Parallel Iterative Solvers for Ill-Conditioned Problems with Reordering

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1. Preconditioned Iterative Solvers with Multicoloring

In the previous work [1], author developed an efficient parallel iterative solver for finite-element applications on the Earth Simulator (ES) [2] using multi-level *hybrid* parallel programming model with MPI and OpenMP. The method employs three-level hybrid parallel programming model for SMP cluster architectures, consisting of MPI, OpenMP and vectorization. Multicolor-based reordering methods have been applied to distributed data sets on each SMP node in order to achieve optimum performance of Krylov iterative solvers with ILU/IC type preconditioning (Fig.1). Developed method attained 3.8 TFLOPS with 176 SMP nodes of ES, corresponding to more than 33% of the peak performance (11.3 TFLOPS) [1].

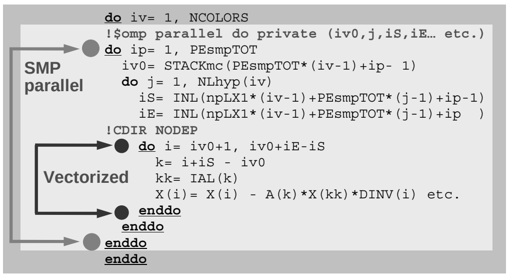


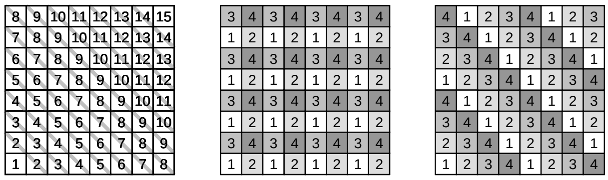
Fig.1 Forward/backward substitution procedure using OpenMP and vectorization directives during ILU(0)/IC(0) preconditioning.

In the cases with many colors, fewer numbers of iterations are required for convergence, because of fewer incompatible nodes [3]. This effect is more significant for ill-conditioned problems. But the performance is worse due to the smaller loop length and greater overhead on vector processors. On ES, hybrid parallel programming model is very sensitive to number of colors. On the contrast, performance of scalar processors is improved according to increase of the number of colors due to data locality. Effect of overhead with more than 500 colors is not negligible, but the drop of performance is not so significant as vector processors [4]. Therefore, different strategy for reordering method should be adopted for vector and scalar processors.

In the previous works [1,4], author adopted multicoloring, mainly because highly parallel performance and load balancing can be obtained easily. In the present work, Reverse Cuthill-McKee (RCM) type approach is also tested and compared with multicoloring on a single SMP node of the Earth Simulator and Hitachi SR11000/J1 [5] using OpenMP.

2. Reordering Methods

Reverse Cuthill-McKee (RCM) method is a typical level set reordering method [1,3,4]. In Cuthill-McKee reordering, the elements of a level set are traversed from the nodes of lowest degree to those of highest degree according to dependency relationships, where the degree refers to the number of nodes connected to each node. Multicoloring (MC) is much simpler than RCM. MC is based on an idea where no two adjacent nodes have the same color. In both methods, elements located on the same color (or level set) are independent. Therefore, parallel operation is possible for the elements in the same color (or level set) and the number of elements in the same color (or level set) should be as large as possible in order to obtain high granularity for parallel computation or sufficiently large length of innermost loops for vectorization.



(a) RCM (b) MC: 4 colors (c) CM-RCM: 4 colors

Fig. 2 Example of RCM, multicoloring and CM-RCM reordering for 2D geometry [1,4]

RCM (Fig. 2(a)) reordering provides fast convergence of IC/ILU-preconditioned Krylov iterative solvers, yet with irregular numbers of the elements in each level set. For example in Fig. 2(a), the 1st level set is of size 1, while the 8th level set is of size 8. Multicoloring provides a uniform element number in each color (Fig. 2(b)). However, it is widely known that the convergence of IC/ILU-preconditioned Krylov iterative solvers with multicoloring is rather slow [3]. Convergence can be improved by increasing the number of colors because of fewer incompatible local graphs [3], but this reduces the number of elements in each color. The solution for this trade-off is cyclic-multicoloring (CM) on RCM [6]. In this method, the elements are renumbered in a cyclic manner. Figure 2(c) shows an example of CM-RCM reordering. In this case, there are 4 colors; the 1st, 5th, 9th and 13th colors in Fig. 2(a) are classified into the 1st color. There are 16 elements in each color. In CM-RCM, the number of colors should be large enough to ensure that elements in the same color are independent.

3. Examples

In this work, IC(0)-CG iterative solvers with reordering have been applied to ill-conditioned matrices derived from finite-element applications for linear-elastic problems with heterogeneous mesh size. Adaptive mesh refinement (AMR) has been applied, and maximum ratio of size for adjacent elements is more than 30 (Fig.3). Table 1 shows features a single SMP node of ES and Hitachi SR11000.

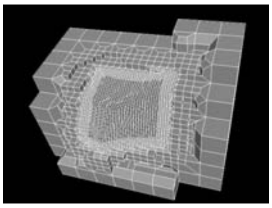


Fig.3 Example of mesh with AMR

Table 1 Node comparisons of the Earth Simulator and Hitachi SR11000/J1 [2,5]

	Earth Simulator	Hitachi SR11000/J1
PE#/node	8	8
Clock rate	500 MHz	1900 MHz
Peak performance/PE	8.00 GFLOPS	7.60 GFLOPS
Memory-PE Bandwidth	32 GB/sec/PE	12.5 GB/sec/PE

Figure 4 shows number of incompatible nodes and iterations for convergence with heterogeneous linear-elastic problem with 10^6 FEM nodes (i.e. 3×10^6 DOF). Convergence is improved according to increase of number of colors, and number of iterations for convergence of RCM and CM-RCM is much fewer than that of MC.

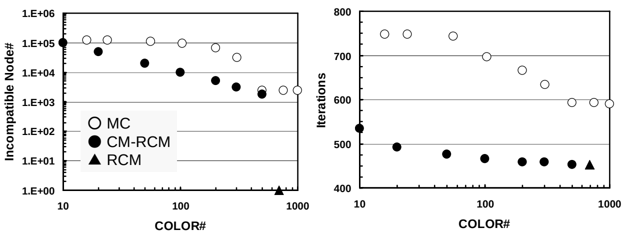


Fig. 4 Number of incompatible nodes and iterations for convergence of heterogeneous linear-elastic problem with 10^6 FEM nodes (i.e. 3×10^6 DOF).

Figures 5 and 6 show elapsed time for linear solvers and GFLOPS rate of a single SMP node of ES and Hitachi SR11000/J1 using OpenMP. Effect of number of colors for each architecture is very clear. Performance on ES is decreasing as number of colors increases. Therefore, computation time increases, although iteration number for convergence decreases, as shown in Fig.4. Fewer numbers of colors are preferable for ES. Performance of CM-RCM for elapsed computation time is 50% better than that of MC for best cases. On the contrast, more colors provide better performance on Hitachi SR1100/J1. Performance of CM-RCM is 33% better than that of MC for best cases.

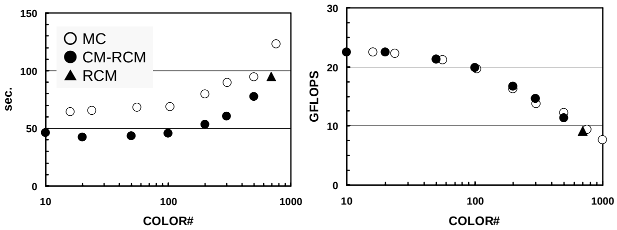


Fig. 5 Elapsed time for linear solvers and GFLOPS rate of a single SMP node of ES for heterogeneous linear-elastic problem with 3×10^6 DOF using OpenMP

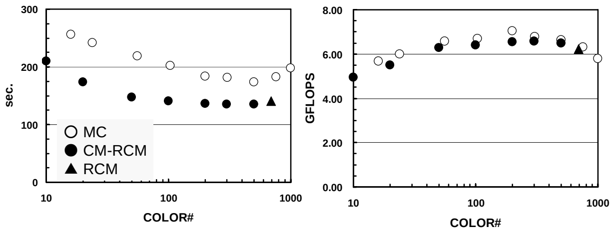


Fig. 6 Elapsed time for linear solvers and GFLOPS rate of a single SMP node of Hitachi SR11000/J1 for heterogeneous linear-elastic problem with 3×10^6 DOF using OpenMP

4. Summary

Parallel IC(0)-CG methods with various types of reordering methods have been applied to ill-conditioned problems with AMR on a single SMP node of ES and Hitachi SR11000/J1 using OpenMP. CM-RCM provides more robust convergence and better performance than MC. More results of applications for complicated geometries will be demonstrated in the presentation.

References

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