Robinson Crusoe example

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Consider an island economy with one agent (Robinson Crusoe) who has an endowment of $\bar{L}$ units of an input that can be used to produce 2 final goods, x and y. Although this example is probably not very interesting (presumably Robinson Crusoe can take care of himself and does not need to rely on a market mechanism, or any mechanism for that matter, to determine how much x and y to consume), it is nonetheless the simplest example for a production economy we can imagine that is still non-trivial. For instance, if $\bar{L}$ could have been used to produce only one final good, say x, instead of both x and y, then Robinson would have simply converted all of $\bar{L}$ into x and there would be no problem to analyze. Yet, with two final goods that can be produced, we can ask how many units of x and how many units of y will be produced and consumed by Robinson.

To make the problem more concrete, suppose that the production functions for x and y are given by

$$x = \sqrt{L_x}, \qquad y = \frac{\sqrt{L_y}}{2}, \tag{1}$$

where L_x and L_y are the quantities of L used in the production of x and y, respectively. In addition suppose that Robinson's utility function is given by

$$U(x,y) = \sqrt{x}\,\sqrt{y}. \tag{2}$$

Given the description of the economy we will now characterize the set of Pareto efficient allocations and compute the Walrasian equilibrium and show that the two coincide.

Pareto efficiency

First we need to determine the Production Possibilities Frontier (PPF). In other words, find all combinations of x and y that can be produced efficiently. That is, all combinations of x and y such that there is no way to get more of both x and y by reallocating $\bar{L}$. To derive the PPF, note that since Robinson does not want to consume L, he will use all of $\bar{L}$ in the production of either x or y. Hence, it must be the case that

$$L_x + L_y = \bar{L}. \tag{3}$$

Using this expression, we can express the production functions for x and y as follows:

$$x = \sqrt{L_x}, \qquad y = \frac{\sqrt{\bar{L} - L_x}}{2}, \tag{4}$$

or equivalently,

$$L_x = x^2, \qquad L_x = \bar{L} - 4y^2. \tag{5}$$

Using the two expressions in equation (5) we get:

$$x^2 = \bar{L} - 4y^2, \ \Leftrightarrow \ T(x,y) \equiv x^2 + 4y^2 - \bar{L} = 0. \tag{6}$$

The equality T(x,y) = 0 characterizes then the efficient combinations of x and y and therefore defines the PPF.

Having derived the PPF we are now ready to solve for the Pareto efficient allocations. Noting that the economy here contains only one individual, the set of Pareto efficient allocations is determined by the following maximization problem:

$$\begin{aligned} &\underset{x,y}{Max} \quad \sqrt{x}\sqrt{y} \\ &s.t. \quad T(x,y) = 0. \end{aligned} \tag{7}$$

The Lagrangian that corresponds to this problem is given by

$$\underset{x,y,\lambda}{Max} \quad \mathfrak{L}(x,y,\lambda) \equiv \sqrt{x}\sqrt{y} + \lambda\left(T(x,y) - 0\right). \tag{8}$$

The first order conditions for the problem are:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} + 2\lambda x = 0, \qquad \textbf{(9)}$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} + 8\lambda y = 0, \qquad \textbf{(10)}$$

and

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = x^2 + 4y^2 - \bar{L} = 0. \qquad \textbf{(11)}$$

Solving equations (9)-(11), the Pareto efficient allocation is given by:

$$x^* = \frac{\sqrt{\bar{L}}}{\sqrt{2}}, \qquad y^* = \frac{\sqrt{\bar{L}}}{\sqrt{8}}. \qquad \textbf{(12)}$$

Walrasian equilibrium

To characterize the Walrasian equilibrium, suppose that Robinson establishes a firm, buys the input $\bar{L}$ from himself for a price of w per unit, then as the owner of the firm, decides how many units of x and how many units of y to produce to maximize profits given the prices of the two goods, p_x and p_y, and then sells the firm's output to himself. Moreover, the prices w, p_x, p_y, are called by an auctioneer (question: if Robinson lives on the island all by himself, who is the auctioneer?) with Robinson responding by submitting his demands and supplies until all three markets, the market for x, the market for y, and the market for L, are cleared. This obviously sounds not only schizophrenic (Robinson deals with himself twice at arm-length: as a provider of the inputs and as a consumer of the final goods), but rather silly: why should Robinson rely on an auctioneer to determine the market clearing prices? Yet again, this is an example of how the market mechanism works and how it leads to Pareto efficient allocations.

To characterize the Walrasian equilibrium, note that what we are looking for is a list, (p_x*, p_y*, w) such that the following conditions are met:

$$x^D(p_x^*, p_y^*, w^*) = x^S(p_x^*, p_y^*, w^*), \tag{13}$$

$$y^D(p_x^*, p_y^*, w^*) = y^S(p_x^*, p_y^*, w^*), \tag{14}$$

and

$$L_x(p_x^*, p_y^*, w^*) + L_y(p_x^*, p_y^*, w^*) = \bar{L}, \tag{15}$$

where x^D and y^D are Robinson's demands for goods x and y given the prices, p_x*, p_y*, and w*; x^S and y^S are the firm's supplies of goods x and y given the prices, p_x*, p_y*, and w*; and L_x and L_y are the quantities of input used in the production of x and y. Therefore, equation (13) is the market clearing condition for good x, equation (14) is the market clearing condition for good y, and equation (15) is the market clearing condition for the input.

To derive the L_x and L_y, note that the firm's profit, given the production functions of x and y is given by

$$\pi = p_x \sqrt{L_x} + p_y \frac{\sqrt{L_y}}{2} - w L_x - w L_y. \tag{16}$$

Hence, the firm's demands for input are defined by the following first order conditions:

$$\frac{\partial \pi}{\partial L_x} = \frac{p_x}{2\sqrt{L_x}} - w = 0. \tag{17}$$

and

$$\frac{\partial \pi}{\partial L_y} = \frac{p_y}{4\sqrt{L_y}} - w = 0. \qquad \textbf{(18)}$$

Solving these two conditions reveals that

$$L_x(p_x,p_y,w) = \frac{p_x^2}{4w^2}, \qquad L_y(p_x,p_y,w) = \frac{p_y^2}{16w^2}. \qquad \textbf{(19)}$$

Given L_x and L_y, the supplies of goods x and y, respectively are:

$$x^S(p_x,p_y,w) = \sqrt{L_x(p_x,p_y,w)} = \frac{p_x}{2w}, \qquad y^S(p_x,p_y,w) = \frac{\sqrt{L_y(p_x,p_y,w)}}{2} = \frac{p_y}{8w}. \qquad \textbf{(20)}$$

Having solved for the firm's demands for inputs and supplies of final products, the firm's profits given the prices p_x, p_y, and w are:

$$\pi = p_x\sqrt{L_x} + p_y\frac{\sqrt{L_y}}{2} - wL_x - wL_y = \frac{4p_x^2 + p_y^2}{16w}. \qquad \textbf{(21)}$$

Since Robinson owns the firm, his income is equal to the firm's profits plus his income from selling $\bar{L}$ units of input to the firm at a price of w per unit. Hence, Robinson's income is:

$$M = \pi + w\bar{L} = \frac{4p_x^2 + p_y^2}{16w} + w\bar{L}. \qquad \textbf{(22)}$$

Now we are ready to characterize Robinson's demand for goods x and y. The demands for x and y are determined by the solution to the following problem:

$$\begin{aligned} &\underset{x,y}{Max} \quad \sqrt{x}\sqrt{y} \\ &s.t. \quad p_x x + p_y y = M, \end{aligned} \qquad \textbf{(23)}$$

where M is defined in equation (22). The Lagrangian associated with this problem is

$$\underset{x,y,\lambda}{Max} \quad \mathcal{L}(x,y,\lambda) \equiv \sqrt{x}\sqrt{y} + \lambda\left(M - p_x x - p_y y\right). \tag{24}$$

The first order conditions for the problem are:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} - \lambda p_x = 0, \tag{25}$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} - \lambda p_y = 0, \tag{26}$$

and

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = M - p_x x - p_y y = 0. \tag{27}$$

Solving equations (25)-(27), the demands of Robinson are given by:

$$x^D(p_x,p_y,w) = \frac{M}{2p_x}, \qquad y^D(p_x,p_y,w) = \frac{M}{2p_y}. \tag{28}$$

Since we already solved for the firm's demands for inputs in equation (19), the firm's supply of x and y in equation (20) and Robinson's demand for goods x and y in equation (28), we can determine the Walrasian equilibrium by substituting from equations (19), (20), and (28), and recalling that Robinson's income is given by equation (22) into the equilibrium conditions in equations (13)-(15). This leads to the following 3 equations that must hold in equilibrium:

$$\frac{\dfrac{4p_x^{*2} + p_y^{*2}}{16w^*} + w^*\bar{L}}{2p_x^*} = \frac{p_x^*}{2w^*}, \tag{29}$$

$$\frac{\dfrac{4p_x^{*2} + p_y^{*2}}{16w^*} + w^*\bar{L}}{2p_y^*} = \frac{p_y^*}{8w^*}, \tag{30}$$

and

$$\frac{p_x^{*2}}{4w^{*2}} + \frac{p_y^{*2}}{16w^{*2}} = \bar{L}. \tag{31}$$

The Walrasian equilibrium, (p_x*, p_y*, w*) is the solution to equations (30)-(32). However, instead of trying to solve the system of 3 equations directly, we can note that by Walras' law we can normalize one of the prices to 1. Since it does not matter which price we normalize to 1, let's pick w* = 1. Now, we need to solve the system with w* = 1 and find p_x* and p_y*.

To this end, note that equations (29) and (30) can be rewritten as follows:

$$4p_x^{*2} + p_y^{*2} + 16\bar{L} = 16p_x^{*2}, \tag{32}$$

and

$$4p_x^{*2} + p_y^{*2} + 4\bar{L} = 4p_y^{*2}. \tag{33}$$

But since the left side of the two equations is the same, it follows that

$$16\,p_x^{*2} \;=\; 4\,p_y^{*2}, \;\;\Leftrightarrow\;\; 2\,p_x^{*} \;=\; p_y^{*}. \qquad \textbf{(34)}$$

Substituting for $2p_x^* = p_y^*$ and $w^* = 1$ in equation (31), the equation becomes:

$$\frac{p_x^{*2}}{4} + \frac{4\,p_x^{*2}}{16} \;=\; \bar{L}, \;\;\Leftrightarrow\;\; p_x^{*} \;=\; \sqrt{2\,\bar{L}}. \qquad \textbf{(35)}$$

Hence,

$$p_y^{*} \;=\; 2\,p_x^{*} \;=\; \sqrt{8\,\bar{L}}. \qquad \textbf{(36)}$$

That is, the walrasian equilibrium is given by:

$$p_x^{*} \;=\; \sqrt{2\,\bar{L}}, \qquad p_y^{*} \;=\; \sqrt{8\,\bar{L}}, \qquad w^{*} \;=\; 1. \qquad \textbf{(37)}$$

To verify that the resulting allocation is Pareto efficient, note that given the equilibrium prices, the quantities of x and y that Robinson will consume are

$$x^{*} \;=\; \frac{p_x}{2\,w^{*}} \;=\; \frac{\sqrt{\bar{L}}}{\sqrt{2}}, \qquad y^{*} \;=\; \frac{p_y^{*}}{8\,w^{*}} \;=\; \frac{\sqrt{\bar{L}}}{\sqrt{8}}, \qquad \textbf{(38)}$$

which is exactly the Pareto efficient allocation we found earlier.

The conclusion is that the Walrasian mechanism is a way to implement Pareto efficient allocations in a decentralized manner: Robinson does not need to see the "big" picture. When he acts as a manager of a firm he responds to the price of the input and the prices of the two goods and decides how much to produce. As a buyer he simply decides how much to buy given the prices and his income. Yet, despite the fact that he acts in two different roles, the final outcome is Pareto efficient exactly as if he were to think about the whole problem of finding the most beneficial allocation from his point of view.