Technical Appendix to the paper:

Optimal state-contingent regulation under limited liability

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This technical appendix contains detailed proofs for Lemmas 1-4 (in the paper we just provide sketches of proofs) and the proof of Proposition whose proof was omitted from the paper.

Lemma 1. *At the optimum, EIR_θ and IC_θ, $\theta = \ell, h$ cannot be both slack.*

Proof: Assume by way of negation that both EIR_ℓ and IC_ℓ are slack. Since EIR_ℓ is slack, either $IR_{\ell g}$ or $IR_{\ell b}$ or both are also slack, so it is possible to slightly lower $t_{\ell g}$ or $t_{\ell b}$ or both. This lowers the right-hand side of IC_h and hence relaxes it, without affecting IR_{hg} and IR_{hb}. At the same time, the value of the regulator's objective function is enhanced since $\alpha < 1$, thereby contradicting the assumption that the solution is optimal. The proof that EIR_h and IC_h cannot be both slack is analogous. ■

Lemma 2: *At the optimum, $t_{\ell b}$ and t_{hg} can be set such that $IR_{\ell b}$ and IR_{hg}, respectively, will be binding while $IR_{\ell g}$ and IR_{hb} are slack.*

Proof: Suppose that at the optimum $IR_{\ell b}$ is slack. Now consider an alternative allocation in which $t_{\ell b}$ is lowered by $\varepsilon_{\ell b} > 0$ until $IR_{\ell b}$ is just binding and $t_{\ell g}$ is increased by $\frac{p_{\ell b}\varepsilon_{\ell b}}{p_{\ell g}}$ to ensure that EIR_ℓ and IC_ℓ remain intact. These changes relax $IR_{\ell g}$ (since $t_{\ell g}$ is increased) but have no effect on EIR_h, IR_{hg}, IR_{hb}, and on the regulator's objective function. At the same time the right-hand side of IC_h changes by

$$p_{hg}\frac{p_{\ell b}\varepsilon_{\ell b}}{p_{\ell g}} - p_{hb}\varepsilon_{\ell b} = p_{hg}\varepsilon_{\ell b}\left[\frac{p_{\ell b}}{p_{\ell g}} - \frac{p_{hb}}{p_{hg}}\right] \leq 0,$$

where the inequality follows because the assumption that $J \equiv p_{hb}p_{\ell g} - p_{hg}p_{\ell b} \geq 0$ implies that $\frac{p_{\ell b}}{p_{\ell g}} \leq \frac{p_{hb}}{p_{hg}}$. Hence, IC_h is relaxed. Altogether, this implies that the new allocation also solves the regulator's problem. Since $IR_{\ell b}$ is binding, EIR_ℓ implies that $IR_{\ell g}$ must be slack. The proof concerning t_{hg} is completely analogous. ■

Lemma 3. *If the optimal production levels are strictly monotonic with respect to the firm's type in each state of nature, i.e., $q_{\ell s} > q_{hs}$ for $s = g, b$, then, EIR_ℓ and EIR_h cannot be both slack.*

Proof: Assume by way of negation that at the optimum, EIR_ℓ and EIR_h are both slack. Then, Lemma 1 implies that IC_ℓ and IC_h must be both binding, while Lemma 2 implies that $\pi_{\ell b} = \pi_{hg} = M$. Hence, we can write IC_ℓ and IC_h respectively, as

$$p_{\ell g}\pi_{\ell g} + p_{\ell b}M = p_{\ell g}\left[M + \Delta_g(q_{hg})\right] + p_{\ell b}\left[\pi_{hb} + \Delta_b(q_{hb})\right], \tag{1}$$

and

$$p_{hg}M + p_{hb}\pi_{hb} = p_{hg}\left[\pi_{\ell g} - \Delta_g(q_{\ell g})\right] + p_{hb}\left[M - \Delta_b(q_{\ell b})\right]. \tag{2}$$

Dividing equation (1) by $p_{\ell g}$, dividing equation (2) by p_{hg}, and adding the two equations yields,

$$\left[\frac{p_{hb}}{p_{hg}} - \frac{p_{\ell b}}{p_{\ell g}}\right](\pi_{hb} - M) + \left[\Delta_g(q_{\ell g}) - \Delta_g(q_{hg})\right] + \left[\frac{p_{hb}}{p_{hg}}\Delta_b(q_{\ell b}) - \frac{p_{\ell b}}{p_{\ell g}}\Delta_b(q_{hb})\right] = 0. \tag{3}$$

The first term on the left-hand side of (3) is nonnegative since $J \equiv p_{hb}p_{\ell g} - p_{hg}p_{\ell b} \geq 0$, and since by Lemma 2, IR_{hb} is slack, so that $\pi_{hb} > M$. The second term is strictly positive given the assumption that output is strictly monotonic (recall that $\Delta_s(q)$ is assumed strictly increasing with q). Finally, the third term is strictly positive as $J \geq 0$ implies that $\frac{p_{hb}}{p_{hg}} \geq \frac{p_{\ell b}}{p_{\ell g}}$ and as output is strictly monotonic. The left-hand side of (3) must therefore be strictly positive, a contradiction. We conclude that EIR_ℓ and EIR_h cannot be both slack. ■

Lemma 4. *If the optimal production levels are strictly monotonic with respect to the firm's type in each state of nature, i.e., $q_{\ell s} > q_{hs}$ for $s = g, b$, then EIR_h is binding.*

Proof: Assume be way of negation that EIR_h is slack. Then IC_h is binding by Lemma 1 and EIR_ℓ is binding by Lemma 3. Since $\pi_{\ell b} = M$ by Lemma 2, EIR_ℓ implies that $\pi_{\ell g} = -\frac{p_{\ell b}}{p_{\ell g}}M$; hence, IC_h can be rewritten as

$$\begin{aligned} p_{hg}M + p_{hb}\pi_{hb} &= p_{hg}\left[\pi_{\ell g} - \Delta_g(q_{\ell g})\right] + p_{hb}\left[M - \Delta_b(q_{\ell b})\right] \\ &= p_{hg}\left[-\frac{p_{\ell b}}{p_{\ell g}}M - \Delta_g(q_{\ell g})\right] + p_{hb}\left[M - \Delta_b(q_{\ell b})\right] \\ &= p_{hg}M\left[\frac{p_{hb}}{p_{hg}} - \frac{p_{\ell b}}{p_{\ell g}}\right] - p_{hg}\Delta_g(q_{\ell g}) - p_{hb}\Delta_b(q_{\ell b}). \end{aligned} \tag{4}$$

Since EIR_h is slack, the left-hand side of (4) is strictly positive. The right-hand side is strictly negative since $M < 0$, since $J \geq 0$ implies $\frac{p_{hb}}{p_{hg}} \geq \frac{p_{\ell b}}{p_{\ell g}}$, and since $\Delta_g(q_{\ell g})$ and $\Delta_b(q_{\ell b})$ are both positive. This contradicts the assumption that at the optimum EIR_h is slack. ■

Proposition 3. *Suppose that Assumptions 1-3 hold and the conditional hazard rate* $\frac{f(\theta|n)}{F(\theta|n)}$ *is nonincreasing with* θ. *Then, for all* M, *there exists a* $\delta > 0$ *such that if* $|p_s'(\theta)| < \delta$ *for all* s *in* $\{1, ..., n\}$, *the solution characterized by (22)-(24) (in the paper) satisfies the* EIR_θ *constraints and is globally incentive compatible.*

Proof of Proposition 3. We begin with the EIR_θ constraints. The proof of Lemma 6 shows that $\sum_s r_s(\overline{\theta})\pi_s(\overline{\theta}) = 0$. Hence, $EIR_{\overline{\theta}}$ is binding. To show that EIR_θ holds for $\theta < \overline{\theta}$, note from equation (24) in the paper that the profit of type $\theta < \overline{\theta}$ in state n is

$$\begin{aligned}\pi_n^{**}(\theta) &= t_n^{**}(\theta) - c_n(\theta)q_n^{**}(\theta) \\ &= \int_\theta^{\overline{\theta}} \sum_s r_s(x)c_s'(x)q_s^{**}(x)dx - \frac{M(1-p_n(\overline{\theta}))}{p_n(\overline{\theta})}.\end{aligned}$$

Since by Lemma 6, $\pi_s^{**}(\theta) = M$, $\forall s \neq n$, and recalling that $r_s(x) \equiv \frac{p_s(x)}{p_n(x)}$, the expected profit of type $\theta < \overline{\theta}$ is

$$\begin{aligned}\sum_s p_s(\theta)\pi_s^{**}(\theta) &= (1-p_n(\theta))\,M + p_n(\theta)\left[\int_\theta^{\overline{\theta}} \sum_s r_s(x)c_s'(x)q_s^{**}(x)dx - \frac{M(1-p_n(\overline{\theta}))}{p_n(\overline{\theta})}\right] \\ &= \left(1 - \frac{p_n(\theta)}{p_n(\overline{\theta})}\right) M + p_n(\theta)\int_\theta^{\overline{\theta}} \frac{B^{**}(x)}{p_n(x)}dx,\end{aligned}$$

where $B^{**}(x) \equiv \sum_s p_s(x)c_s'(x)q_s^{**}(x)$. Differentiating the expected profit expression, we get

$$\frac{d}{d\theta}\left(\sum_s p_s(\theta)\pi_s^{**}(\theta)\right) = -\frac{p_n'(\theta)}{p_n(\overline{\theta})}M + p_n'(\theta)\int_\theta^{\overline{\theta}} \frac{B^{**}(x)}{p_n(x)}dx - B^{**}(\theta). \tag{5}$$

Since $EIR_{\overline{\theta}}$ is binding (so $\sum_s p_s(\overline{\theta})\pi_s^{**}(\overline{\theta}) = 0$), it is sufficient to show that the derivative in (5) is negative in order to establish that the EIR_θ constraints are satisfied for all $\theta \in \Theta$. Our strategy will be to show that the right-hand side of (5) is bounded from above and its upper bound is negative for small enough δ.

To find an upper bound for the right-hand side of (5), note from the proof of Lemma 6 that,

$$\frac{F(\theta \mid n)}{f(\theta \mid n)} = \frac{\int_{\underline{\theta}}^{\theta} p_n(x)f(x)dx}{p_n(\theta)f(\theta)}.$$

But since by Assumption 1, $p_n(\theta)$ is increasing with θ,

$$\frac{F(\theta \mid n)}{f(\theta \mid n)} \leq \frac{\int_{\underline{\theta}}^{\theta} p_n(\theta)f(x)dx}{p_n(\theta)f(\theta)} = \frac{F(\theta)}{f(\theta)}.$$

On the other hand, since by Assumption 1, $p_s(\theta) \geq \epsilon > 0$ for all $s \in \{1, ..., n\}$ and all $\theta \in \Theta$,

$$\frac{F(\theta \mid n)}{f(\theta \mid n)} \geq \frac{\int_{\underline{\theta}}^{\theta} \epsilon f(x)dx}{f(\theta)} = \frac{\epsilon F(\theta)}{f(\theta)}.$$

Therefore, equation (22) in the paper implies that for all $s \in \{1, ..., n\}$ and all $\theta \in \Theta$, $\underline{q}_s^{**}(\theta) \leq q_s^{**}(\theta) \leq \overline{q}_s^{**}(\theta)$, where $\underline{q}_s^{**}(\theta)$ and $\overline{q}_s^{**}(\theta)$ are defined implicitly by

$$S'\left(\underline{q}_s^{**}(\theta)\right) = c_s(\theta) + (1-\alpha)\, c_s'(\theta)\frac{F(\theta)}{f(\theta)},$$

and

$$S'\left(\overline{q}_s^{**}(\theta)\right) = c_s(\theta) + (1-\alpha)\, c_s'(\theta)\frac{\epsilon F(\theta)}{f(\theta)}.$$

Using these expressions, the definition of $B^{**}(\cdot)$, and the assumption that $p_n'(\theta) < \delta$ for all $\theta \in \Theta$, we get

$$\begin{aligned} \frac{d}{d\theta}\left(\sum_s p_s(\theta)\pi_s^{**}(\theta)\right) &< \delta\left[\frac{-M}{p_n(\overline{\theta})} + \int_{\theta}^{\overline{\theta}} \frac{\sum_s p_s(x)c_s'(x)\overline{q}_s^{**}(x)}{p_n(x)}dx\right] - \sum_s p_s(x)c_s'(x)\underline{q}_s^{**}(x) \\ &\leq \delta\left[\frac{-M}{\epsilon} + \int_{\theta}^{\overline{\theta}} \frac{\sum_s c_s'(x)\overline{q}_s^{**}(x)}{\epsilon}dx\right] - \sum_s \epsilon c_s'(x)\underline{q}_s^{**}(x), \end{aligned} \tag{6}$$

where the second line follows because $\epsilon \leq p_s(\theta) \leq 1$ for all $s \in \{1, ..., n\}$ and all $\theta \in \Theta$. For sufficiently low δ, the right-hand side of (6) is negative, so $\frac{d}{d\theta}\left(\sum_s p_s(\theta)\pi_s^{**}(\theta)\right) < 0$. Hence, for sufficiently low δ, the EIR_θ constraints are satisfied for all $\theta \in \Theta$.

It now remains to check that the solution to the regulator's relaxed problem satisfies $IC_{\theta,\widehat{\theta}}$ for all $\theta, \widehat{\theta} \in \Theta$. Substituting the transfers defined by equations (23) and (24) in the paper into the $IC_{\theta,\widehat{\theta}}$, recalling that $r_s(\theta) \equiv \frac{p_s(\theta)}{p_n(\theta)}$ and simplifying, the constraint becomes

$$\begin{aligned} &\int_{\theta}^{\overline{\theta}} \sum_s r_s(x)c_s'(x)q_s^{**}(x)dx - \int_{\widehat{\theta}}^{\overline{\theta}} \sum_s r_s(x)c_s'(x)q_s^{**}(x)dx \\ \geq\ &\sum_s r_s(\theta)\left[c_s(\widehat{\theta}) - c_s(\theta)\right] q_s^{**}(\widehat{\theta}), \qquad \forall \theta, \widehat{\theta} \in \Theta. \end{aligned} \tag{7}$$

Now suppose that $\widehat{\theta} \neq \theta$. Then, (7) becomes

$$\int_{\theta}^{\widehat{\theta}} \sum_s r_s(x)c_s'(x)q_s^{**}(x)dx \geq \sum_s r_s(\theta)\left[c_s(\widehat{\theta}) - c_s(\theta)\right] q_s^{**}(\widehat{\theta}), \qquad \forall \theta, \widehat{\theta} \in \Theta.$$

Integrating the left-hand side by parts, we get

$$\begin{aligned}
&\sum_s \left[r_s(\widehat{\theta})c_s(\widehat{\theta})q_s^{**}(\widehat{\theta}) - r_s(\theta)c_s(\theta)q_s^{**}(\theta)\right] - \int_\theta^{\widehat{\theta}} \sum_s c_s(x)\left[r_s'(x)q_s^{**}(x) + r_s(x)q_s^{**\prime}(x)\right]dx \\
\geq\ &\sum_s r_s(\theta)\left[c_s(\widehat{\theta}) - c_s(\theta)\right]q_s^{**}(\widehat{\theta}), \qquad \forall \theta, \widehat{\theta} \in \Theta.
\end{aligned}$$

Rearranging terms and using the fact that $r_s(\widehat{\theta}) - r_s(\theta) = \int_\theta^{\widehat{\theta}} r_s'(x)dx$ and $q_s^{**}(\widehat{\theta}) - q_s^{**}(\theta) = \int_\theta^{\widehat{\theta}} q_s^{**\prime}(x)dx$,

$$\begin{aligned}
&\int_\theta^{\widehat{\theta}} \sum_s r_s'(x)c_s(\widehat{\theta})q_s^{**}(\widehat{\theta})dx + \int_\theta^{\widehat{\theta}} \sum_s r_s(\theta)c_s(\theta)q_s'^{**}(x)dx \\
\geq\ &\int_\theta^{\widehat{\theta}} \sum_s c_s(x)\left[r_s'(x)q_s^{**}(x) + r_s(x)q_s^{**\prime}(x)\right]dx, \qquad \forall \theta, \widehat{\theta} \in \Theta.
\end{aligned}$$

Rearranging terms once again and multiplying both sides of the inequality by $\frac{2}{\left(\widehat{\theta}-\theta\right)^2}$,

$$\begin{aligned}
&\frac{2}{\left(\widehat{\theta}-\theta\right)^2}\int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x)\left[r_s(x)c_s(x) - r_s(\theta)c_s(\theta)\right]dx \qquad (8) \\
\geq\ &\frac{2}{\left(\widehat{\theta}-\theta\right)^2}\int_\theta^{\widehat{\theta}} \sum_s -r_s'(x)\left[c_s(\widehat{\theta})q_s^{**}(\widehat{\theta}) - c_s(x)q_s^{**}(x)\right]dx, \qquad \forall \theta, \widehat{\theta} \in \Theta.
\end{aligned}$$

We now establish that as δ goes to 0, the left-hand side of (8) converges to a strictly positive term while the right-hand side has an upper bound that converges to 0. We begin with the right-hand side of (8). Since $r_s(x) = \frac{p_s(x)}{p_n(x)}$, $-r_s'(x) = \frac{p_s(x)p_n'(x) - p_s'(x)p_n(x)}{(p_n(x))^2}$. But since by assumption, $|p_s'(\cdot)| < \delta$ and $\epsilon \leq p_s(\theta) \leq 1$, it follows that $-r_s'(x) \leq \frac{2\delta}{\epsilon^2}$. Using this inequality and the fact that $c'_s(\cdot) \geq 0$ and $q_s^{**\prime}(\cdot) \leq 0$, yields

$$\begin{aligned}
& \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -r'_s(x) \left[c_s(\widehat{\theta}) q_s^{**}(\widehat{\theta}) - c_s(x) q_s^{**}(x)\right] dx \\
= & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -r'_s(x) \left[\int_x^{\widehat{\theta}} \left[c'_s(z) q_s^{**}(z) + c_s(z) q_s^{\prime **}(z)\right] dz\right] dx \\
\leq & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s \frac{2\delta}{\epsilon^2} \left[\int_x^{\widehat{\theta}} c'_s(z) q_s^{**}(z) dz\right] dx \\
\leq & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s \frac{2\delta}{\epsilon^2} \left[c'_s(\overline{\theta}) q_s^{**}(\underline{\theta}) \int_x^{\widehat{\theta}} dz\right] dx \\
= & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s \frac{2\delta}{\epsilon^2} \left[c'_s(\overline{\theta}) q_s^{**}(\underline{\theta}) \left(\widehat{\theta}-x\right)\right] dx \\
= & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \sum_s \frac{2\delta}{\epsilon^2} c'_s(\overline{\theta}) q_s^{**}(\underline{\theta}) \frac{\left(\widehat{\theta}-\theta\right)^2}{2} \\
= & \frac{2\delta}{\epsilon^2} \sum_s c'_s(\overline{\theta}) q_s^{**}(\underline{\theta}).
\end{aligned}$$

Clearly, this expression converges to 0 as δ goes to 0. As for the left-hand side of (8), recall that for all $s \in \{1, ..., n\}$ and all $\theta \in \Theta$, $\underline{q}_s^{**}(\theta) \leq q_s^{**}(\theta) \leq \overline{q}_s^{**}(\theta)$. Since $q_s^{**\prime}(\cdot) \leq 0$, it follows that $q \in \left[\underline{q}_s^{**}(\overline{\theta}), \overline{q}_s^{**}(\underline{\theta})\right]$. Let

$$k \equiv \max_s \max_q \left\{ |S"(q)| \mid q \in \left[\underline{q}_s^{**}(\overline{\theta}), \overline{q}_s^{**}(\underline{\theta})\right] \right\},$$

be the upper bound on $|S"(q^{**}(\theta))|$. Then, equation (22) in the paper implies that

$$-q_s^{\prime **}(\theta) = \frac{c'_s(\theta) + (1-\alpha)\left(c"_s(\theta) \frac{F(\theta|n)}{f(\theta|n)} + c'_s(\theta) \frac{d}{d\theta}\left(\frac{F(\theta|n)}{f(\theta|n)}\right)\right)}{|S"\left(q_s^{**}(\theta)\right)|} \geq \frac{c'_s(\underline{\theta})}{k}.$$

Using this inequality, noting that $r_s(\cdot) \equiv \frac{p_s(\cdot)}{p_n(\cdot)} \geq \epsilon$, and recalling that $-r'_s(\cdot) \leq \frac{2\delta}{\epsilon^2}$ and

$c_s'(\cdot) > 0$, we get

$$\begin{aligned}
& \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x) \left[r_s(x)c_s(x) - r_s(\theta)c_s(\theta) \right] dx \\
= \; & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x) \left[\int_\theta^x \left[r_s'(z)c_s(z) + r_s(z)c_s'(z) \right] dz \right] dx \\
\geq \; & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x) \left[\int_\theta^x \left[-\frac{2\delta}{\epsilon^2} c_s(z) + \epsilon c_s'(z) \right] dz \right] dx \\
\geq \; & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x) \left[\int_\theta^x \left[-\frac{2\delta}{\epsilon^2} c_s(\overline{\theta}) + \epsilon c_s'(\underline{\theta}) \right] dz \right] dx \\
= \; & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \int_\theta^{\widehat{\theta}} \sum_s -q_s'^{**}(x) \left[-\frac{2\delta}{\epsilon^2} c_s(\overline{\theta}) + \epsilon c_s'(\underline{\theta}) \right] (x-\theta) dx \\
\geq \; & \frac{2}{\left(\widehat{\theta}-\theta\right)^2} \sum_s \frac{c_s'(\underline{\theta})}{k} \left[-\frac{2\delta}{\epsilon^2} c_s(\overline{\theta}) + \epsilon c_s'(\underline{\theta}) \right] \frac{\left(\widehat{\theta}-\theta\right)^2}{2} \\
= \; & \sum_s \frac{c_s'(\underline{\theta})}{k} \left[-\frac{2\delta}{\epsilon^2} c_s(\overline{\theta}) + \epsilon c_s'(\underline{\theta}) \right].
\end{aligned}$$

As δ goes to 0, this expression converges to $\sum_s \frac{(c_s'(\underline{\theta}))^2 \epsilon}{k} > 0$. Hence, for a sufficiently small δ, (8) holds, implying that $IC_{\theta,\widehat{\theta}}$ holds for all $\theta, \widehat{\theta} \in \Theta$. ■