Support-Graph Preconditioners for 2-Dimensional Trusses

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Abstract

We use support theory, in particular the fretsaw extensions of Shklarski and Toledo [ST06a], to design preconditioners for the stiffness matrices of 2-dimensional truss structures that are stiffly connected. Provided that all the lengths of the trusses are within constant factors of each other, that the angles at the corners of the triangles are bounded away from 0 and π, and that the elastic moduli and cross-sectional areas of all the truss elements are within constant factors of each other, our preconditioners allow us to solve linear equations in the stiffness matrices to accuracy ϵ in time $O(n^{5/4}(\log^2 n \log\log n)^{3/4} \log(1/\epsilon))$.

1 Truss Structures

Definition 1.1. *A* **2-dimensional truss** *is an undirected weighted planar graph $\mathcal{T} = \langle V, E, \gamma \rangle$ with vertices $V = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\} \subset \mathbb{R}^2$, such that every edge belongs to a triangular face. An edge $e = (\boldsymbol{v}_i, \boldsymbol{v}_j)$, called a* **truss element**, *represents an idealized bar, with weight $\gamma(e)$ denoting the product of the bar's cross-sectional area and the elastic modulus of its material.*

For each truss element $e = (\boldsymbol{v}_i, \boldsymbol{v}_j)$, we define a length $2n$ column vector $\boldsymbol{u}_e = [u_e^1 \; ... \; u_e^{2n}]^T$ with 4 nonzero entries satisfying $[u_e^{2i-1} \; u_e^{2i}]^T = -[u_e^{2j-1} \; u_e^{2j}]^T = \frac{\boldsymbol{v}_i - \boldsymbol{v}_j}{|\boldsymbol{v}_i - \boldsymbol{v}_j|}$

We then define the $2n \times 2n$ **stiffness matrix**

$$A_{\mathcal{T}} = \sum_{e=(\boldsymbol{v}_i,\boldsymbol{v}_j)\in E} \frac{\gamma(e)}{|\boldsymbol{v}_i - \boldsymbol{v}_j|} \boldsymbol{u}_e \boldsymbol{u}_e^T$$

The **rigidity graph** *$Q_{\mathcal{T}}$ of a truss $\mathcal{T}$ is the graph with vertex set given by the set of triangular faces of the truss, and with edges connecting triangles that share an edge.*

We say that a truss $\mathcal{T}$ is **stiffly connected** *if (1) $Q_{\mathcal{T}}$ is connected, and (2) for every $\boldsymbol{v} \in V$, $Q_{\mathcal{T}}^{\boldsymbol{v}}$ is connected, where $Q_{\mathcal{T}}^{\boldsymbol{v}}$ is the graph induced by $Q_{\mathcal{T}}$ on the set of triangles containing $\boldsymbol{v}$.*

Theorem 1.2. *Let $\mathcal{T} = \langle V, T, \gamma \rangle$ be an n-vertex stiffly connected truss such that*

- *All triangle edges have lengths in the range $[l_{min}, l_{max}]$*
- *All triangle angles are in the range $[\theta_{min}, \pi - \theta_{min}]$.*
- *All triangle weights are in the range $[\gamma_{min}, \gamma_{max}]$.*

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for positive constants $l_{min}, l_{max}, \theta_{min}, \gamma_{min}, \gamma_{max}$.

Then linear systems in matrix $A_{\mathcal{T}}$ can be solved in time $O\left(n^{5/4}(\log^2 n \log\log n)^{3/4} \log(1/\epsilon)\right)$ within relative error ϵ.

We prove Theorem 1.2 by constructing a preconditioner for the stiffness matrix, and then solving the linear system by applying the preconditioned conjugate gradient method. The preconditioner is in the family of *fretsaw preconditioners*, introduced by [ST06a]. A fretsaw preconditioner, a picture of which appears below, can be interpreted as arising by duplicating some of the vertices and truss elements of the truss structure, so that the rigidity graph of the fretsaw extension is a connected subgraph of the original. By extending techniques used in [ST06b] to add edges to the low-stretch spanning trees of [EEST06], we design a fretsaw extension whose rigidity graph looks like a spanning tree plus k edges, and such that the relative condition number between the stiffness matrix and its preconditioner is at most $(\frac{n}{k} \log^2 n \log\log n)^3$.

To bound the condition number, we extend combinatorial techniques used in [ST06b], and apply the *path lemma*, which provides a bound on how well a stiffly connected truss whose rigidity graph is a path supports another truss element between two of its vertices.

Lemma 1.3. *Let $\mathcal{T} = \langle V, T, \gamma \rangle$ be an t-triangle stiffly connected truss with constant-bounded edge, angle, and weight ranges, as in Theorem 1.2. For any $\boldsymbol{v}_i, \boldsymbol{v}_j \in V$, define a truss element $e = (\boldsymbol{v}_i, \boldsymbol{v}_j)$ and let $A_{i,j} = \boldsymbol{u}_e \boldsymbol{u}_e^T$. Then we have*

$$\lambda_{max}(A_{ij}, A_{\mathcal{T}}) = O(t^3)$$

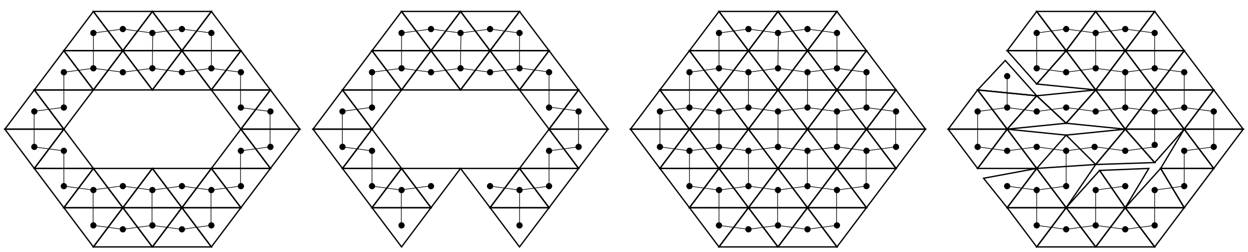

Figure 1: From left-to-right: a stiffly connected truss, a truss that is not stiffly connected, another truss, and a fretsaw extension of that truss. The rigidity graph is drawn inside each truss structure.

References

[EEST06] Michael Elkin, Yuval Emek, Daniel A. Spielman, and Shang-Hua Teng. Lower-stretch spanning trees. *SIAM Journal on Computing*, 2006. To appear.

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