Using Multiple Generalized Cross-Validation as a Method for Varying Smoothing Effects

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Classical Tikhonov Regularization Method

The most commonly used method for the solution of ill-posed problems is Tikhonov regularization method. The major concept of the Tikhonov regularization scheme is replacement of the original ill-posed system of,

$$\min_x \|Kx - d\|_2^2 \tag{1}$$

with a well-posed problem of;

$$\min_x(\|Kx - d\|_2^2 + \lambda^2\|Lx\|_2^2) \tag{2}$$

The solution of this regularization method depends on the choice of the priori, L, and the regularization parameter, λ.

we show that rewriting the Tikhonov eq of (2) in a mutilevel-regularization approach would result in:

$$\min_x(||Kx - d||_2^2 + \sum_{i=1}^{q}\lambda_i^2||L_i x||_2^2) \tag{3}$$

Where q is the number of subdomains of the solution and L_i is the local regularization matrix and the regularization vector Λ is a diagonal matrix with q diagonal elements as;

$$\Lambda = \begin{pmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_q \end{pmatrix} \tag{4}$$

The major difficulty in the solution of(3) is the determination of the regularization parameter, λ. For the case of 1-D regularization parameter[1], there are two popular methods of L-curve[3] and Generalized Cross Validation(GCV)[1,2,4]. In this work, we use multiple GCV algorithm, as the method of determination of the regularization parameters.

Indeed, the evaluation of the GCV function(in order to determine the regularization parameters) of,

$$GCV(\Lambda) = \frac{\| \left(I - K(K^T K + \Lambda^2 I)^{-1} K^T\right) d\|_2^2}{\frac{1}{m}(trace((I - K(K^T K + \Lambda^2 I)^{-1} K^T)))^2} \tag{5}$$

, where

$$x_\Lambda = K^{\neq} d = (K^T K + \Lambda^2 I)^{-1} K^T d \tag{6}$$

, is numerically an arduous work. Here we explore the idea of decoupling of GCV function(5), so (5) would be dependent on only one regularization vector components λ_i at a time. Thus we could apply the standard regularization algorithm(for the case that the regularization parameter is a scalar (1-D) case).

The following table is the results of the application of multiple GCV method for an image reconstruction example. The results shows improvement over the case of simple GCV (1-D GCV) method.

Table 1: solution errors for both cases of regularization methods

std of the added noise	1-D GCV :($\times 10^{-2}$)	Multiple GCV:($\times 10^{-2}$)
.01	1.91062	1.38063
.05	1.62866	2.27616
.1	2.47110	1.93282
1	5.81025	4.35003
3	7.67133	5.28412

References

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