The incentive to participate in open source projects: a signalling approach*

Yossi Spiegel†


Abstract

This paper examines the incentives of programmers to contribute to open source software projects on a voluntary basis. In particular, the paper looks at how this incentive changes as (i) performance becomes more visible to the relevant audience, (ii) effort has a stronger impact on performance, and (iii) performance becomes more informative about talent. In all three cases, it is shown that these conjectures may or may not hold depending, among other things on the stability of equilibrium, as well as the nature of the probability that successful actions will be observed by prospective employers.

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†Recanati Graduate School of Business Administration, Tel Aviv University, email: spiegel@post.tau.ac.il, http://www.tau.ac.il/~spiegel
1 Introduction

Open source software (OSS) is a computer program whose source code - the instructions for the program, written in a human readable format - is distributed free of charge and can be modified, extended, adapted, and incorporated into other programs with relatively few restrictions. OSS is a rapidly expanding phenomenon: some OSS such as the Apache web server, dominate their product categories. In the personal computer market, some OSS such as the operating system Linux and the web browser Firefox gain rapid popularity. It is estimated that there are currently 29 million users of Linux worldwide, and as of April 2006, over 165 million downloads of Firefox were counted.\(^1\)

Apart from having millions of OSS users, there are also tens of thousands of participating programmers who contribute to various OSS projects, and there is also a growing number of firms who sell services, support, and documentation for OSS. The majority of the programmers who participate in OSS projects are unpaid volunteers. For example, Hars and Ou (2002) have surveyed 81 individuals involved in open source projects and found that only 16% received any direct monetary compensation for their contribution. This raises obvious questions about the incentives and motivations of the participating programmers. There are three main, mostly complimentary, explanations for the willingness of programmers to contribute to OSS projects. The first two involve intrinsic motivations while the third involves extrinsic motivations.

The first explanation is that programmers simply like to be involved in open source projects, either because they enjoy being creative, or due to a sense of obligation or community related reasons, or simply due to sheer altruism.\(^2\) Indeed, a web-based survey conducted by Lakhani and Wolf (2003) reveals that the responding programmers were mainly driven by enjoyment-based intrinsic motivations.

The second explanation involves another type of intrinsic motivation. According to this explanation, individual users such as system managers (e.g., users of Apache), who


\(^2\)Athey and Ellison (2006) consider a dynamic model of the evolution of open source software projects, in which altruistic programmers who have used the software in the past are motivated to publish their own improvements for the benefit of other users.
make all sorts of software improvements for their own benefit, are willing to share these improvements with other users in their community. A model along these lines is offered by Johnson (2002), who views participation in OSS projects as a private provision of a public good (see Bessen, 2006, and Bitzer, Schrettl, and Schröder, 2007, for related models).

The third explanation, suggested by Lerner and Tirole (2002), is that programmers are willing to contribute to OSS projects in order to signal their ability to potential employers, venture capitalists, or to peers and thereby boost their human capital or enhance their social status within the programmers’ community. Fershtman and Gandal (2004) examine a large data set on programmers’ participation in OSS projects and argue that their findings are consistent with the hypothesis that participants in OSS projects are indeed driven by such extrinsic motivations. Hann et al (2004) examine a panel data on contributions to three Apache OSS projects for the period 1998 to 2002. They find that while an increase in the number of contributions to the Apache OSS projects does not result in wage increases for contributors, credentials earned through the merit-based ranking system within the Apache open source community are associated with a 13%–27% increase in wages, depending on the rank attained. These findings suggest that status within the Apache Software Foundation serves as a credible signal of the contributor’s productivity.

Drawing on the “career concerns” literature (e.g., Holmström, 1999), Lerner and Tirole (2002) conjecture that the signalling incentive to participate in OSS projects will become stronger as (i) performance becomes more visible to the relevant audience, (ii) effort has a stronger impact on performance, and (iii) performance becomes more informative about talent. While these conjectures are intuitively appealing, it is possible to think about the opposite conjectures. For instance, if effort has a greater impact on performance and/or if performance becomes more visible, then even a small amount of effort may enable talented programmers to produce a visible positive signal about their talent.³ Likewise, if performance is more informative about talent, then it might be that even limited amount of performance may be enough to demonstrate high ability.

³When effort has a weak effect on performance or when performance has low visibility, talented programmers would have to exert a higher amount of effort to produce the same amount of visible positive signals about their talent.
The purpose of this paper is to examine the Lerner and Tirole conjectures in the context of a formal model. In this model, programmers are privately informed about their types: some are “talented” and have high productivity, while others are “untalented” and have low productivity. To signal their types to prospective employers, programmers participate in OSS projects and their participation is either “successful” (i.e., the programmer “solves a problem”) or it “fails” (i.e., the programmer “fails to come up with satisfactory results”). The main question that I ask is how is the effort that agents exert when participating in OSS projects affected by (i) the probability that successful participation will be observed by prospective employers, (ii) an increase in the marginal effect of effort on the probability of success, and by (iii) a decrease in the probability that an untalented programmer will succeed.

I show that the model always admits a no-effort equilibrium in which firms do not expect programmers to exert effort in OSS projects, and programmers indeed do not exert such effort. However, the model may also admit interior equilibria in which programmers contribute to OSS projects and observed success in such projects translates into higher wages. When these equilibria exist, then generically their number is even, with half being stable and half being unstable. The analysis shows that the three conjectures may or may not be correct depending on whether we start from a stable or an unstable interior equilibrium and depending on the shape of the marginal productivity of the effort of talented agents. These results suggest that in general it is hard to say which factors will boost the signalling incentive of talented agents and which factors will weaken it.

There are two closely related papers that also argue that programmers participate in OSS projects in order signal their abilities to prospective employers. These papers differ from mine both in terms of their set up and in terms of their main focus. Lee, Moisa, and Weiss (2003) consider a model in which programmers need to choose between joining closed source software firms and OSS projects. If they join software firms, their wage reflects the expected productivity of all programmers who join closed-source software firms (more and less talented ones). On the other hand, if they join OSS projects, they forgo current wages, but can signal their productivity to software firms and thereby boost their future wages. The main focus of their analysis is on the relative sizes of the closed-source and the open-source
systems. They show that since mediocre programmers, who cannot benefit from signaling their talent, always prefer to join software firms, an open-source system can exist only if there are sufficiently many talented programmers.

Leppämäki and Mustonen (2004) consider a model in which programmers signal their talent to software firms by choosing how many lines of code to contribute to an OSS project. As in the traditional Spence signalling model, talented programmers have a lower cost of writing lines of code. Consequently, in a separating equilibrium, only talented programmers contribute to the OSS project and their contribution is sufficiently large to deter untalented programmers from mimicking. Their model departs from the traditional Spence signalling model in that the freely available OSS project imposes either a positive or a negative externality on commercial software, depending on whether it is a substitute or a complement for the commercial software. This externality in turn affects the wages that software firms are willing to offer and hence the marginal benefit to signalling. Leppämäki and Mustonen focus on the effect of the externality on the incentive of talented agents to contribute to the OSS project. They show that if the OSS is a substitute (complement) for the commercial software then the contribution of talented programmers will be lower (higher) than in the case where OSS and the commercial software are independent of each other.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 shows that the model can give rise to multiple equilibria and characterizes them. Section 4 illustrates the equilibria in the model and Section 5 studies the comparative static properties of the model and examines how the incentive to contribute to OSS projects is affected by the visibility of the contribution to prospective employers, by the sensitivity of performance to effort, and by how informative is the performance about talent. Section 6 examines the effect of intrinsic motivation to contribute to OSS projects. Concluding remarks are in Section 7.

2 The model

Consider a competitive job market with a large number of agents, each of whom is either “talented” (i.e., has a high productivity) or “untalented” (i.e., has a low productivity). If hired by a firm, the marginal productivity of a talented agent is $w$, while the marginal
productivity of an untalented agent is normalized to 0. Under full information, the wage of each agent is equal to his marginal product. Hence, the wage of talented agents is \( w \) while the wage of untalented agents is 0.

Under asymmetric information, it is common knowledge that the fraction of talented agents in the population is \( \alpha \), but firms cannot tell the agents’ types before hiring them (agents however know their own types). Before the labor market opens up, agents participate in an OSS project in the hope of convincing prospective employers that they are talented.\(^4\) I assume that each agent either succeeds (i.e., “solves a problem”) or fails (i.e., “fails to come up with satisfactory results”). The probability that an untalented agent succeeds is exogenous and equal to \( p_0 \). Talented agents by contrast can boost their probability of success by exerting effort: if a talented agent exerts effort \( e \) in the OSS project, his probability of success increases from \( p_0 \equiv p(0) \) to \( p(e) \).

In and of itself, the OSS project does not benefit the firms nor the agents directly (for now I ignore intrinsic motivations to participate in OSS projects). The only advantage of participation is that it generates a signal on the agents’ types. Firms cannot directly observe if and how much effort agents exert; rather they can only (imperfectly) observe successful actions.\(^5\) In particular, firms observe a successful action with probability \( \beta \). With probability \( 1 - \beta \), as well as when the activity fails, firms observe nothing. Hence, \( \beta \) is a measure of the visibility of the agents’ performance to potential employers. Whenever firms observe nothing, they cannot discern whether the agent has exerted effort but failed or whether he did not exert any effort.

Using subscripts to denote partial derivatives, I make the following assumptions on the probability that a talented agent will succeed:

\[
\begin{align*}
\text{A1} & \quad p'(e) > 0 > p''(e) \\
\text{A2} & \quad p(0) = p_0 \geq 0, \quad \lim_{e \to \infty} p(e) = 1
\end{align*}
\]

Assumption A1 says that effort raises the probability of success but does so at a decreasing

\(^4\)To simplify matters, I assume that the cost of participation is sufficiently low to ensure that all agents participate. Relaxing this assumption complicates the analysis without adding new insights.

\(^5\)My model therefore involves “noisy” signalling since the mapping from efforts to observed success is imperfect. This approach is different from the approach in Leppämäki and Mustonen (2004) where agents choose how many lines of code to write and prospective employers can perfectly observe this choice.
rate. Assumption A2 says that if talented agents do not exert effort, then their probability of success is equal to the success probability of untalented agents. Moreover, it says that the success probability of talented agents goes to 1 as their effort goes to infinity; this assumption will ensure the existence of a solution to the maximization problem of agents.

The payoff of each agent is increasing with his expected wage, \( Ew \), and decreasing with his effort level, \( e \):

\[
U = Ew - e.
\]

3 Equilibrium

I now look for a perfect Bayesian equilibrium in which talented agents exert effort, untalented agents do not exert effort, and the beliefs of firms are consistent with the agents’ strategies. To characterize this equilibrium, suppose that firms believe that the effort of talented agents is \( \hat{e} \). Recalling that the fraction of talented agents is \( \alpha \), it follows that conditional on observing a successful action, firms believe that the agent is talented with probability

\[
q(\hat{e} | s) = \frac{\alpha p(\hat{e})}{\alpha p(\hat{e}) + (1 - \alpha)p_0}.
\]

On the other hand, if firms do not observe a success, then they cannot tell whether (i) the agent is talented, exerted effort, and either failed or his success was unobserved, (ii) the agent is untalented and either failed or his success was not observed. The probability of events (i) is \( \alpha(1 - \beta p(\hat{e})) \), while the probability of event (ii) is \( (1 - \alpha)(1 - \beta p_0) \). Hence, conditional on not observing a successful action, firms believe that the agent is talented with probability

\[
q(\hat{e} | n) = \frac{\alpha (1 - \beta p(\hat{e}))}{\alpha (1 - \beta p(\hat{e})) + (1 - \alpha)(1 - \beta p_0)}.
\]

Note that given Assumption A2, \( q(0 | s) = q(0 | n) = \alpha \): if firms expect talented agents to exert no effort, then success or failure is not an informative signal about the agent’s talent. Moreover, note that \( q(\hat{e} | s) \) approaches 1 as \( p_0 \) approaches 0: if untalented agents cannot

\(^6\)Untalented agents do not exert effort because their success probability is independent of their effort and equal to \( p_0 \).
succeed then success is a sure sign that the agent is talented.

The next step is to find the effort level that talented agents will exert. To this end, note that since the labor market is competitive, the wage of agents is $q(\widehat{e} \mid s)w$ following an observed success and $q(\widehat{e} \mid n)w$ otherwise. Hence, the expected payoff of talented agents given their effort level, $e$, and given the belief of firms, $\widehat{e}$, is

$$U(e, \widehat{e}) = \beta p(e)q(\widehat{e} \mid s)w + (1 - \beta p(e))q(\widehat{e} \mid n)w - e.$$  \hspace{1cm} (3)

The first term on the right-hand side of (3) reflects the idea that with probability $\beta p(e)$, a talented agent succeeds and his success is observed by firms. The second term reflects the idea that with probability $1 - \beta p(e)$, a talented agent either succeeds but his success is not observed by firms or the agent simply fails. In both cases, firms cannot tell whether the agent is talented or not and hence they pay him $q(\widehat{e} \mid n)w$. The last term on the right-hand side of (3) is the agent’s cost of effort.

Assuming that there is a large number of talented agents, each will ignore the effect of his own effort level on $\widehat{e}$. Since Assumption A1 ensures that $U(e, \widehat{e})$ is strictly concave in $e$, the effort level that each talented agent will choose given the firms’ beliefs, $\widehat{e}$, is defined implicitly by the following first order condition:

$$\frac{\partial U(e, \widehat{e})}{\partial e} = \beta p'(e)\Delta(\widehat{e})w - 1 \leq 0, \hspace{1cm} e \frac{\partial U(e, \widehat{e})}{\partial e} = 0,$$  \hspace{1cm} (4)

where

$$\Delta(\widehat{e}) \equiv q(\widehat{e} \mid s) - q(\widehat{e} \mid n)$$  \hspace{1cm} (5)

$$= \frac{\alpha (1 - \alpha) (p(\widehat{e}) - p_0)}{(\alpha p(\widehat{e}) + (1 - \alpha)p_0) (1 - \beta (\alpha p(\widehat{e}) + (1 - \alpha)p_0))},$$

is the increase in the probability that firms assign to an agent being talented following an observed success. The expression $\beta p'(e)\Delta(\widehat{e})w$ represents the marginal benefit from effort which is equal to the marginal effect of effort on the probability that a successful action will be observed, $\beta p'(e)$, times the extra expected wage that an agents gets in this event, $\Delta(\widehat{e})w$. At an interior optimum, this marginal benefit must be equal to the marginal cost of effort,
which is 1. But, if $\beta p'(e) \Delta(\tilde{e}) w < 1$ for all positive effort levels, then talented agents will not exert any effort. Assumption A2 ensures that $\frac{\partial U(e, \tilde{e})}{\partial e}$ must be negative for large values of $e$ because the fact that $\lim_{e \to \infty} p(e) = 1$ implies that $\lim_{e \to \infty} p'(e) = 0$. Before proceeding, I establish two important properties of $\Delta(\tilde{e})$ in the next lemma:

**Lemma 1:** $\Delta(\tilde{e})$ is an increasing function with $\Delta(0) = 0$.

**Proof:** Straightforward differentiation reveals that

$$
\Delta'(\tilde{e}) = \frac{\alpha (1 - \alpha) p'(\tilde{e}) \left[ \beta \alpha^2 (p(\tilde{e}) - p_0)^2 + p_0 (1 - \beta p_0) \right]}{(\alpha p(\tilde{e}) + (1 - \alpha) p_0)^2 (1 - \beta (\alpha p(\tilde{e}) + (1 - \alpha) p_0))^2} > 0. 
$$

(6)

Since Assumption A2 ensures that $q(0 | s) = q(0 | n) = \alpha$, we get $\Delta(0) = 0$. ■

Recalling that $\Delta(\tilde{e}) w$ is the extra expected wage that an agent receives following an observed success, Lemma 1 implies that when firms believe that talented agents exert more effort, then they are willing to pay higher wages to agents who were observed to be successful as these agents are more likely to be talented. Moreover, if firms believe that talented agents do not exert effort, then observed success is not informative about talent.

Next, let $BR(\tilde{e})$ denote the solution of (4). This function is the best-response of each talented agent against the firms’ beliefs about his effort level. In equilibrium, the firms’ beliefs must be consistent with the true efforts of talented agents. Hence, the equilibrium effort level, $e^*$, is defined implicitly by the equation

$$
e^* = BR(e^*).
$$

(7)

In other words, the equilibrium is defined by the intersection of the best response function, $BR(\tilde{e})$, with the $45^\circ$ line in the $(\tilde{e}, e)$ space. Given its central role in what follows, I now study the properties of $BR(\tilde{e})$ in the next lemma. To establish this lemma, I first impose the following assumption on $p'(0)$:

A3 The marginal effect of effort on the probability of success is large for $e = 0$:

$$
p'(0) > \frac{(\alpha + (1 - \alpha)p_0) (1 - \beta (\alpha + (1 - \alpha)p_0))}{\beta \alpha (1 - \alpha) (1 - p_0) w}.
$$

(8)
If Assumption A3 fails, then it never pays talented agents to exert effort, no matter how high \( \hat{e} \) is, so \( BR(\hat{e}) = 0 \) for all \( \hat{e} \).

**Lemma 2:** Suppose that Assumption A3 holds. Then, the best response of talented agents against the firms’ beliefs about their effort levels, \( BR(\hat{e}) \), has the following properties:

(i) \( BR(\hat{e}) = 0 \) for all \( 0 < \hat{e} \leq \hat{e}_1 \) (talented agents do not exert effort if firms do not expect them to exert much effort) and \( BR(\hat{e}) > 0 \) for all \( \hat{e} > \hat{e}_1 \) (talented agents exert effort if firms expect them to exert a large enough effort), where \( \hat{e}_1 \) is implicitly defined by \( \beta p_0 \Delta(\hat{e}_1) w = 1 \).

(ii) For all for all \( \hat{e} > \hat{e}_1 \),

\[
BR'(\hat{e}) = -\frac{p'(e) \Delta'(\hat{e})}{p''(e) \Delta(\hat{e})} > 0, \tag{9}
\]

and \( \lim_{\hat{e} \to \infty} BR'(\hat{e}) = 0 \).

**Proof:** See the Appendix.

Using Lemma 2, I can now characterize the equilibrium effort level of talented agents. To this end, recall that the equilibrium effort level, \( e^* \), is attained at the intersection of the best response function, \( BR(\hat{e}) \), with the 45° line in the \((\hat{e}, e)\) space (the 45° line reflects the requirement that in equilibrium, firms must hold correct beliefs about the efforts of talented agents). Since \( BR(\hat{e}) \) passes through the origin, \( e^* = 0 \) is an equilibrium effort level. Hence, there always exists a no-effort equilibrium in which talented agents are not expected to exert effort and indeed they do not exert effort.\(^7\) The question is whether there also exist interior equilibria with \( e^* > 0 \)?

To address this question, I present \( BR(\hat{e}) \) in Figure 1, using Lemma 2. As the figure shows, \( BR(\hat{e}) \) coincides with the horizontal axis for sufficiently small values of \( \hat{e} \). Assumption A3 however ensures that there exists a critical value of \( \hat{e} \), denoted \( \hat{e}_1 \), above which \( BR(\hat{e}) \)

\(^7\)Interestingly, the Athey and Ellison (2006) model also admits a “no-effort” equilibrium in which programmers do not contribute to open source projects. In their dynamic model, this equilibrium is driven by the fact that potential contributors to open source projects are former users. An open source software with 0 quality attracts no users, and hence has no future contributors. As a result, its quality can never improve.
becomes positive. As part (ii) of Lemma 2 shows, $BR(\hat{c})$ increases in this range with $\hat{c}$ although eventually its slope becomes flat. Recalling that the equilibrium effort level of talented agents, $e^*$, is determined by the intersection of $BR(\hat{c})$ with the $45^0$ line, it follows from Figure 1 that in general, there are two possibilities.

The first possibility, illustrated in Figure 1a, arises when $BR(\hat{c})$ intersects the $45^0$ line only at $e = 0$. In this case, the model does not admit interior equilibria with $e^* > 0$. A sufficient (though not necessary) condition for this case is that $BR'(\hat{c}) < 1$ for all $\hat{c} > \hat{c}_1$.

The second possibility, illustrated in Figure 1b, arises when $BR(\hat{c})$ intersects the $45^0$ line at least once from below at some $\hat{c} > \hat{c}_1$. In this case, the model admits interior equilibria with $e^* > 0$. Since $\lim_{\hat{c} \to \infty} BR'(\hat{c}) = 0$, $BR(\hat{c})$ must also intersect the $45^0$ line from above at least once. Hence, if there are interior equilibria, then generically, their number must be even. A necessary (but not sufficient) condition for the existence of only two interior equilibria (apart from the no-effort equilibrium) is that $BR''(\hat{c}) < 0$. Using (9), it follows that

$$BR''(\hat{c}) = -\frac{p'(\hat{c})}{p''(\hat{c})} \frac{d}{d\hat{c}} \left[ \frac{\Delta'(\hat{c})}{\Delta(\hat{c})} \right].$$

\[8\] It is also possible that $BR(\hat{c})$ is just tangent to the $45^0$ line. Such tangency point is also an equilibrium, but this equilibrium in non generic in the sense that it will vanish if we introduce small perturbations that may shift $BR(\hat{c})$ either upward or downward. In the rest of the paper I will therefore focus exclusively on generic equilibria.
Since \( p''(e) < 0 \), it follows that \( BR''(\hat{e}) < 0 \) if and only if \( \frac{d}{de} \left[ \frac{\Delta'(\hat{e})}{\Delta(\hat{e})} \right] < 0 \).

Notice that when \( BR'(e^*) < 1 \), the best response function of talented agents, evaluated at the equilibrium point, is flatter than the 45° line and hence must cut it from above. The resulting equilibria \((e_2^* \text{ and } e_4^* \text{ in Figure 1b})\) are then stable in the sense that a Cournot tatônnement process will lead to convergence to the equilibrium point starting from any close neighborhood of the equilibrium point. On the other hand, whenever \( BR'(e^*) > 1 \), \( BR(e^*) \) is steeper than the 45° line and hence must cut it from below. Consequently, the resulting equilibria \((e_1^* \text{ and } e_3^* \text{ in Figure 1b})\) are unstable. I summarize this discussion in the following Proposition:

**Proposition 1:** The model always admits a no-effort equilibrium in which \( e^* = \hat{e}^* = 0 \). A sufficient (but not necessary) condition for the no-effort equilibrium to be unique is that \( BR'(\hat{e}) < 1 \) for all \( \hat{e} > \hat{e}_1 \). However, if the model admits interior equilibria with \( e^* > 0 \), then generically, their number is even, with half being stable and half being unstable. A necessary (but not sufficient) condition for the model to admit only two interior equilibria is that \( \frac{d}{de} \left[ \frac{\Delta'(\hat{e})}{\Delta(\hat{e})} \right] < 0 \).

Next, suppose that interior equilibria exist. Recalling that in equilibrium \( \hat{e} = e^* \), it follows from equation (4) that the equilibrium effort level, \( e^* \), is implicitly defined by

\[
\beta G(e^*)w = 1, \quad G(e^*) \equiv p'(e^*)\Delta(e^*). \tag{10}
\]

It should be noted that the left-hand side of (10) differs from the left-hand side of (4) because in (4), the beliefs of firms about the efforts of talented agents are arbitrary, while in (10) they are consistent with the true efforts of talented agents.\(^9\)

In Section 5 below, I will study the comparative statics properties of \( e^* \). Since the slope of \( G(e^*) \) plays a key role in that analysis, I now establish the relationship between the slope of \( G(e^*) \) and equilibrium stability.

**Proposition 2:** Suppose that the model admits interior equilibria with \( e^* > 0 \). Then, a

\(^9\)Put differently, (4) characterizes the best response of talented agents against any beliefs, while (10) characterizes the intersection of the best response of talented agents with the 45° line.
given interior equilibrium is stable if \( G'(e^*) < 0 \) and unstable if \( G'(e^*) \geq 0 \).

**Proof:** Note from (10) that

\[
G'(e^*) = p''(e^*)\Delta(e^*) + p'(e^*)\Delta'(e^*) = [1 - BR'(e^*)]p''(e^*)\Delta(e^*),
\]

where the second equality follows from (9). Since \( p''(e^*)\Delta(e^*) < 0 \), it follows that \( G'(e^*) \geq 0 \) if \( BR'(e^*) > 1 \) (i.e., the equilibrium is unstable) and \( G'(e^*) < 0 \) if \( BR'(e^*) < 1 \) (the equilibrium is stable). \( \blacksquare \)

### 4 An example

To illustrate the results in the previous section, I will now assume that \( p(e) = 1 - \left(\frac{1}{2}\right)^{t+e} \) and \( p_0 \equiv p(0) = 1 - \left(\frac{1}{2}\right)^t \). It is easy to verify that Assumptions A1-A2 are satisfied. To ensure that Assumption A3, which ensures that \( BR(\hat{e}) > 0 \) for sufficiently large values of \( \hat{e} \), holds I will assume that

\[
w > \frac{\left[1 - (1 - \alpha) \left(\frac{1}{2}\right)^t\right] \left[1 - \beta \left(1 - (1 - \alpha) \left(\frac{1}{2}\right)^t\right)\right]}{\left(\frac{1}{2}\right)^{2t} \ln(2) \alpha \beta (1 - \alpha)}.
\]

(11)

Substituting for \( p(e) \) in equation (4) and rearranging terms, the best-response function of talented agents for large enough values of \( \hat{e} \), is given by

\[
BR(\hat{e}) = \frac{\ln(\beta \ln(2) \Delta(\hat{e}) w - t \ln(2))}{\ln(2)},
\]

(12)

where

\[
\Delta(\hat{e}) \equiv \frac{\alpha (1 - \alpha) \left(\frac{1}{2}\right)^t \left(1 - \left(\frac{1}{2}\right)^{t+\hat{e}}\right)}{\left[\alpha \left(1 - \left(\frac{1}{2}\right)^{t+\hat{e}}\right) + (1 - \alpha) \left(1 - \left(\frac{1}{2}\right)^t\right)\right] \left[1 - \beta \left(\alpha \left(1 - \left(\frac{1}{2}\right)^{t+\hat{e}}\right) + (1 - \alpha) \left(1 - \left(\frac{1}{2}\right)^t\right)\right)\right]}.
\]

For small values of \( \hat{e} \), the right-hand side of (12) is negative so \( BR(\hat{e}) = 0 \).

Setting \( \alpha = \beta = 0.5 \) and \( t = 0.01 \), I now show \( BR(\hat{e}) \) in Figure 2 for two values of \( w \).
When $w = 20$, there exist two interior equilibria - a stable equilibrium in which $e^* = 2.065$ and an unstable equilibrium in which $e^* = 0.008$. When $w = 6$, $BR(\hat{\alpha})$ lies everywhere below the $45^0$ line and interior equilibria fail to exist.

Using this example, I can now examine how changes in $\alpha$ and $w$ affect $e^*$. Figure 3a shows the given that $\beta = 0.5$, $t = 0.01$, and $w = 20$, there are no interior equilibria if $\alpha < 0.0055$ or $\alpha > 0.833$. Otherwise, there exist two interior equilibria for every value of $\alpha$: a stable equilibrium with a high $e^*$ and an unstable equilibrium with a low $e^*$. Focusing on the stable equilibria, the figure shows that $e^*$ first increases and then decreases with $\alpha$. Hence, an increase in the likelihood that a randomly recruited agent is talented may lead to either more or less effort even when attention is restricted to stable equilibria.

Figure 3b shows that holding $\alpha = \beta = 0.5$, and $t = 0.01$, interior equilibria exist only when $w > 7.124$. In this range, there exists for every value of $w$ one stable equilibrium with a high $e^*$ and one unstable equilibrium with a low $e^*$. In stable equilibria, $e^*$ increases with $w$, implying that talented agents exert more effort in stable equilibria when their marginal productivity increases.
5 Comparative statics

Having characterized the equilibrium in my model, I can now examine the conjectures of Lerner and Tirole (2002) that the signalling incentive of agents is stronger when:

(i) performance becomes more visible to the relevant audience,

(ii) effort has a stronger impact on performance, and

(iii) performance becomes more informative about talent.

5.1 The effect of the visibility of performance

To examine conjecture (i), recall that $\beta$ is a measure of the visibility of the agents’ performance to firms. Hence, I examine conjecture (i) by looking at how $e^*$ is affected by an increase in $\beta$:

**Proposition 3:** An increase in $\beta$ which measures the visibility of the agents’ performance to firms, induces talented agents to exert more effort in stable interior equilibria and exert less effort in unstable interior equilibria.
Proof: Differentiating equation (10) with respect to $e^*$ and $\beta$ and rearranging terms, yields

$$\frac{\partial e^*}{\partial \beta} = -\frac{G(e^*) + \beta \frac{\partial G(e^*)}{\partial \beta}}{\beta G'(e^*)},$$

where

$$\frac{\partial G(e^*)}{\partial \beta} = p'(e^*) \frac{\partial \Delta(e^*)}{\partial \beta} = \frac{p'(e^*) \alpha (1 - \alpha) (p(e^*) - p_0)}{\alpha p(e^*) + (1 - \alpha)p_0} \frac{\alpha p(e^*) + (1 - \alpha)p_0}{(1 - \beta (\alpha p(e^*) + (1 - \alpha)p_0))^2} = \frac{p'(e^*) \Delta(e^*)}{(1 - \beta (\alpha p(e^*) + (1 - \alpha)p_0))} > 0.$$

Hence, the sign of $\frac{\partial e^*}{\partial \beta}$ is equal to the sign of $-G'(e^*)$, which by Proposition 2 is positive in stable interior equilibria and negative in unstable interior equilibria. □

Proposition 3 is illustrated in Figure 4. The equilibrium effort level, $e^*$, is attained when $\beta G(e)w$, which is the marginal benefit of effort in equilibrium (i.e., when firms hold correct beliefs about the agent’s effort) is equal to 1 which is the marginal cost of effort. An increase in $\beta$ shifts $\beta G(e)w$ upward. Whether this leads to a higher or a lower $e^*$ depends on whether $G(e)$ is upward sloping (the equilibrium is unstable) or downward sloping (the equilibrium is stable). When $G(e)$ is upward sloping, $e^*$ increases with $\beta$ while when $G(e)$ is downward sloping, $e^*$ decreases with $\beta$.

Proposition 3 shows that Lerner and Tirole’s (2002) conjecture that the signalling incentive of agents will become stronger as their performance becomes more visible to the relevant audience is true only if the model admits interior equilibria and only if these interior equilibria are stable.

So far, I have assumed that the visibility of successful performance is $\beta p(e)$ if the agent is talented and $\beta p_0$ if the agent is untalented. This assumption implies that the effort that talented agent exert in boosting their probability of success and the visibility of success are complements in the production of signals about success. While this assumption is reasonable, one can also imagine case in which effort and visibility of successful performance are substitutes rather than complements. For example, if effort contributes not only to
success but is also required in order to attract attention to the agent’s performance, then an exogenous increase in visibility may allow agents to attract the same amount of attention with less effort.

To illustrate this point, suppose that the probability of observing a successful action is \( p(e, \beta) = 1 - \left(\frac{1}{2}\right)^{\beta + e} \) if the agent is talented and \( p_0(\beta) \equiv p(0, \beta) = 1 - \left(\frac{1}{2}\right)^\beta \) if the agent is untalented and. It is easy to verify that \( p(e, \beta) \) satisfies Assumptions A1-A2, is increasing with \( \beta \), and \( \frac{\partial^2 p(e, \beta)}{\partial e \partial \beta} < 0 \), so that \( e \) and \( \beta \) are indeed substitutes in the production of positive signals. Substituting for \( p(e, \beta) \) and \( \frac{\partial p(e, \beta)}{\partial e} \) in equation (4) and rearranging terms, the best-response function of talented agents for large enough values of \( \hat{\epsilon} \), is given by

\[
BR(\hat{\epsilon}) = \frac{\ln \left( \ln \left( \frac{\Delta(\hat{\epsilon})}{w} \right) \right) - \beta \ln \left( 2 \right)}{\ln \left( 2 \right)},
\]

where

\[
\Delta(\hat{\epsilon}) \equiv \frac{\alpha \left( 1 - \alpha \right) \left( 1 - \left(\frac{1}{2}\right)^{\hat{\epsilon}} \right)}{\left[ 1 - \alpha + \alpha \left(\frac{1}{2}\right)^{\hat{\epsilon}} \right] \left[ 1 - \alpha \left(\frac{1}{2}\right)^{\beta + \hat{\epsilon}} - (1 - \alpha) \left(\frac{1}{2}\right)^\beta \right]}.
\]

Differentiating \( BR(\hat{\epsilon}) \) with respect to \( \beta \),

\[
\frac{\partial BR(\hat{\epsilon})}{\partial \beta} = -\frac{1}{1 - \alpha \left(\frac{1}{2}\right)^{\beta + \hat{\epsilon}} - (1 - \alpha) \left(\frac{1}{2}\right)^\beta}.
\]
Hence, an increase in $\beta$ shifts $BR(\bar{c})$ downward, implying that the effort of talented agents decreases in stable interior equilibria but increases in unstable interior equilibria. This result, which is the opposite of Proposition 3, suggests that conjecture (i) depends not only on the stability of equilibrium, but also on whether effort and visibility are complements in the production of positive signals (as implicitly assumed in Proposition 3) or substitutes (as in the current example).

5.2 The effect of the sensitivity of performance to effort

Next, I examine conjecture (ii) which states that the signalling incentive of agents will become stronger as effort has a stronger impact on performance. To this end, I will introduce a new shift parameter, $\gamma$, which increases the probability of talented agents to succeed at each effort level. That is, I will assume that the probability that a talented agent will succeed in an OSS project is given by $p(e, \gamma)$, where $\frac{\partial p(e, \gamma)}{\partial \gamma} > 0$. To keep the notation simple, I will continue to denote the derivative of $p(e, \gamma)$ with respect to $e$ by $p'(e, \gamma)$. I can now examine conjecture (ii) by studying the effect of an increase in $\gamma$ on $e^*$:

**Proposition 4**: Let $\varepsilon_{p', \gamma} \equiv \frac{\partial p'(e^*, \gamma)}{\partial \gamma} \gamma p'(e^*, \gamma)$ be the elasticity of $p'(e^*, \gamma)$ with respect to $\gamma$ and $\varepsilon_{\Delta \gamma} \equiv \frac{\partial \Delta(e^*)}{\partial \gamma} \frac{\gamma}{\Delta(e^*)}$ be the elasticity of $\Delta(e^*)$ with respect to $\gamma$. Then, an increase in $\gamma$ which implies that effort has a stronger effect on performance induces talented agents to exert more effort in interior equilibria if either (i) $\frac{\varepsilon_{p', \gamma}}{\varepsilon_{\Delta \gamma}} + 1 > 0$ and the equilibrium is stable, or (ii) $\frac{\varepsilon_{p', \gamma}}{\varepsilon_{\Delta \gamma}} + 1 < 0$ and the equilibrium is unstable. Otherwise, an increase in $\gamma$ induces talented agents to exert less effort in interior equilibria.

**Proof**: Differentiating equation (10) with respect to $e^*$ and $\gamma$ and rearranging terms,

$$\frac{\partial e^*}{\partial \gamma} = \frac{\frac{\partial p'(e^*, \gamma)}{\partial \gamma} \Delta(e^*) + p'(e^*, \gamma) \frac{\partial \Delta(e^*)}{\partial \gamma}}{-G'(e^*)} = \left[\frac{\varepsilon_{p', \gamma}}{\varepsilon_{\Delta \gamma}} + 1\right] \frac{p'(e^*, \gamma) \frac{\partial \Delta(e^*)}{\partial \gamma}}{-G'(e^*)},$$
where \( \varepsilon_{\gamma} \) and \( \varepsilon_{\Delta \gamma} \) are defined in the proposition. The proof follows by noting that

\[
\frac{\partial \Delta(e^*)}{\partial \gamma} = \frac{\alpha (1 - \alpha) \frac{\partial p(e^*; \gamma)}{\partial \gamma} \left[ \beta \alpha^2 (p(e^*, \gamma) - p_0)^2 + p_0 (1 - \beta p_0) \right]}{(\alpha p(e^*, \gamma) + (1 - \alpha)p_0)^2 (1 - \beta (\alpha p(e^*, \gamma) + (1 - \alpha)p_0))^2} > 0,
\]

and recalling that \(-G'(e^*)\) is positive in stable interior equilibria and negative in unstable interior equilibria.

Proposition 4 is more involved than Proposition 3 because an increase in \( \gamma \) can shifts \( G(e) \) either upward or downward in Figure 4, depending on the sign of \( \frac{\partial p(e^*; \gamma)}{\partial \gamma} + 1 \) (which is the case when \( \frac{\partial p(e^*; \gamma)}{\partial \gamma} \) is either positive or not too negative), then \( G(e) \) shifts upward when \( \gamma \) increases. As Figure 4 shows, this upward shift in \( G(e) \) leads to a higher \( e^* \) if \( G(e) \) is downward sloping (i.e., the equilibrium is stable) but to a lower \( e^* \) if \( G(e) \) is upward sloping (i.e., the equilibrium is unstable). Otherwise, if \( \frac{\partial p(e^*; \gamma)}{\partial \gamma} + 1 < 0 \), then \( G(e) \) shifts downward and hence, \( e^* \) increases if \( G(e) \) is upward sloping (i.e., the equilibrium is unstable) and decreases if \( G(e) \) is downward sloping (i.e., the equilibrium is stable).

The implication of Proposition 4 is that an increase in \( \gamma \) can either lead to more effort by talented agents, as Lerner and Tirole’s (2002) conjecture, or to less effort, contrary to their conjecture. An increase in \( \gamma \) leads to more effort either when we start from a stable equilibrium and \( p'(e^*, \gamma) \), which is the marginal effect of effort on the probability of success, increases with \( \gamma \), or when we start from an unstable equilibrium and \( p'(e^*, \gamma) \) decreases with \( \gamma \). Otherwise, the Lerner and Tirole conjecture does not hold and an increase in \( \gamma \) induces talented agents to exert less rather than more effort.

To illustrate Proposition 4, I will now modify the example from Section 4 by assuming that \( p(e, \gamma) = 1 - \left( \frac{1}{2} \right)^{t+\gamma e} \), where \( \gamma > 0 \). It is easy to verify that this function satisfies Assumptions A1-A2, is increasing with \( \gamma \), and \( p'(e, \gamma) = \gamma \ln(2) \left( \frac{1}{2} \right)^{t+\gamma e} \) is first increasing and then decreasing with \( \gamma \). Assumption A3 which ensures that \( BR(\tilde{e}) > 0 \) for sufficiently large values of \( \tilde{e} \) can be written in this case as follows:

\[
w > \frac{\left[ 1 - (1 - \alpha) \left( \frac{1}{2} \right)^t \right] \left[ 1 - \beta \left( 1 - (1 - \alpha) \left( \frac{1}{2} \right)^t \right) \right]}{\left( \frac{1}{2} \right)^{2t} \ln(2) \alpha \beta \gamma (1 - \alpha)}.
\]
In what follows, I will assume that $w$ satisfies (11).

Substituting for $p(e, \gamma)$ and $p'(e, \gamma)$ in equation (4) and rearranging terms, the best-response function of talented agents for large enough values of $\widehat{e}$, is given by

$$BR(\widehat{e}) = \frac{\ln (\beta \gamma \ln (2) \Delta(\widehat{e}) w) - t \ln (2)}{\gamma \ln (2)},$$

where

$$\Delta(\widehat{e}) \equiv \frac{\alpha (1 - \alpha) \left( \frac{1}{2} \right)^t \left( 1 - \left( \frac{1}{2} \right)^{\gamma \widehat{e}} \right)}{\left[ \alpha \left( 1 - \left( \frac{1}{2} \right)^{t+\gamma \widehat{e}} \right) + (1 - \alpha) \left( 1 - \left( \frac{1}{2} \right)^t \right) \right] \left[ 1 - \beta \left( \alpha \left( 1 - \left( \frac{1}{2} \right)^{t+\gamma \widehat{e}} \right) + (1 - \alpha) \left( 1 - \left( \frac{1}{2} \right)^t \right) \right) \right]}.$$

To examine the effect of $\gamma$ on $e^*$, let $\alpha = \beta = 0.5$, $t = 0.01$, and $w = 20$. Figure 5a shows that when $\gamma$ increases from 0.5 to 1, $BR(\widehat{e})$ shift upward and the effort of talented agents in the stable interior equilibrium increases from 1.856 to 2.066. However, when $\gamma$ increases from 0.5 to 2, $BR(\widehat{e})$ rotates clockwise, and the effort of talented agents in the stable equilibrium decreases from 1.856 to 1.563. Consequently, the relationship between $\gamma$ and $e^*$ is non-monotonic.

To examine this nonmonotonicity further, Figure 6 shows $e^*$ as a function of $\gamma$ for $\alpha = \beta = 0.5$, $t = 0.01$, and $w = 20$. When $\gamma$ is small, there do not exist interior equilibria.
When $\gamma > 0.356$, there exist for each value of $\gamma$ two interior equilibria: a stable equilibrium with a high $e^*$ and an unstable equilibrium with a low $e^*$. Focusing on stable equilibria (the upper contour in Figure 6), one can see that $e^*$ increases as $\gamma$ increases from 0.356 to 0.754. However, once $\gamma > 0.754$, further increases in $\gamma$ lead to a decrease in $e^*$. Hence, as Proposition 4 shows, the effect of $\gamma$ on $e^*$ can be positive or negative even when attention is restricted to stable equilibria.

\[ \text{Figure 6: the effect of } \gamma \text{ on } e^* \]
Parameter values: $\alpha = \beta = 0.5, t = 0.01, w = 20$

5.3 The effect of the informativeness of performance about talent

Conjecture (iii) of Lerner and Tirole states that the signalling incentive of agents will become stronger as performance becomes more informative about talent. This conjecture can be examined by studying the effect of $p_0$ on the equilibrium effort level of talented agents, $e^*$, because a decrease in $p_0$ implies that successful agents are more likely to be talented. That is, when $p_0$ decreases towards 0, the probability that a successful agent is talented, $q(\hat{c} | s)$, increases towards 1.

**Proposition 5:** A decrease in $p_0$, which implies that performance is more informative about talent, induces talented agents to exert more effort in stable interior equilibria and exert less effort in unstable interior equilibria.
Proof: Differentiating equation (10) with respect to $e^*$ and $p_0$, recalling that $\Delta(\hat{e}) \equiv q(\hat{e} \mid s) - q(\hat{e} \mid n)$ and rearranging terms,

$$\frac{\partial e^*}{\partial p_0} = \frac{p'(e^*) \frac{\partial \Delta(e^*)}{\partial p_0}}{-G'(e^*)} = \frac{p'(e^*)}{-G'(e^*)} \left[ \frac{\partial q(\hat{e} \mid s)}{\partial p_0} - \frac{\partial q(\hat{e} \mid n)}{\partial p_0} \right],$$

where the square bracketed expression is negative because (1) implies that $\frac{\partial q(\hat{e} \mid s)}{\partial p_0} < 0$ while (2) implies that $\frac{\partial q(\hat{e} \mid n)}{\partial p_0} > 0$. Hence, the sign of $\frac{\partial e^*}{\partial p_0}$ is equal to the sign of $G'(e^*)$, which by Proposition 2 is negative in stable interior equilibria and positive in unstable interior equilibria. Consequently, a decrease in $p_0$ raises $e^*$ in stable interior equilibria and lowers $e^*$ in unstable interior equilibria.

Like Propositions 3, a decrease in $p_0$ shifts $G(e)$ upward. When $G(e)$ is decreasing (increasing) with $e$, which is the case in stable (unstable) interior equilibria, this shifts induces more (less) effort. As in the case of Proposition 3 then, the conjecture is true only in stable interior equilibria but not in unstable equilibria.

6 Intrinsic motivation for participation in OSS projects

Up to now I have only considered extrinsic motivations for participation in OSS projects: talented agents take part in OSS projects in the hope of generating positive signals about their talent and thereby boosting their prospects in the labor market. However, this view is obviously too narrow given that many programmers contribute to OSS projects for other reasons, like their sense of creativity, or their desire to solve problems that they face in performing daily tasks (e.g., system managers who fix bugs or add new functions to an existing software), or acquiring programming skills. The question is how such intrinsic motivations affect matters.\footnote{Bitzer, Schrettl, and Schröder (2006) study a dynamic model that involves both intrinsic and extrinsic motives (i.e., signaling) for participation in open source projects and explore the interaction between them.}

To address this question, suppose that apart from their ability to boost their prospects...
in the labor market, agents also draw a positive utility \( v \) from successful contributions to OSS projects (this utility is independent on whether the success is or is not observed by firms).\(^{11}\) Given \( v \), the utility of talented agents becomes

\[
U(e, \hat{e}) = p(e) \left[ v + \beta q(\hat{e} \mid s)w + (1 - \beta p(e)) q(\hat{e} \mid n)w - e \right].
\]

The effort level that each talented agent will choose given the firms’ beliefs, \( \hat{e} \), is now defined implicitly by the following first order condition:

\[
\frac{\partial U(e, \hat{e})}{\partial e} = p'(e) \left[ v + \beta \Delta(\hat{e})w \right] - 1 \leq 0, \quad e \frac{\partial U(e, \hat{e})}{\partial e} = 0.
\] (13)

It is easy to see from (13) that \( v \) raises the marginal benefit from effort and hence, other things being equal, it expands the set of parameters for which the model admits interior equilibria. Moreover, \( v \) shifts the best response function of talented agents upward in the sense that holding the belief of firms, \( \hat{e} \), constant, an increase in \( v \) leads to an increase in \( BR(\hat{e}) \). Consequently, it is clear from the analysis in the previous sections that an increase in \( v \) will lead to more effort in stable interior equilibria but less effort in unstable equilibria.

7 Conclusion

The main finding in this paper is that the extrinsic motivation of programmers to contribute to OSS projects is more complex than it would seem at first glance. The reason for this is that the efforts of programmers are strategic complements in the sense that in equilibrium, the efforts of other programmers affect the expectations of firms about the programmers’ talent and therefore affect the marginal benefit of participation in OSS projects. This complementarity has several important implications. First, there may not exist interior equilibria in which programmers exert effort in OSS projects. This implies in turn that OSS projects may never take off. Second, when interior equilibria exist, we generically have an even number

\(^{11}\) Obviously, agents can also benefit from unsuccessful participation. However, in that case their utility will simply increase by a constant and hence their effort level will not be affected. To make things more interesting, I therefore assume that agents receive \( v \) only when their actions succeed.
of interior equilibria. Assuming that an OSS project grows larger when programmers exert more effort, this means that in general, an OSS project may be small or large depending on the prevailing equilibrium. Finally, the comparative static properties of interior equilibria are more complex than might be thought at a first glance. In particular, shifts in exogenous parameters, like an increase in the visibility of performance and an increase in the marginal productivity of effort may either boost the signalling incentive of talented agents or weaken it even when attention is restricted to stable interior equilibria.

Of course, there are still a number of open questions interesting extensions that must be addressed before we have a good understanding of the incentives to engage in licensing of interim R&D knowledge and their implication....

Mark Rysman: Suppose that participation in OSS boosts the human capital of agents by giving them experience. How can one discriminate the hypothesis that agents join in order to obtain human capital from the competing hypothesis that they join to signal their existing ability and cannot boost it by participation? Perhaps a way to address this is by assuming that without participation you get a wage $w$ but with participation you get a wage $w' > w$. This higher wage is attained irrespective of whether participation is observed.

Dynamic consideration: Heski: Suppose that we have programmer heterogeneity and 2 periods. With a high prob., talented programmers succeed in period 1. If $\beta$ increases, they exit the system in period 1 (a successful agent will stay if his success is not observed) so the average quality of the pool of participants declines in period 2. The incentive to stay in period 2 is therefore lowered and hence the incentive to exert effort will be affected. What will happen then? In particular, could it be that we get unravelling? If you stay in period 2 its not worthwhile to exert effort and then this lowers the overall benefit from joining and may lower the incentive to join in period 1.

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8 Appendix

Following is the proof of Lemma 2.

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Proof: (i) Since $p''(e) < 0$, it is easy to see from equation (4) that $\frac{\partial U(e, \tilde{e})}{\partial e}$ is a strictly decreasing function of $e$ for all $\tilde{e} > 0$. Moreover, Assumption A2 implies that $\lim_{e \to \infty} \frac{\partial U(e, \tilde{e})}{\partial e} = -1$, because the fact that $\lim_{e \to \infty} p(e) = 1$ implies that $\lim_{e \to \infty} p'(e) = 0$. Hence, there exists a unique value of $e$ at which $\frac{\partial U(e, \tilde{e})}{\partial e} = 0$ if and only if

$$\frac{\partial U(0, \tilde{e})}{\partial e} = \beta p'(0) \Delta(\tilde{e})w - 1 > 0. \quad (14)$$

Since Lemma 1 implies that $\Delta(0) = 0$, condition (14) clearly fails when $\tilde{e} = 0$, and by continuity, it also fails for sufficiently small values of $\tilde{e}$. Hence, $BR(0) = 0$ for small values of $\tilde{e}$. On the other hand, since $\Delta'(<\tilde{e}) > 0$, it follows that $\frac{\partial U(0, \tilde{e})}{\partial e}$ is increasing with $\tilde{e}$ and moreover,

$$\lim_{\tilde{e} \to \infty} \frac{\partial U(0, \tilde{e})}{\partial e} = \beta p'(0) w \left( \lim_{\tilde{e} \to \infty} \Delta(\tilde{e}) \right) - 1 = \frac{\beta p'(0) w \alpha (1 - \alpha) (1 - p_0)}{(\alpha + (1 - \alpha)p_0) (1 - \beta (\alpha + (1 - \alpha)p_0))} - 1 > 0. \quad (15)$$

The second equality in equation (15) follows because $\lim_{e \to \infty} p(e) = 1$ by Assumption A2, and the inequality follows by Assumption A3. Therefore, there exists a unique value of $\tilde{e}$, denoted $\tilde{e}_1$, such that $\frac{\partial U(0, \tilde{e})}{\partial e} > 0$ for all $\tilde{e} > \tilde{e}_1$ and $\frac{\partial U(0, \tilde{e})}{\partial e} < 0$ otherwise, where $\tilde{e}_1$ is implicitly defined by the equation

$$\frac{\partial U(0, \tilde{e})}{\partial e} = \beta p'(0) w \Delta(\tilde{e}) - 1 = 0.$$

In sum, whenever $\tilde{e} \leq \tilde{e}_1$, $\frac{\partial U(e, \tilde{e})}{\partial e} < 0$ for all $e$, so $BR(\tilde{e}) = 0$. On the other hand, whenever $\tilde{e} > \tilde{e}_1$, there exists a unique value of $e$ that solves the equation $\frac{\partial U(e, \tilde{e})}{\partial e} = 0$. Hence, $BR(\tilde{e}) > 0$ for all $\tilde{e} > \tilde{e}_1$.

(ii) As part (i) shows, $BR(\tilde{e})$ is defined implicitly by the equation $\frac{\partial U(BR(\tilde{e}), \tilde{e})}{\partial e} = 0$ for all $\tilde{e} > \tilde{e}_1$. Fully differentiating this equation with respect to $\tilde{e}$ and using Assumption A1 and Lemma 1, reveals that $BR'(<\tilde{e})$, defined in equation (9), is positive.
Using equations (5) and (6), it follows that

\[
\lim_{\bar{e} \to \infty} BR'(\bar{e}) = - \frac{p'(\bar{e})}{p''(\bar{e})} \lim_{\bar{e} \to \infty} \frac{\Delta'(\bar{e})}{\Delta(\bar{e})} = - \frac{p'(\bar{e})}{p''(\bar{e})} \lim_{\bar{e} \to \infty} \left[ \frac{p'(\bar{e}) \left[ \beta \alpha^2 (p(\bar{e}) - p_0)^2 + p_0 (1 - \beta p_0) \right]}{(\alpha p(\bar{e}) + (1 - \alpha) p_0)^2 (1 - \beta (\alpha p(\bar{e}) + (1 - \alpha) p_0))} \times \frac{1}{(p(\bar{e}) - p_0)} \right] = 0,
\]

where the last equality follows from Assumption A2. ■

9 References


