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Solution to Problem set 2

Problem 1

(a) If the entrepreneur finances the project out of his own pocket, his expected payoff is given by:

$$U^E(p,0) = pY + Y(1-p^2) - 1. \tag{1}$$

The first order condition for p is given by:

$$\frac{\partial U^E(p,0)}{\partial p} = Y - 2pY = 0. \tag{2}$$

The first term in this condition is the marginal benefit from choosing a higher probability of success while the second term is the marginal cost to the entrepreneur from choosing a less "attractive" project. The first order condition implies that at the optimum, p* = 1/2. The entrepreneur's expected payoff at the optimum is:

$$U^E(p^*,0) = \frac{Y}{2} + Y\left(1 - \frac{1}{4}\right) - 1 = \frac{5Y}{4} - 1. \tag{3}$$

(b) Debtholders receive D with probability p and K (the value of the collateral) with probability 1-p. Hence their expected payoff is pD + (1-p)K. This payoff must be at least 1+r, which is the amount of money that debtholders can earn on the money they give the entrepreneur. Hence,

$$pD + (1-p)K \geq 1 + r. \tag{4}$$

Solving for D:

$$D \geq \frac{1 + r - (1-p)K}{p} \tag{5}$$

(c) Given a debt level D, and recalling that with probability p the entrepreneur can repay his debt and with probability 1-p he can't, the expected payoff of the entrepreneur is given by:

$$U^E(p,D) = p(Y - D) - (1-p)K + Y(1-p^2). \tag{6}$$

The first order condition for p (holding D fixed) is given by:

$$\frac{\partial U^E(p,D)}{\partial p} = Y - D + K - 2pY = 0. \tag{7}$$

This condition implies that the optimal p for the entrepreneur is

$$p^*(D) = \frac{Y - D + K}{2Y}. \tag{8}$$

Clearly, the entrepreneur now chooses projects with lower probability of success but higher private benefits. This is exactly the "cost of outside equity" first identified by Jensen and Meckling (1976), although here it arises with debt financing rather than with equity (can you think for a reason why debt financing here creates the same type of distortion that equity financing creates in Jensen and Meckling?).

(d) Since the capital market is competitive, (4) will hold with equality. Substituting p*(D) from (8) into the right hand side of (4), and solving for D yields:

$$D^* = \frac{2K + Y - \sqrt{Y(Y - 8(1 + r - K))}}{2}. \tag{9}$$

This equation has a solution only if Y > 8(1+r-K). Otherwise, there does not exist a D that ensures that outside investors break even so the entrepreneur cannot finance investment with debt.

(e) Differentiating (9) with respect to K yields

$$\frac{\partial D^*}{\partial K} = 1 - \frac{8Y}{4\sqrt{Y(Y - 8(1 + r - K))}}.$$

The derivative is negative if

$$4\sqrt{Y(Y - 8(1 + r - K))} < 8Y \quad \Leftrightarrow \quad Y(Y - 8(1 + r - K)) < 4Y^2,$$

or, since Y > 0:

$$Y - 8(1 + r - K) < 4Y \quad \Leftrightarrow \quad -8(1 + r - K) < 3Y.$$

The last inequality must hold because by assumption, K < 1+r, so the left-hand side of the inequality is negative while the right-hand side is positive. Hence, D* is decreasing with K. The reason is simple: D is the compensation of debtholders when the project succeeds and K is their compensation when the project fails. Since n expectation debtholders should receive 1+r, it is clear that the higher K is, the smaller is the compensation that debtholders require when the project succeeds.

Problem 2

(a) If the manager invests optimally and the firm can raise funds in period 1 after $\tilde{X}$ is realized, then if $\tilde{X} \leq I^*$, the firm will raise $I^* - \tilde{X}$ dollars to make sure that it can invest exactly I* dollars which is the efficient level of investment. If $\tilde{X} > I^*$, the firm will not raise new funds and will invest just I*. Hence, the market value of the firm in period 0 is

$$\begin{aligned} V &= \int_0^{I^*} \left(z_1 I^* - (I^* - \tilde{X})\right) dF(\tilde{X}) + \int_{I^*}^{X} \left(z_1 I^* + (\tilde{X} - I^*)\right) dF(\tilde{X}) \\ &= \hat{X} + I^*(z_1 - 1). \end{aligned} \tag{10}$$

That is, the market value of the firm is just equal to the mean of $\tilde{X}$ plus the net return on the efficient level of investment, $(z_1-1)I^*$.

(b) Since the manager of the firm invests all of $\tilde{X}$ in the project, the market value of the firm when it cannot raise external funds in period 1 is

$$V = \int_0^{I^*} z_1 \tilde{X} dF(\tilde{X}) + \int_{I^*}^{X} \left(z_1 I^* + z_2(\tilde{X} - I^*)\right) dF(\tilde{X}). \tag{11}$$

The first integral represents the expected value of the firm when $\tilde{X} \leq I^*$ and the second integral represents the expected value of the firm when $\tilde{X} > I^*$. Together, the two integrals represent the expected earnings in period 2. The market value of the firm in period 0 is equal to the expected earnings in period 2 because this is the amount the investors can get if they buy the firm's shares.

(c) To rewrite V, let's add and subtract V^{fb} from V. Rearranging terms we obtain:

$$V = V^{fb} - \int_0^{I^*} (I^* - \tilde{X})(z_1 - 1) dF(\tilde{X}) - \int_{I^*}^{X} (\tilde{X} - I^*)(1 - z_2) dF(\tilde{X}). \tag{12}$$

The first integral represents the expected cost of underinvestment that occurs when the firm does not have enough funds to reach the efficient level of investment I*. The second integral represents the expected cost of overinvestment that arises when $\tilde{X} > I^*$, in which case the manager invests even though investment is inefficient.

(d) If the firm can raise N dollars from outsiders then it has $\tilde{X}$ +N dollars to invest in period 1.

Thus, if $\tilde{X}$ +N ≤ I*, or $\tilde{X}$ ≤ I*-N, the manager invests less than I* and the return on the investment is z_1. If however, $\tilde{X}$ +N > I*, or $\tilde{X}$ > I*-N, the manager invests more than I* and the return on the investment beyond I* is z_2. Hence the expected earnings in period 2 are:

$$Y = \int_{0}^{I^*-N} (\tilde{X} + N)z_1 dF(\tilde{X}) + \int_{I^*-N}^{X} \left(I^* z_1 + \left(\tilde{X} + N - I^*\right)z_2\right)dF(\tilde{X}). \tag{13}$$

Since new equityholders anticipate that in period 2 the earnings will be E they require an equity fraction such that αY = N.

(e) The value of the existing shareholders is (1-α)Y. But since αY = N, we can write it as E-N. Using equation (13) we therefore have:

$$V(N) = \int_{0}^{I^*-N} (\tilde{X} + N)z_1 dF(\tilde{X}) + \int_{I^*-N}^{X} \left(I^* z_1 + \left(\tilde{X} + N - I^*\right)z_2\right)dF(\tilde{X}) - N. \tag{14}$$

The first order condition for N* when N* > 0 is given by:

$$\begin{aligned} V'(N) &= \int_{0}^{I^*-N} z_1 dF\left(\tilde{X}\right) + \int_{I^*-N}^{X} z_2 dF\left(\tilde{X}\right) - 1 \\ &= z_1 F(I^* - N) + z_2\left(1 - F\left(I^* - N\right)\right) - 1 = 0. \end{aligned} \tag{15}$$

To understand this condition, note that F(I*-N) is the likelihood that the firm will underinvest and 1-F(I*-N) is the likelihood that it will overinvest. When the firm raises another dollar from outside investors it can raise the amount of investment in states in which investment is efficient and in states in which it is not. The first term in the first order condition is the marginal benefit from relaxing the underinvestment problem slightly. The second term is the marginal cost from exacerbating the overinvestment problem slightly. The last term is the marginal cost of external funds.

To find a condition that ensures that N* > 0, let's rewrite V'(N) as follows:

$$\begin{aligned} V'(N) &= z_1 F(I^* - N) + z_2\left(1 - F(I^* - N)\right) - 1 \\ &= (z_1 - z_2)F(I^* - N) - (1 - z_2). \end{aligned} \tag{16}$$

Now, N* > 0 if V'(0) > 0. Hence the condition that ensures that N* > 0 is:

$$F(I^*) > \frac{1 - z_2}{z_1 - z_2}. \tag{17}$$

This condition says that the firm will issue new equity only if the likelihood of underinvestment is large (i.e., F(I*) is large) and the likelihood of overinvestment is small (a large F(I*) implies that 1-F(I*) is small). The condition is more likely to hold when the benefit from lowering the likelihood of underinvestment is large (i.e., z_1 is large) and the cost of overinvestment is small (z_2 is not too small).

(f) The value of debt is given by:

$$B(D) = \int_0^D \tilde{X}\, dF(\tilde{X}) + \int_D^X D\, dF(\tilde{X}). \tag{18}$$

Since the firm has to pay out D dollars, it has $\tilde{X}$ -D dollars left for investment in period 1 (provided that this is positive; if it is not the firm is liquidated). Thus, if $\tilde{X}$ - D ≤ I*, or $\tilde{X}$ ≤ I*+D, the manager invests less than I* and the return on the investment is z_1. If however, $\tilde{X}$ -D > I*, or $\tilde{X}$ > I*+D, the manger invests more than I* and the return on the investment beyond I* is z_2. Hence the market value of equity is given by

$$E(D) = \int_D^{I^*+D} (\tilde{X} - D)\, z_1\, dF(\tilde{X}) + \int_{I^*+D}^X \left(I^* z_1 + \left(\tilde{X} - D - I^*\right) z_2\right) dF(\tilde{X}). \tag{19}$$

Adding equations (18) and (19), the value of the firm is given by:

$$\begin{aligned} V(D) &= \int_0^D \tilde{X}\, dF(\tilde{X}) + \int_D^{I^*+D} \left(D + (\tilde{X} - D) z_1\right) dF(\tilde{X}) \\ &+ \int_{I^*+D}^X \left(D + I^* z_1 + \left(\tilde{X} - D - I^*\right) z_2\right) dF(\tilde{X}). \end{aligned} \tag{20}$$

(g) The first order condition for D* when D* > 0 is given by:

$$\begin{aligned} V'(D) &= -\int_D^{I^*+D} (z_1 - 1)\, dF(\tilde{X}) + \int_{I^*+D}^X (1 - z_2)\, dF(\tilde{X}) \\ &= -(z_1 - 1)\left(F(I^*+D) - F(D)\right) + (1 - z_2)\left(1 - F(I^*+D)\right) = 0. \end{aligned} \tag{21}$$

To understand this condition, note that as D increases, the firm has less money to invest, both in states in which investment is efficient and in states in which it is inefficient. The first term in the first order condition is the marginal cost from exacerbating the underinvestment problem. This cost arises because by increasing D, the firm lowers the amount of money that can be (efficiently) invested at a return of $z_1 > 1$ and increases instead the transfer to debtholders that pays for the debt only 1. The second term is the marginal benefit from relaxing the overinvestment problem: when D increases, the firm lowers the amount of money that can be (inefficiently) invested at a return of $z_2 < 1$ and increases instead the transfer to debtholders that pays for the debt 1.

$D^* > 0$ if $V'(0) > 0$. Substituting $D = 0$ in $V'(D)$ and rearranging term, the condition that ensures that $D^* > 0$ is:

$$F(I^*) < \frac{1 - z_2}{z_1 - z_2}. \tag{22}$$

This condition is exactly the opposite of the condition that ensures that $N^* > 0$. This is not surprising since debt takes money out of the manager's hands in period 1 while new equity puts money in the manager's hand. Thus, if the likelihood of underinvestment is large (i.e., $F(I^*)$ is large) it is better to issue new equity and relax the underinvestment problem. On the other hand, if the probability of overinvestment is large (i.e., $F(I^*)$ is large), the shareholders prefer to issue debt that relaxes the overinvestment problem.

Problem 3

(a) Given z, the firm will choose I to maximize $X + 2zI^{1/2} - I$. The optimal investment level is $I^* = z^2$ and the cash flow of the firm in date 2 given I^* is $X + 2z\,z - z^2 = X + z^2$. Since this is positive for all I, the firm will invest in all states of nature and its expected value at date 0 is:

$$V(0) = \int_0^1 \left(X + z^2\right)dz = X + \int_0^1 z^2 dz. \tag{23}$$

(b) Given that the cash flow of the firm in date 2 is $X + z^2$, the payoff of equityholders is $\text{Max}\{X + z^2 - D, 0\}$. The firm will invest only this payoff is positive, i.e., whenever $z^2 \geq D - X$. Hence, $z^* = 0$ if $D < X$ and $z^* = (D - X)^{1/2}$ if $D \geq X$. That is, so long as $D < X$, debt is completely safe and hence irrelevant (the firm invests in this case for all $z \geq 0$ exactly as in (a)). When $D \geq X$, debt is risky and hence there are states of nature in which the firm will not invest.

(c) If $D < X$, debt is irrelevant and the market value of the firm is $V(0)$. When $D \geq X$, the firm will invest if $z \geq z^* = (D - X)^{1/2}$ and its cash flow will be $X + z^2$ and will not invest if $z < z^* = (D - X)^{1/2}$ and its cash flow will be simply X. Hence, given z^*, the market value of equity at date 0 is

$$E(D) = \int_{\sqrt{D-X}}^{1} \left(X + z^2 - D\right)dz, \tag{24}$$

and the market value of debt is

$$B(D) = \int_0^{\sqrt{D-X}} X dz + \int_{\sqrt{D-X}}^{1} D dz. \tag{25}$$

Combining the two expressions, the market value of the firm is

$$V(D) = E(D) + B(D) = X + \int_{\sqrt{D-X}}^{1} z^2 dz. \tag{26}$$

(d) It is easy to see that risky debt lowers the value of the firm since it raises the lower bound of the integral. The reason why risky debt lowers the value of the firm is that it induces the firm to forgo profitable investment opportunities in some states of nature.

Problem 4

See the paper.