i_1,i_2}}{B_{j_1,j_2}}\ .$$

Note that stretch is a summation of the squares of the quantities that constitute dilation, and similarly for crowding and congestion. Papers in the support-preconditioning literature are not consistent about these terms, so check the definitions carefully when you consult a paper that deals with congestion, dilation, and so on.

LEMMA 2.2. *Let A and B be weighted Laplacians with zero row sums, and let π be a path embedding of G_A into G_B. Then*

$$\begin{aligned}
\sigma(A,B) &\le \sum_{(i_1,i_2)\in G_A} \mathit{stretch}_\pi(i_1,i_2) \\
\sigma(A,B) &\le \sum_{(j_1,j_2)\in G_B} \mathit{crowding}_\pi(j_1,j_2) \\
\sigma(A,B) &\le \left(\max_{(i_1,i_2)\in G_A} \mathit{dilation}_\pi(i_1,i_2)\right) \\
&\quad \cdot \left(\max_{(j_1,j_2)\in G_B} \mathit{congestion}_\pi(j_1,j_2)\right) .
\end{aligned}$$

PROOF. Let U and V be the canonical incidence factors of A and B, and let W be the matrix representation of the embedding π, so $U = VW$. We have

$$\begin{aligned}
\sigma(A,B) &\le \|W\|_2^2 \\
&\le \|W\|_F^2 \\
&= \sum_{c=1}^{m}\sum_{r=1}^{k} W_{r,c}^2 \\
&= \sum_{(i_1,i_2)\in G_A} \sum_{\substack{(j_1,,j_2)\\ (j_1,,j_2)\in\pi(i_1,i_2)}} \frac{A_{i_1,i_2}}{B_{j_1,j_2}} \\
&= \sum_{(i_1,i_2)\in G_A} \mathrm{stretch}_\pi(i_1,i_2) .
\end{aligned}$$

The proof for crowding is identical. We also have

$$\begin{aligned}
\sigma(A,B) &\leq \|W\|_1 \|W\|_\infty \\
&= \left(\max_{c=1}^{m} \sum_{r=1}^{k} |W_{r,c}|\right)\left(\max_{r=1}^{k} \sum_{c=1}^{m} |W_{r,c}|\right) \\
&= \left(\max_{(i_1,i_2)\in G_A} \sum_{\substack{(j_1,,j_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \sqrt{\frac{A_{i_1,i_2}}{B_{j_1,j_2}}}\right) \\
&\quad \cdot \left(\max_{(j_1,j_2)\in G_A} \sum_{\substack{(i_1,,i_2) \\ (j_1,,j_2)\in\pi(i_1,i_2)}} \sqrt{\frac{A_{i_1,i_2}}{B_{j_1,j_2}}}\right) \\
&= \left(\max_{(i_1,i_2)\in G_A} \text{dilation}_\pi(i_1,i_2)\right) \\
&\quad \cdot \left(\max_{(j_1,j_2)\in G_A} \text{congestion}_\pi(j_1,j_2)\right) .
\end{aligned}$$

□

We now describe one way to deal with matrices with some positive row sums. The matrix B is a preconditioner. In many applications, the matrix B is not given, but rather constructed. One simple way to deal with positive row sums in A is to define$\pi(i_1) = (i_1)$. That is, vertex vectors in the canonical incidence factor of A are mapped into the same vertex vectors in the incidence factor of B. In other words, we construct B to have exactly the same row sums as A. With such a construction, the rows of W that correspond to vertex vectors in U are columns of the identity.

Lemma 2.3. *Let A and B be weighted Laplacians with the same row sums, let π be a path embedding of G_A into G_B, and let ℓ be the number of rows with positive row sums in A and B. Then*

$$\begin{aligned}
\sigma(A,B) &\leq \ell + \sum_{(i_1,i_2)\in G_A} \textit{stretch}_\pi(i_1,i_2) \\
\sigma(A,B) &\leq \left(\max\left\{1, \max_{(i_1,i_2)\in G_A} \textit{dilation}_\pi(i_1,i_2)\right\}\right) \\
&\quad \cdot \left(\max\left\{1, \max_{(j_1,j_2)\in G_B} \textit{congestion}_\pi(j_1,j_2)\right\}\right) .
\end{aligned}$$

Proof. Under the hypothesis of the lemma, the rows and columns of W can be permuted into a block matrix

$$W = \begin{pmatrix} W_Z & 0 \\ 0 & I_{\ell\times\ell} \end{pmatrix} ,$$

where W_Z represents the path embedding of the edges of G_A into paths in G_B. The bounds follow from the structure of W and from the proof of the previous lemma. □

The sparse bounds on the 2-norm of a matrix lead to tighter combinatrial bounds.

Lemma 2.4. *Let A and B be weighted Laplacians with zero row sums, and let π be a path embedding of G_A into G_B. Then*

$$\begin{aligned}
\sigma(A,B) &\leq \max_{(j_1,j_2)\in G_B} \sum_{\substack{(i_1,i_2)\in G_A \\ (j_1,j_2)\in\pi(i_1,i_2)}} \mathit{stretch}_\pi(i_1,i_2) , \\
\sigma(A,B) &\leq \max_{(i_1,i_2)\in G_A} \sum_{\substack{(j_1,j_2)\in G_A \\ (j_1,j_2)\in\pi(i_1,i_2)}} \mathit{crowding}_\pi(j_1,j_2) .
\end{aligned}$$

We can derive similar bounds for the other sparse 2-norm bounds.

3. Subset Preconditioners

Normally, to obtain a bound on $\kappa(A,B)$ we need a bound on both $\sigma(A,B)$ and $\sigma(B,A)$. But in one common case, bounding $\sigma(B,A)$ is trivial. Many support preconditioners construct G_B to be a subgraph of G_A, with the same weights. That is, V is constructed to have a subset of the columns in U. If we denote by $\bar{V}$ the set of columns of U that are *not* in V, we have

$$\begin{aligned}
B &= VV^T \\
A &= UU^T \\
&= VV^T + \bar{V}\bar{V}^T \\
&= B + \bar{V}\bar{V}^T .
\end{aligned}$$

This immediately implies $x^TAx \geq x^TBx$ for any x, so $\lambda_{\min}(A,B) \geq 1$.

4. Notes and References

The low-stretch forests are from Elkin-Emek-Spielman-Teng, STOC 2005. This is an improvement over Alon-Karp-Peleg-West.