# Multi-Level Approach to Numerical Solution of Inverse Problems

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Mathematical modeling of an engineering system often leads to such formulations for which one can not obtain a closed form solution/analysis, and thus numerical methods are to be used. In the process, we need to transform the system from an infinite dimensional space to a finite dimensional one(discretization). The result is usually a system of linear equations[5] for which the linear least squares method[1,6] is used to compute the solution. The difficulty is that the real engineering problems are ill-posed and consequently their discretization lead to an ill-conditioned system of equations, and thus the method of least squares produce irrelevant solutions. As a result, application of some types of regularization techniques would be necessary for the solution of ill-conditioned systems. Tikhonov regularization method[3,8] is one of the most popular regularization techniques. In this scheme the original ill-conditioned system of,

$$\min_x \|Kx - d\|_2^2 \tag{1}$$

is transformed to a well-conditioned system of;

$$\min_x(\|Kx - d\|_2^2 + \lambda^2\|Lx\|_2^2) \tag{2}$$

The solution of this regularization method depends on the choice of the priori, L, and the regularization parameter, $\lambda$[2,4,7].

In this work we like to explore the idea of multilevel regularization. This approach enables us to apply different regularization levels for different subdomains of the problem. Since $K$ and $d$ are considered to be given values, so the way to make any adjustment in(2) would be through making changes in the regularization term of $\|LX\|_2$ , i.e,

$$x_\lambda = f(\lambda, L) \tag{3}$$

Using the normal equation formulation of (2),

$$K^T K x + \lambda S^{'}(x) = K^T d \tag{4}$$

Where $S(x)$ is the smoothing operator(in continous form) of,

$$S(x) = \int_{t_o}^{t_f} (\frac{d^k x}{dt^k})^2 dt \tag{5}$$

We show that we can rewrite the Tikhonov eq of (2) in a multilevel formulation of:

$$\min_x(||Kx - d||_2^2 + \sum_{i=1}^{q} \lambda_i^2 ||L_i x||_2^2) \tag{6}$$

Where q is the number of subspace of the solution and $1 \leq q \leq n$ . $L_i$ is the localized regularization matrix. We use different algorithms for the solutions of the multi-level approach(eq. 6)and we compare the advantages and disadvantages of these algorithms. Then we show the application of the algorithms for the solution of a two different applications, image reconstruction and helioseismology, resulting in improvement in the accuracy of the solutions.

# References


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