The Herfindahl-Hirschman Index and the distribution of social surplus *

Yossi Spiegel[†]

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Abstract

I show that in a broad range of oligopoly models the Herfindahl-Hirschman index (HHI) reflects the ratio of producer surplus to consumer surplus and therefore the division of surplus between firms' owners and consumers.

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[†]Coller School of Management, Tel Aviv University, CEPR; and ZEW. email: spiegel@post.tau.ac.il, http://www.tau.ac.il/~spiegel.

1 Introduction

The Herfindahl-Hirschman Index (HHI) of concentration, calculated as the sum of the squared market shares of firms, is by now a standard structural measurement tool for assessing the intensity of competition.[1] Although antitrust and merger-analysis policymakers often draw inferences about social welfare and changes in social welfare from this measurement, it is not entirely clear why this approach is justified and whether the HHI is related to any measures of welfare.[2]

In this paper I show that in a broad range of oligopoly models, the HHI reflects the ratio of producer surplus to consumer surplus, with higher values of the HHI being associated with a lower share of consumer surplus in the total surplus. In other words, the HHI reflects the distribution of total surplus between firms' owners and consumers. Although this result pertains to the distribution of total welfare rather than its level, it is nonetheless interesting and policy relevant – especially in an era when income distribution issues have gained great prominence.[3]

More specifically, I begin by showing that in a Cournot model, where firms have (not necessarily identical) constant marginal costs, the ratio of producer surplus to consumer surplus in equilibrium is such that $\frac{PS^*}{CS^*} = \eta(Q^*) H$, where H is the value of the HHI in equilibrium and $\eta(Q^*)$ is the elasticity of consumer surplus with respect to the equilibrium output level, Q^*. This result generalizes to the case of common ownership with the MHHI (the modified HHI as defined by O'Brien and Salop [2000]) replacing the HHI. When marginal costs are increasing, $\eta(Q^*) H$

[1] The index can be viewed as a weighted sum of the market shares of firms, where the weights are equal to the market shares. The index was independently developed by Hirschman [1945], who used it as a measure of a country's foreign trade concentration, and by Herfindahl [1950], who used it to measure "gross changes" in the concentration of the U.S. steel industry. The index was then used by Stigler [1964] in his seminal paper on collusion, and became popular after William Baxter introduced it in the Department of Justice when he served as the Assistant Attorney General in charge of the Antitrust Division in the early 1980's, and especially after it was included in the 1982 horizontal merger guidelines. For a history of the HHI, see Calkins [1983].

[2] For instance, Farrell and Shapiro [1990b] write that "Oligopoly policy in the United States can be succinctly, if roughly, described as 'try not to allow measured concentration [typically in terms of the HHI] to become too high'." They argue that this policy is based on the presupposition that "since economic welfare is maximized under perfect competition, movements "away from" atomistic competition and "toward" a monopoly market structure will reduce welfare." Moreover, in Farrell and Shapiro [1990a] they argue that the use of the HHI as a diagnostic tool to screen horizontal mergers "reflects a view that anticompetitive harm is an increasing function of concentration...," although they show that "traditional merger analysis can be misleading in its use of the Herfindahl Index."

[3] See for instance, Baker and Salop [2015], Hovenkamp (2017), and Lyons [2017]. In particular, Baker and Salop [2015] write that "antitrust law and regulatory agencies could address inequality more broadly by treating the reduction of inequality as an explicit antitrust goal."

becomes the lower bound on $\frac{PS^*}{CS^*}$, and when marginal costs are decreasing, $\eta(Q^*)H$ becomes the upper bound on $\frac{PS^*}{CS^*}$. The relationship between the HHI and the distribution of total surplus can also be rewritten as $\frac{CS^*}{W^*} = \frac{1}{1+\eta(Q^*)H}$, where W^* is the sum of CS^* and PS^*. Stated in this way, the HHI, along with $\eta(Q^*)$, reflect $\frac{CS^*}{W^*}$, which is the share of consumer surplus in the total surplus.

Although it is tempting to conclude from the above equation that an increase in the HHI is associated with a decrease in $\frac{CS^*}{W^*}$, one should bear in mind that the HHI is endogenously determined, so exogenous shocks (demand or cost shocks, or changes in the number of firms following entry, exit, or mergers) which cause an increase in the HHI may also affect $\eta(Q^*)$ both directly (when the demand function changes) and indirectly by affecting the equilibrium output level, Q^*. Hence, an increase in the HHI is accompanied by a decrease of $\frac{CS^*}{W^*}$ only if the HHI and $\eta(Q^*)H$ move in the same direction.

One case where the HHI and $\eta(Q^*)H$ clearly move in the same direction is when $\eta(Q)$ is constant. It turns out that $\eta(Q)$ is constant if and only if demand is ρ-linear, in which case the inverse demand function is given by $p = A - bQ^\delta$, where $A \geq 0$ and $b\delta > 0$.[4] Then, $\eta(Q) = 1 + \delta$, where $1+\delta$ is also the inverse of the cost pass-through rate and is also equal to $2-\sigma(Q)$, where $\sigma(Q)$ is the curvature of the demand function. The family of ρ-linear demand functions is very broad and includes linear, constant elasticity, and log-linear inverse demand functions as special cases. Since $\eta(Q)$ is constant, an increase in the HHI (even when it is due to a change in δ) is necessarily accompanied by a decrease in $\frac{CS^*}{W^*}$. When demand is not ρ-linear, $\eta(Q)$ is no longer constant, so exogenous shocks which cause an increase in the HHI may also affect $\eta(Q^*)$. Nonetheless, I show that when firms are symmetric, $\eta(Q^*)H$ still moves in the same direction as H. I also provide sufficient conditions for $\eta(Q^*)H$ to move in the same direction as H when firms are not symmetric, and show examples that satisfy these sufficient conditions.

Turning to differentiated products, I show that in models with linear demand systems (e.g., Spence [1976] or Shubik and Levitan [1980]), and constant marginal costs, the HHI can be expressed as an increasing function of the ratio of producer to consumer surplus, both under quantity and price competition. Hence, as in the Cournot case, larger values of the HHI are still associated with distributions of total surplus which are more favorable to firms' owners and less favorable to

[4] A function f is called ρ-linear if f^ρ is linear. Anderson and Renault [2003] refer to inverse demand functions that satisfy $p = A - bQ^\delta$, as ρ-linear because the associated demand function, $Q = \left(\frac{A-p}{b}\right)^{\frac{1}{\delta}}$, is ρ-linear when $\rho = \delta$. The family of ρ-linear demand functions was first used by Bulow and Pfleiderer [1983]. The same functional form was also used by Genesove and Mullin [1998] to explore the methodology of using demand information to infer market conduct and unobserved cost components under static oligopoly behavior.

consumers.[5] Moreover, I show that whenever the HHI increases, $\frac{PS^*}{CS^*}$ necessarily increases as well, provided that the increase in the HHI is caused by an increase in the intercepts of the demand functions, a decrease in the marginal costs, or a decrease in the number of firms when demand is given by the Spence [1976] specification, or is given by the Shubik-Levitan [1980] specification and firms are symmetric.

The literature has already provided several interpretations of the HHI. By and large, these interpretations are based on the Cournot model.[6] Cowling and Waterson [1976] show that when each firm has a constant marginal cost, the HHI equals $\varepsilon\frac{PS^*}{R^*}$, where ε is the elasticity of demand and R^* is the equilibrium aggregate revenue.[7] Dansby and Willig [1979] consider a more general setting where firms do not necessarily have constant marginal costs and show that the HHI equals $(\varepsilon\phi^*)^2$, where ϕ^* is the "industry performance gradient," which reflects the rate of change in welfare as output is adjusted by moving within a fixed distance from the equilibrium point. Kwoka [1985] considers a similar setting and shows that the HHI equals εL^*, where $L^* \equiv \sum_{i=1}^{n} s_i^* L_i^*$ is a weighted average of the equilibrium Lerner indices of individual firms, with s_i^* being the equilibrium market share of firm i, and $L_i^* = \frac{p^* - c_i'}{p}$ its equilibrium Lerner index. The three papers then imply that if we hold ε constant, an increase in the HHI is associated with an increase in (i) the ratio of producer surplus to aggregate revenues, (ii) the industry performance gradient, and (iii) the average price-cost margin in the industry. Corchón [2008] also considers a Cournot model where firms have constant marginal costs and shows that when the demand function is ρ-linear, the percentage welfare loss (the gap between the levels of welfare in equilibrium and at the social optimum divided by the latter) is a decreasing function of the HHI when the market share of the largest firm is held fixed. Farrell and Shapiro [1990a,b] also consider a general Cournot model and show that an increase in the HHI may be associated with an increase in welfare even when output falls. The reason is that

[5] To the best of my knowledge, the only other paper that studies the relationship between the HHI and welfare when products are differentiated is Nocke and Schutz [2018]. They study the welfare implications of horizontal mergers in a model of multiproduct-firm price competition with nested CES or nested logit demands. Using a Taylor approximation around small market shares or around monopolistic competition conduct, they show that the difference in the outcomes of consumer surplus and aggregate surplus under oligopoly and monopolistic competition is proportional to the HHI.

[6] Exceptions are Stigler [1964] that studies collusion with secrete price cuts and shows that the HHI reflects the likelihood of sustaining collusion, and Nocke and Schutz [2018].

[7] Multiplying the first-order conditions for profit maximization of each firm i, $p(Q) + p'(Q)\, q_i - k_i = 0$, by q_i, summing the product over all firms and rearranging, yields $\sum_{i=1}^{n} (p(Q) - k_i)\, q_i = -p'(Q) \sum_{i=1}^{n} (q_i)^2$. Noting that $\sum_{i=1}^{n} (p(Q) - k_i)\, q_i = PS^*$, and dividing both sides of the equation by $R^* = p(Q)\, Q$, yields $\frac{PS^*}{R^*} \equiv \frac{H}{\varepsilon}$.

in a Cournot equilibrium, larger firms have lower marginal costs, so if production shifts from small to large firms (and hence the HHI increases), the cost savings from more efficient production may outweigh the negative effect of the reduction in total output.[8] While these results are useful, they do not tell us how the HHI is related to the distribution of the total surplus between firms' owners and consumers, which is the main focus of this paper.

My paper is also related to papers that study the distribution of total surplus in oligopoly models. Anderson and Renault [2003] derive lower and upper bounds on the ratios of deadweight loss and consumer surplus to producer surplus in the context of the Cournot model. Among other things, they show that when firms have symmetric costs, consumers get a smaller share in the total surplus when demand is more concave. Weyl and Fabinger [2013] use the principle of tax incidence to study a range of economic questions; among other things, they show that the ratio of consumer to producer surplus is, all else equal, smaller when large firms have the least competitive conduct.[9]

The rest of the paper is organized as follows. In Section 2, I present the main result, which I establish in the context of the Cournot model. In Section 3, I show that the main result generalizes to the case of common ownership; the only difference is that the MHHI replaces the HHI. In Section 4, I show that the main insight from Section 2 also generalizes to the case of differentiated products with linear demands. Concluding remarks are in Section 5. The Appendix contains technical proofs and derivations.

2 The HHI in the Cournot model

Before going into the model, it is worth considering the following simple example. Consider the textbook Cournot model with n firms, a linear inverse demand function $p = A - bQ$, and constant marginal costs. From the first-order conditions for a Nash equilibrium, the equilibrium margin of each firm i, given its marginal cost k_i, is $p^* - k_i = bq_i^*$. The producer surplus of each firm i is then $PS_i = b\left(q_i^*\right)^2$. Summing over all firms, aggregate producer surplus is $PS^* = b\sum_{i=1}^n \left(q_i^*\right)^2$. Meanwhile, consumer surplus in this linear model is simply $CS^* = \frac{b}{2}\left(Q^*\right)^2$.[10] Dividing the first equation by the second, yields $\frac{PS^*}{CS^*} = \frac{2\sum_{i=1}^n \left(q_i^*\right)^2}{Q^*} \equiv 2H$. This equation implies that in equilibrium,

[8] Nocke and Whinston [2020] show that in a general Cournot model, only the naïvely-computed change in the HHI due to a merger (twice the product of the per-merger market shares of the merging firms), but not the level of the HHI, is useful in screening mergers for whether their unilateral effects will harm consumers.

[9] Atkin and Donaldson [2015] extend results from an earlier version of Weyl and Fabinger [2013] and estimate the distribution of the gains from globalization; they show that intermediaries capture the majority of the gains.

[10] It is the area of a triangle whose width is Q^* and whose height is $A - (A - bQ^*) = bQ^*$.

the HHI is proportional to the ratio of producer surplus to consumer surplus, with higher values of the HHI being associated with distributions of total surplus that are more favorable to firms' owners and less favorable to consumers. The question however is whether this result is merely an artifact of the linear demand assumption or is more general. A related question is why H is multiplied by 2; that is, what does the 2 represent. In this paper, I answer these questions and push the analysis as far as I can.

2.1 The setting and preliminary analysis

Consider a Cournot model with n firms. The marginal cost of each firm i is constant and equals $k_i > 0$. The inverse demand function is $p(Q)$, where $Q = \sum_{i=1}^{n} q_i$ is aggregate output and q_i is firm i's output. I will assume that $p'(Q) < 0$ and $p'(Q) + p''(Q)Q \leq 0$; these assumptions are standard (see e.g., Farrell and Shapiro, 1990a] and ensure that the model is well behaved.[11] Each firm i chooses its output, q_i, to maximize its respective profit

$$\pi_i = p(Q)\, q_i - F_i - k_i q_i,$$

where F_i is fixed cost.

An interior Nash equilibrium is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:[12]

$$p(Q) + p'(Q)\, q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n. \tag{1}$$

The price-cost margin of each firm i in an interior Nash equilibrium is given by

$$p(Q^*) - k_i = -p'(Q^*)\, q_i^*,$$

where $Q^* = \sum_{i=1}^{n} q_i^*$. Using this expression, the equilibrium producer surplus of each firm i (its profit gross of fixed cost) can be written as

$$PS_i^* = (p(Q^*) - k_i)\, q_i^* = -p'(Q^*)\,(q_i^*)^2\,.$$

[11] In particular, the latter assumption implies that the marginal revenue of each firm is downward sloping, i.e., $2p'(Q) + p''(Q)\, q_i \leq 0$ for all q_i. If $p''(Q) \leq 0$, the result follows trivially; otherwise, $p'(Q) + p''(Q)\, q_i < p'(Q) + p''(Q)\, Q \leq 0$, which ensures that $2p'(Q) + p''(Q)\, Q < 0$. The last inequality ensures the existence and uniqueness of equilibrium (see Anderson and Renault [2003]).

[12] The equilibrium is interior if the price when the $n-1$ most efficient firms produce, exceeds k_i for the least efficient firm.

The equilibrium value of consumer surplus is

$$CS^* = \int_0^{Q^*} p(z)\,dz - p(Q^*)\,Q^*.$$

Noting that $(CS^*)' = -p'(Q^*)\,Q^*$, (aggregate) producer surplus is given by

$$PS^* \equiv \sum_{i=1}^{n} PS_i^* = \frac{(CS^*)' \sum_{i=1}^{n} (q_i^*)^2}{Q^*}. \tag{2}$$

To establish the main result, I first define $\eta(Q) \equiv \frac{Q(CS)'}{CS}$, which is the elasticity of consumer surplus with respect to output. As we shall see, $\eta(Q)$, plays an important role in what follows. In the next lemma, which may be of independent value, I establish two useful properties of $\eta(Q)$. The proof of the lemma, as well as the proofs of other results, appears in the Appendix.

Lemma 1: *The elasticity of consumer surplus,* $\eta(Q) \equiv \frac{Q(CS)'}{CS}$*, has the following properties:*

(i) $\eta(Q) \geq 1$ *if* $p'(Q) + p''(Q)\,Q \leq 0$ *and* $\eta(Q) \in (0,1)$ *otherwise.*

(ii) $\eta'(Q)$ *has the same sign as* $2 - \sigma(Q) - \eta(Q)$*, where* $\sigma(Q) \equiv -\frac{p''(Q)Q}{p'(Q)}$ *is the curvature of the demand function (or the elasticity of the slope of the inverse demand function).*

The first part of Lemma 1 shows that when the model is well-behaved in the sense that $p'(Q) + p''(Q)\,Q \leq 0$, a 1% increase in output implies at least 1% increase in consumer surplus. Intuitively, note that $CS'' = -(p'(Q) + p''(Q)\,Q)$; hence, $p'(Q) + p''(Q)\,Q \leq 0$ is equivalent to $CS'' \geq 0$, i.e., consumer surplus is convex in output. But then if CS is a convex function of Q, it is obvious that its elasticity, $\eta(Q)$, which reflects the ratio of the marginal to the average value, is above 1. In other words, Part (i) of Lemma 1 simply says that $\eta(Q) \geq 1$ if CS is convex in Q and conversely if it is concave. In the latter case, $\eta(Q)$ is below 1 but is still positive because CS is the integral under the inverse demand function from 0 to Q, so an increase in Q must increase CS. Part (ii) of Lemma 1 shows that an increase in Q makes CS more elastic if $\eta(Q)$ is not too large (i.e., below $2 - \sigma(Q)$) and less elastic otherwise.[13]

[13] Note that $p'(Q) + p''(Q)\,Q \leq 0$ is also equivalent to $\sigma(Q) \equiv -\frac{p''(Q)Q}{p'(Q)} < 1$. Consequently, Lemma 1 implies that whenever $p'(Q) + p''(Q)\,Q \leq 0$, $\eta'(Q) \geq 0$ for $\eta(Q) \in [1, 2-\sigma(Q)]$ and $\eta'(Q) < 0$ for $\eta(Q) > 2 - \sigma(Q)$, and whenever $p'(Q) + p''(Q)\,Q \geq 0$, $\eta'(Q) \geq 0$ for $\eta(Q) \in [0, 2-\sigma(Q)]$ and $\eta'(Q) < 0$ for $\eta(Q) \in [2-\sigma(Q), 1]$.

2.2 The level of the HHI and the distribution of surplus

Given a Nash equilibrium, the market share of firm i is simply $\frac{q_i^*}{Q^*}$. Hence, the HHI is given by

(3) $$H = \sum_{i=1}^{n} \left(\frac{q_i^*}{Q^*} \right)^2 = \frac{\sum_{i=1}^{n} \left(q_i^*\right)^2}{\left(Q^*\right)^2}.$$

Substituting for $\sum_{i=1}^{n} \left(q_i^*\right)^2$ from (2) into (3), and using the definition of $\eta\left(Q^*\right)$, yields the following result:

Proposition 1: *In an n-firm Cournot model, where firms have (possibly different) constant marginal costs,*

(4) $$\frac{PS^*}{CS^*} = \eta\left(Q^*\right) H.$$

Proposition 1 implies that the HHI, along with the elasticity of consumer surplus, $\eta\left(Q^*\right)$, reflect the ratio of producer surplus to consumer surplus and therefore the distribution of surplus between firms' owners and consumers. This result shows that the linear demand example considered at the top of this section generalizes to the case of general demand functions. Moreover, when demand is linear, $\eta\left(Q^*\right) = 2$ (a 1% increase in output leads to a 2% increase in consumer surplus), hence in the linear demand example, H is multiplied by 2.

Another way to think about Proposition 1 is to denote the total surplus by $W^* = PS^* + CS^*$; then, equation (4) can be rewritten as

(5) $$\frac{CS^*}{W^*} = \frac{1}{1 + \eta\left(Q^*\right) H}.$$

Expressed in this way, the HHI reflects the share of consumer surplus in the total surplus: consumers obtain a larger share in the total surplus when $\eta\left(Q^*\right)$ is lower, i.e., when consumer surplus becomes more inelastic with respect to output. Lemma 1 implies that under the common assumption that $p'\left(Q\right) + p''\left(Q\right) Q \leq 0$ (i.e., CS is convex in output), $\eta\left(Q^*\right) \geq 1$; hence (5) implies that the share of consumer surplus in the total surplus is bounded from above by $\frac{1}{1+H}$. For instance, under monopoly, where $H = 1$, consumers obtain no more than 50% of the total surplus.

In Figure 1, I illustrate equation (5) for two values of $\eta\left(Q^*\right)$: $\eta\left(Q^*\right) = 1$ and $\eta\left(Q^*\right) = 2$.[14] Below, I show that $\eta\left(Q\right) = 1$ for all Q when demand is log-linear and $\eta\left(Q\right) = 2$ for all Q when demand is linear.

[14] Note that in the standard usage of the HHI in merger analysis, market shares are measured in terms of percentage points, so the HHI varies from 0 to 10,000.

'Place Figure 1 about here.'

To interpret Figure 1, note that the HHI plays an important role in horizontal merger analysis (see e.g., DOJ and FTC, 2006].[15] The 2010 Horizontal Merger Guidelines of the DOJ and the FTC define markets as unconcentrated if the HHI is below 1,500 and state that "Mergers resulting in unconcentrated markets are unlikely to have adverse competitive effects and ordinarily require no further analysis."[16] The Guidelines also define markets as highly concentrated if the HHI is above 2,500 and state that "Mergers resulting in highly concentrated markets that involve an increase in the HHI of more than 200 points will be presumed to be likely to enhance market power." Equation (5) shows that if $\eta(Q^*) = 1$, an HHI below 1,500 implies that consumers obtain more than $1/(1 + 0.15) = 87\%$ of the total surplus, while an HHI above 2,500 implies that consumers obtain at most $1/(1 + 2 \times 0.25) = 80\%$ of the total surplus. When $\eta(Q^*) \geq 1$, these shares are lower; for instance, if $\eta(Q^*) = 2$, an HHI below 1,500 implies that consumers obtain more than $1/(1 + 2 \times 0.15) = 77\%$ of the total surplus, while an HHI above 2,500 implies that consumers obtain at most $1/(1 + 2 \times 0.25) = 67\%$ of the total surplus. Viewed in this way, one can interpret the Guidelines as implying that whenever, say, $\eta(Q^*) = 2$, horizontal mergers raise competitive concerns when consumer surplus is less than 67% of the total surplus, but not when it is more than 77% of the total surplus.

Proposition 1 is obtained under the assumption that all firms have constant marginal costs. One may wonder what happens when this assumption is relaxed. Then, an interior Nash equilibrium is defined by the following system of first-order conditions:

$$p(Q) + p'(Q)\, q_i - c_i'(q_i) = 0, \qquad i = 1, 2, , \ldots, n,$$

where $c_i'(q_i)$ is the marginal cost of firm i. When some, or even all, firms have increasing marginal costs (i.e., $c_i''(q_i) \geq 0$ for all i), $c_i'(q_i) \geq \frac{c_i(q_i)}{q_i}$, so the first-order conditions imply that the producer surplus of each firm i is such that

$$PS_i^* = \left(p(Q^*) - \frac{c_i(q_i^*)}{q_i^*} \right) q_i^* \geq \left(p(Q^*) - c_i'(q_i^*) \right) q_i^* = -p'(Q^*)(q_i^*)^2 .$$

[15] In particular, they write on p. 20 that although market shares and concentration alone are not good predictors of enforcement challenges, they "nevertheless are important in the Agencies' evaluation of the likely competitive effects of a merger."

[16] Indeed, according to the DOJ and FTC [2006], horizontal merger investigations are almost always closed when concentration levels are below these thresholds.

The inequality is reversed when $c_i''(q_i) \leq 0$ for all i. Repeating the same steps as above, yields the following result:

Corollary 1: *In an n-firm Cournot model, where firms have (possibly different) non-decreasing marginal cost functions,*

$$\frac{PS^*}{CS^*} \geq \eta(Q^*) H,$$

or equivalently,

$$\frac{CS^*}{W^*} \leq \frac{1}{1+\eta(Q^*) H}.$$

These inequalities are reversed when firms have non-increasing marginal cost functions.

Corollary 1 implies that when firms have non-decreasing marginal costs, $\eta(Q^*) H$ becomes the lower bound on the ratio of producer to consumer surplus, or equivalently, $\frac{1}{1+\eta(Q^*)H}$ becomes an upper bound on the share of consumer surplus in the total surplus.[17] When firms have non-increasing marginal costs the reverse holds: now $\eta(Q^*) H$ becomes the upper bound on the ratio of producer to consumer surplus, or equivalently, $\frac{1}{1+\eta(Q^*)H}$ becomes an upper bound on the share of consumer surplus in the total surplus.

2.3 Changes in the level of the HHI and the distribution of surplus

So far the analysis was static and only examined the relationship between a given level of the HHI and the distribution of surplus. The question now is whether it is also true that whenever the HHI increases, $\frac{PS^*}{CS^*}$ necessarily increases as well, and hence the share of consumer surplus in the total surplus falls. To address this question, it is important to notice that since the HHI is endogenously determined, changes in the HHI are caused by demand or cost shocks, or by a change in the number of firms following entry, exit, or mergers. These factors, however, may also affect $\eta(Q^*)$ both directly (when the demand function changes) and indirectly (through the effect on Q^*). Therefore, equation (4) implies that $\frac{PS^*}{CS^*}$ is positively related to H only when $\eta(Q^*) H$ moves in the same direction as H.

I now study two cases: one in which $\eta(Q)$ is constant and one in which it is not.

2.3.1 $\eta(Q)$ is constant

The next lemma provides a necessary and sufficient condition for $\eta(Q)$ to be constant. When this condition holds, $\eta(Q^*) H$ moves in the same direction as H, implying that when H increases, $\frac{PS^*}{CS^*}$

[17] I thank Geert van Moer for pointing out this possibility to me.

increases as well.

Lemma 2: *An inverse demand function exhibits a constant elasticity of consumer surplus if and only if it can be expressed as:*

$$p = A - bQ^{\delta}, \tag{6}$$

where $A \geq 0$ and $b\delta > 0$. The constant elasticity of consumer surplus is then given by,

$$\eta(Q) = 1 + \delta,$$

where $1+\delta$ is equal to $\frac{1}{p'(k)}$ which is the the inverse of the cost pass-through rate and is also equal to $2 - \sigma(Q)$, where $\sigma(Q) \equiv -\frac{p''(Q)Q}{p'(Q)}$ is the curvature of the demand function.

As mentioned in the Introduction, Anderson and Renault [2003] refer to demand functions that satisfy (6) as ρ-linear. The family of ρ-linear demand functions, which exhibit a constant $\eta(Q)$, is quite broad. It includes as special cases linear demand functions when $A, b > 0$ and $\delta = 1$; log-linear inverse demand functions when $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$, $b = \frac{\widetilde{b}}{\delta}$, and $\delta \to 0$, in which case the inverse demand function becomes $p = \widetilde{A} - \widetilde{b}\ln(Q)$;[18] and iso-elastic demand functions when $A = 0$, and $b, \delta < 0$, in which case the inverse demand function becomes $p = -bQ^{\delta}$. In the latter case, $-\frac{1}{\delta}$ represents the (constant) elasticity of demand. To ensure that the monopoly price is bounded from above, it must be that $\delta \in (-1, 0)$ (i.e., the elasticity of demand exceeds 1); as δ grows from (just above) -1 to (just below) 0, the elasticity of demand, $-\frac{1}{\delta}$, grows from (just above) 1 to ∞.

Lemma 2 shows that when demand is ρ-linear, there is a simple relationship between $\eta(Q)$, the cost pass-through rate, and the curvature of the demand function.[19] In the Appendix, I show that when demand is not ρ-linear, the relationship between $\eta(Q)$, the cost pass-through rate, and the curvature of the demand function, is no longer simple.

Lemma 2 also implies that $\eta(Q) = 1 + \delta$, where $\delta = 1$ when demand is linear, $\delta \to 0$ when the inverse demand is log-linear, and $\delta \in (-1, 0)$ when demand is iso-elastic.[20] Together with Proposition 1, this implies the next Corollary:

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[18]To see this, note that since $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$ and $b = \frac{\widetilde{b}}{\delta}$, the inverse demand function is $p = \widetilde{A} + \frac{\widetilde{b}(1-Q^{\delta})}{\delta}$. Using L'Hôpital's rule, $\lim_{\delta \to 0}\left(\widetilde{A} + \frac{\widetilde{b}(1-Q^{\delta})}{\delta}\right) = \lim_{\delta \to 0}\left(\widetilde{A} - \frac{\widetilde{b}Q^{\delta}\ln(Q)}{1}\right) = \widetilde{A} - \widetilde{b}\ln(Q)$. The associated demand function is exponential and given by $Q = e^{\frac{\widetilde{A}-p}{\widetilde{b}}}$.

[19]Note that since $\eta(Q) = 2 - \sigma(Q)$, Part (ii) of Lemma 1 implies that $\eta'(Q) = 0$, which is indeed the case as $\eta(Q)$ is constant.

[20]Note that when demand is iso-elastic, $p'(Q) + p''(Q)Q = -b\delta^2 Q^{\delta-1} > 0$, contrary to the assumption I maintain throughout most of the paper. This assumption however is only sufficient, but not necessary, for the existence and

Corollary 2: *In an n-firm Cournot model where firms have (possibly different) constant marginal costs and the inverse demand function is given by (6),*

$$\frac{PS^*}{CS^*} = (1+\delta) H, \tag{7}$$

or equivalently,

$$\frac{CS^*}{W^*} = \frac{1}{1+(1+\delta) H}, \tag{8}$$

where $\delta \in (-1, 0)$ *if demand is is-elastic,* $\delta = 0$ *if the inverse demand function is log-linear, and* $\delta = 1$ *if demand is linear.*

Corollary 2 implies that when demand is ρ-linear, H is proportional to $\frac{PS^*}{CS^*}$, with $1+\delta$ being the factor of proportionality. In particular, this implies in turn that every 100 points increase in the HHI is accompanied by an increase in producer surplus relative to consumer surplus by $\frac{1+\delta}{10}$.

An interesting implication of Corollary 2 is that for a given value of the HHI, the share of consumer surplus in the total surplus is highest in the case of an iso-elastic inverse demand $(1+\delta < 1)$, followed by a log-linear inverse demand $(1+\delta = 1)$, and is lowest when demand is linear $(1+\delta = 2)$.

Another interesting implication of Corollary 2 is that when the inverse demand function is linear or log-linear, knowing H is sufficient to determine how the total surplus is distributed between firms' owners and consumers. In either case, there is no need for any other data to determine the relationship between the HHI and $\frac{PS^*}{CS^*}$ and $\frac{CS^*}{W^*}$. In particular, equation (8) implies that consumer surplus is $1/(1+H)$ of the total surplus when demand is log-linear and $1/(1+2H)$ when demand is linear (see Figure 1 which shows $\frac{CS^*}{W^*}$ as a function of H).

When the inverse demand function is iso-elastic, the relationship between the HHI and $\frac{PS^*}{CS^*}$ also depends on the parameter δ, which in this case is the negative of the inverse elasticity of demand. One may wonder if whenever the HHI increases due to a change in δ, $\frac{PS^*}{CS^*}$ increases as well. To answer this question I first establish the next result:

Lemma 3: *When the inverse demand function is iso-elastic and given by* $p = -bQ^{\delta}$, *where* $b, \delta < 0$, *an increase in* δ *causes an increase in the HHI.*

uniqueness of equilibrium. The profit of each firm i in this case is still concave because $-1 < \delta < 0$ implies that $\pi_i'' = -\delta b Q^{\delta-2}(2Q + (\delta - 1) q_i) < 0$. Hence, a Cournot equilibrium exists. Also note that as $p'(Q) + p''(Q) Q > 0$, $\eta(Q) \equiv \frac{Q(CS)'}{CS} < 1$ by Lemma 1.

Lemma 3 implies that an increase in δ causes an increase in the HHI, and therefore in $(1+\delta)H$. By (8), $\frac{CS^*}{W^*}$ falls, implying that as demand becomes more elastics and δ grows from -1 to 0, the HHI increases, while the share of consumer surplus in the total surplus falls.

2.3.2 $\eta(Q)$ is not constant

Next, I consider cases where the demand function is not ρ-linear, in which case $\eta(Q)$ is no longer constant. Examples for demand functions that exhibit $\eta'(Q) > 0$ include the Logit demand function, $Q = \frac{e^{\frac{A-p}{b}}}{1+e^{\frac{A-p}{b}}}$, with the associated inverse demand function $p = A - b\ln\left(\frac{Q}{1-Q}\right)$, and the Logarithmic demand function, $Q = \ln\left(\frac{A-p}{b}\right)$, with the associated inverse demand function $p = A - e^{bQ}$. An example for an inverse demand function that exhibits $\eta'(Q) < 0$ is $p = A - \frac{Q}{1+Q}$. In all three cases, $\eta(Q)$ depends only on Q, but not on any other parameter that may affect the HHI.

The following result shows that when firms are symmetric and have the same constant marginal cost k, $\eta(Q^*)H$ still moves in the same direction as H, regardless of whether $\eta(Q)$ is constant. As a result, (4) implies that when the HHI increases, $\frac{PS^*}{CS^*}$ increases as well, and (5) implies that the share of consumer surplus in the total surplus falls. That is, under symmetry, the HHI is still informative about the distribution of the total surplus between firms' owners and consumers, even when $\eta(Q)$ is not constant.

Proposition 2: *In an n-firm Cournot model, where firms are symmetric and have the same constant marginal cost, k, an increase in H is accompanied by an increase in $\eta(Q^*)H$, and therefore by a decrease in the share of consumer surplus in the total surplus, even when $\eta(Q)$ is not constant.*

Intuitively, when firms are symmetric, $H = \frac{1}{n}$. Hence, the HHI can increase only if the number of firms, n, falls. The change in n does not affect $\eta(Q^*)$ directly because the demand function does not change but it does affect $\eta(Q^*)$ indirectly through its effect on Q^*. It is not hard to show that a decrease in n causes an increase in Q^*. Therefore, if $\eta'(Q^*) > 0$, it is clear that $\eta(Q^*)$, and hence $\eta(Q^*)H$, also increase. If $\eta'(Q^*) < 0$, there are two conflicting effects: H increases but $\eta(Q^*)$ decreases. Proposition 2 shows however that the first effect dominates, so $\eta(Q^*)H$ increases and hence $\frac{PS^*}{CS^*}$.

Moving beyond the symmetric case, I now provide sufficient conditions for $\eta(Q^*)H$ to move in the same direction as H when $\eta(Q)$ is not constant and firms are possibly asymmetric. To establish these conditions, consider a change in some parameter x, which can be a demand or a cost parameter, and suppose that the change in x causes an increase in H. The resulting effect on $\eta(Q^*) H$ is then given by

$$\frac{\partial}{\partial x}\left(\eta\left(Q^*\right) H\right)=\eta\left(Q^*\right) \frac{\partial H}{\partial x}+H\left[\frac{\partial \eta\left(Q^*\right)}{\partial x}+\eta'\left(Q^*\right) \frac{\partial Q^*}{\partial x}\right].$$

The first term of the derivative is the effect of x on H, which is positive by assumption. The bracketed term is the effect of x on $\eta(Q^*)$. It consists of a direct effect when Q^* is held fixed, $\frac{\partial \eta(Q^*)}{\partial x}$ (this effect can be present only if x is a demand parameter, otherwise $\frac{\partial \eta(Q^*)}{\partial x}=0$), and an indirect effect, $\eta'(Q^*)\frac{\partial Q^*}{\partial x}$, due to the effect of x on Q^*. Noting from (4) that $\frac{PS^*}{CS^*}$ increases if and only if $\frac{\partial}{\partial x}(\eta(Q^*) H) \geq 0$, the next result follows.

Lemma 4: *In an n-firm Cournot model, sufficient conditions for a change in some exogenous parameter x that causes an increase the HHI to also cause an increase in $\frac{PS^*}{CS^*}$ are (i) $\frac{\partial \eta(Q^*)}{\partial x} \geq 0$, and (ii) $\eta'(Q^*)$ and $\frac{\partial Q^*}{\partial x}$ have the same sign.*

The sufficient conditions in Lemma 4 require that a demand or a cost shock that causes an increase in the HHI does not lower $\eta(Q^*)$ either directly or indirectly through its effect on Q^*. When these conditions hold, the shock causes an increase in $\eta(Q^*) H$ and then by (4) also in $\frac{PS^*}{CS^*}$.[21] Building on Lemma 4, I now consider three types of changes in the marginal costs of firms which cause an increase in the HHI and examine the resulting effect on $\frac{PS^*}{CS^*}$. As the changes involve marginal costs, they affect $\eta(Q^*)$ only indirectly through their effect on Q^*.

I begin with a mean-preserving spread of marginal costs, i.e., an increase in the variance of marginal costs across firms in the industry, while keeping the average marginal cost in the industry constant.

Proposition 3: *Consider an n-firm Cournot model, where all firms have (possibly different) constant marginal costs. A mean-preserving spread of marginal costs causes an increase in H and has no effect on $\eta(Q^*)$ and is therefore accompanied by a decrease in the share of consumer surplus in the total surplus.*

The reason for this result is that when firms have constant marginal costs, the aggregate output Q^* depends only on the sum of the marginal costs, but not on their composition. Hence, Q^* is unaffected by a mean-preserving spread of marginal costs and since a cost change does not

[21] A similar argument can be made when the increase in the HHI is caused by a change in the number of firms, although when firms are not symmetric, a change in their number also affects the industry's cost structure.

affect $\eta(Q^*)$ directly, $\eta(Q^*)$ is not affected as well. The HHI however increases because the mean-preserving spread increases the variance of marginal costs and hence the variance of market shares. By Lemma 4 then, $\frac{PS^*}{CS^*}$ increases as well. Another way to see the latter result is to note that since a mean-preserving spread of marginal costs does not affect Q^*, it does not affect CS^*, nor the industry revenue. The mean-preserving spread however lowers the marginal costs of low cost firms (who have high market shares) and raises the marginal costs of high cost firms (who have low market shares), so production becomes more efficient; as a result, PS^* increases and hence $\frac{PS^*}{CS^*}$.[22]

Things are more complex when the marginal costs change, but their mean is not preserved, because then Q^*, and hence $\eta(Q^*)$, are also affected. In the next proposition, I consider two types of changes in marginal costs which cause a increase in the HHI and provide sufficient conditions on the sign of $\eta'(Q^*)$ that ensure that the changes lead to an increase in both the HHI and $\frac{PS^*}{CS^*}$.

Proposition 4: *Consider an n-firm Cournot model, where all firms have (possibly different) constant marginal costs. Then,*

(i) *An increase in the marginal costs of all firms by a constant (which is equivalent to a decrease in the inverse demand function) causes an increase in H and a decrease in Q^*; a sufficient condition for a decrease in the share of consumer surplus in the total surplus is $\eta'(Q^*) < 0$.*

(ii) *A decrease in the marginal cost of the lowest-cost firm causes an increase in H and in Q^*; a sufficient condition for a decrease in the share of consumer surplus in the total surplus is $\eta'(Q^*) > 0$.*

Both changes in marginal costs considered in Proposition 4 cause an increase in the HHI, but have opposite effects on Q^*. An increase in the marginal cost of all firms clearly causes a decrease in Q^* and hence in CS^*.[23] By Lemma 4, the share of consumer surplus in the total surplus also

[22] Proposition 3 is consistent with Février and Linnemer [2004]. Among other things, they show that in an n-firm Cournot model where firms have constant marginal costs, a mean-preserving spread of marginal costs (this case arises in their model when the covariance between marginal costs and market shares is negative) does not affect consumer surplus (Proposition 1 in their paper) but raises aggregate profits (Corollary 4.1 in their paper).

[23] This result is consistent with Proposition 1 in Kimmel [1992] which shows that in an n-firm Cournot model, aggregate output falls when the constant marginal costs of all firms increase by the same amount. Février and Linnemer [2004] show that an increase in the constant marginal costs of all firms by a constant, causes an increase in aggregate profits (despite the fact that firms have higher costs) if and only if $\sigma^* H > \frac{2(1+\sigma^*)}{n}$ (Corollary 4.2 in their paper). When this condition holds, $\frac{PS^*}{CS^*}$ increases, because PS^* increases, while Q^* and hence CS^* fall.

falls in this case if $\eta'(Q^*) < 0$. As mentioned earlier, an example for a demand function for which $\eta'(Q) < 0$ for all Q, is $p = A - \frac{Q}{1+Q}$; to ensure that $p'(Q) + p''(Q)Q \leq 0$ (consumer surplus is convex in output) as assumed in Part (i) of Proposition 4, I assume that $A = 1$ and $\frac{2n-1}{4n} \leq \widehat{k} \leq 1$, where $\widehat{k} \equiv \frac{1}{n}\sum_{i=1}^{n} k_i$ is the average marginal cost in the industry (see the Appendix for details).

By contrast, a decrease in the marginal cost of the lowest-cost firm causes an increase in this firm's output, and due to strategic substitutability, it causes a decrease in the output of all other firms. Due to the assumptions that $p'(Q) < 0$ and $p'(Q) + p''(Q)Q \leq 0$, the first effect dominates the second, so Q^* and hence CS^* increase. Although consumers are better off in absolute terms, the share of consumer surplus in the total surplus may still decrease. In other words, firms' owners may benefit from the cost reduction of the lowest-cost firm more than consumers. Recalling that cost changes do not affect $\eta(Q^*)$ directly, Lemma 4 implies that sufficient conditions for $\frac{PS^*}{CS^*}$ to increase (implying that the share of consumer surplus in the total surplus falls) when Q^* increases is $\eta'(Q^*) > 0$. Examples for demand functions for which $\eta'(Q) > 0$ for all Q, are the Logit or the logarithmic demand functions.

3 The MHHI in the Cournot model with common ownership

In recent years there is a growing concern about the potential anticompetitive effects of common ownership, i.e., the fact that a few large institutional investors such as Berkshire Hathaway, BlackRock, Vanguard, and State Street are the major shareholders of competing firms such as airlines or banks.[24] A common measure of concentration in the presence of common ownership is the MHHI due to O'Brien and Salop [2000]. In this section I show that Proposition 1 above generalizes to the case of common ownership with the MHHI replacing the HHI.

Under common ownership, there are m shareholders who own shares in the various firms. Let α_{jk} be the stake that shareholder k owns in firm j. The wealth of shareholder k is equal to his combined stake in the n firms, $w_k = \sum_{i=j}^{n} \alpha_{jk}\pi_j$. The objective of firm i's manager is to maximize a weighted average of the wealth of the firm's shareholders, where the weight assigned to shareholder k's wealth is λ_{ik} (this weight may reflect the degree of control that shareholder k has over firm i):

$$O_i = \sum_{k=1}^{m} \lambda_{ik} w_k = \sum_{k=1}^{m} \lambda_{ik} \sum_{j=1}^{n} \alpha_{jk}\pi_j = \sum_{j=1}^{n}\sum_{k=1}^{m} \lambda_{ik}\alpha_{jk}\pi_j.$$

[24]See Backus, Conlon, and Sinkinson [2019] for a recent paper that documents the rise of common ownership in the U.S. economy and the increased incentive of firms to internalize the negative competitive externality that they exert on rivals.

It is useful to rewrite the objective function of firm i's manager as

$$O_i = \sum_{j=1}^{n} \pi_j \left(\sum_{k=1}^{m} \lambda_{ik} \alpha_{jk} \right).$$

An interior Nash equilibrium when each manager i chooses his firm's output q_i to maximize his objective function O_i is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$\left(p(Q) + p'(Q) q_i - k_i\right) \left(\sum_{k=1}^{m} \lambda_{ik} \alpha_{ik} \right) + \sum_{j \neq i}^{n} p'(Q) q_j \left(\sum_{k=1}^{m} \lambda_{ik} \alpha_{jk} \right) = 0, \qquad i = 1, 2, , \ldots, n.$$

The price-cost margin of each firm i in an interior Nash equilibrium is given by

$$p(Q^*) - k_i = -p'(Q^*) \sum_{j=1}^{n} q_j^* \underbrace{\left(\frac{\sum_{k=1}^{m} \lambda_{ik} \alpha_{jk}}{\sum_{k=1}^{m} \lambda_{ik} \alpha_{ik}} \right)}_{\kappa_{ij}},$$

where κ_{ij} is the weight that firm i's manager assigns to the profit of firm j relative to firm i (note that $\kappa_{ij} = 1$).[25] Using the last expression, the equilibrium producer surplus of each firm i can be written as

$$PS_i^* = \left(p(Q^*) - k_i\right) q_i^* = -p'(Q^*) \sum_{j=1}^{n} \kappa_{ij} q_j^* q_i^*.$$

Recalling that $(CS^*)' = -p'(Q^*) Q^*$, aggregate producer surplus is given by

$$PS^* \equiv \sum_{i=1}^{n} PS_i^* = \frac{(CS^*)' \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} q_j^* q_i^*}{Q^*}.$$

Dividing and multiplying the right-hand side by Q^*, noting that $\frac{q_i^*}{Q^*} = s_i^*$ and $\frac{q_j^*}{Q^*} = s_j^*$ are the market shares of firms j and i, and recalling that $\eta(Q^*) \equiv \frac{Q^* (CS^*)'}{CS^*}$ is the elasticity of consumer surplus with respect to output, yields

$$PS^* = \eta(Q^*) CS^* \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*,$$

where $\sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*$ is the MHHI as defined by O'Brien and Salop [2000].[26] Hence,

[25] To see this, note that the objective function of firm i's manager can be rewritten as

$$O_i = \pi_i \left(\sum_{k=1}^{m} \lambda_k \alpha_{ik} \right) + \sum_{j \neq i}^{n} \pi_j \left(\sum_{k=1}^{m} \lambda_k \alpha_{jk} \right) = \left(\sum_{k=1}^{m} \lambda_k \alpha_{ik} \right) \left[\pi_i + \sum_{j \neq i}^{n} \pi_j \underbrace{\left(\frac{\sum_{k=1}^{m} \lambda_k \alpha_{jk}}{\sum_{k=1}^{m} \lambda_k \alpha_{ik}} \right)}_{\kappa_{ij}} \right].$$

Hence, κ_{ij} is the weight that firm i's manager assigns to firm j's profit and $\kappa_{ii} = 1$.

[26] $\sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*$ can be written as $\sum_{i=1}^{n} (s_i^*)^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \kappa_{ij} s_j^* s_i^*$, which is similar to the expression in equation (1) in O'Brien and Salop [2000].

Proposition 5: *In an n-firm Cournot model, where firms have (possibly different) constant marginal costs, and the manager of each firm maximizes a weighted average of the wealth of the firm's shareholders (who also hold shares in rival firms),*

$$\frac{PS^*}{CS^*} = \eta(Q^*)\, MHHI.$$

Proposition 5 shows that Proposition 1 generalizes to the case of common ownership, with the MHHI replacing the HHI. The implication is that under common ownership, the value of the MHHI reflects the distribution of the total surplus between firms' owners and consumers, with the share of consumers being inversely related to $\eta(Q^*)$.

4 The HHI in differentiated products models

I now show that the key insight from the Cournot model carries over to models of differentiated products.[27] To this end, suppose that the n firms produce differentiated products and each firm i is facing an inverse demand function $p_i(q_1, \ldots, q_n)$ and has a cost function $c_i(q_i) = F_i + k_i q_i$, where $k_i < p_i(q_1, \ldots, q_n)$ when $q_i = 0$. The profit of each firm i is given by

$$\pi_i = (p_i(q_1, \ldots, q_n) - k_i)\, q_i - F_i.$$

An interior Nash equilibrium when firms compete by setting quantities is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$p_i(q_1, \ldots, q_n) + \frac{\partial p_i(q_1, \ldots, q_n)}{\partial q_i} q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n.$$

Since in a Nash equilibrium, $p_i^* - k_i = -\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} q_i^*$, where $p_i^* = p_i^*(q_1^*, \ldots, q_n^*)$, the equilibrium producer surplus is

$$PS^* = \sum_{i=1}^{n} (p_i^* - k_i)\, q_i^* = -\sum_{i=1}^{n} \frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} (q_i^*)^2. \tag{9}$$

Note that PS^* is a linear function of $\sum_{i=1}^{n} (q^*)^2$, which is the numerator of the HHI, only if $\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i}$ is identical across firms. This holds only if $p_i(q_1, \ldots, q_n) = A_i - \delta q_i - G(Q)$, where $G(Q)$ is such that $\frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i} = -\delta - G'(Q) < 0$ (to ensure that demand is downward sloping).

[27] As mentioned in the Introduction, the only other paper that I am aware of which studies the relationship between the HHI and welfare when products are differentiated is Nocke and Schutz [2018].

In what follows, I will assume that $\delta = \beta - \gamma$ and $G(Q) = \gamma Q$; hence, the inverse demand system is linear and symmetric and given by

$$p_i(q_1, \ldots, q_n) = A_i - \beta q_i - \gamma \sum_{j \neq i}^{n} q_j, \qquad i = 1, 2, , \ldots, n, \tag{10}$$

where $A_1, \ldots, A_n$ and β, are positive parameters, and $0 < \gamma < \beta$ is a measure of the degree of product differentiation, with lower values of γ representing a larger degree of differentiation.[28] This inverse demand system corresponds to the Spence [1976] specification (Dixit [1979] and Singh and Vives [1984] also use this specification), but if $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, where $\tau > 0$, it corresponds to the Shubik-Levitan [1980] specification.[29] In the latter case, the parameter τ reflects the degree of product differentiation, with lower values of τ representing a larger degree of differentiation.[30]

In the Appendix, I show that given (10), consumer surplus is given by

$$CS = \frac{(\beta - \gamma) \sum_{i=1}^{n} (q_i)^2 + \gamma (Q)^2}{2}. \tag{11}$$

In the Shubik-Levitan specification, consumer surplus is given by the same expression except that now $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$.

4.1 Quantity competition

Noting from (10) that $\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} = -\beta$ for all i, (9) implies that $\sum_{i=1}^{n} (q_i^*)^2 = \frac{PS^*}{\beta}$. Evaluating (11) at the equilibrium values, substituting for Q^* from (11) into (3), and rearranging, yields the following result:

Proposition 6: *In an n-firm differentiated products oligopoly with quantity competition, where firms have (possibly different) constant marginal costs and face a linear inverse demand system (10),*

$$\frac{PS^*}{CS^*} = \frac{2H}{\widehat{\gamma} + (1 - \widehat{\gamma}) H}, \tag{12}$$

where $\widehat{\gamma} \equiv \frac{\gamma}{\beta}$*. In the Shubik-Levitan case, the equation is similar, except that* $\widehat{\gamma} = \frac{\tau}{n+\tau}$*.*

[28] Obviously, γ cannot be too low relative to β, otherwise the products are not in the same market in which case HHI becomes meaningless.

[29] See Choné and Linnemer [2020] for an overview of linear demand systems for differentiated products and a discussion of their exact origin.

[30] A third notable example for a differentiated products oligopoly model with linear demands is the Vickery-Salop circular city model (Vickery [1964] and Salop [1979]).

Proposition 6 implies that, similarly to the Cournot case, $\frac{PS^*}{CS^*}$ is positively related to the HHI, implying that higher values of the HHI are associated with a lower share of consumer surplus in the total surplus. Notice from (12) that as $\widehat{\gamma} \to 1$ (products become homogeneous), the right-hand side of (12) approaches $2H$, which is the right-hand of equation (7) when demand is linear (in which case $\delta = 1$).[31]

4.2 Price competition

To study the relationship between the HHI and $\frac{PS^*}{CS^*}$ under price competition, I first show in the Appendix that the demand system associated with (10) is given by:

$$q_i\left(p_1, \ldots, p_n\right) = \mu\left(A_i - p_i\right) - \theta \sum_{j \neq i}^{n} \left(A_i - p_j\right), \qquad i = 1, 2, , \ldots, n, \tag{13}$$

where

$$\mu \equiv \frac{\beta + (n-2)\gamma}{(\beta - \gamma)(\beta + (n-1)\gamma)}, \qquad \theta \equiv \frac{\gamma}{(\beta - \gamma)(\beta + (n-1)\gamma)}.$$

Now, the profit of each firm i is given by

$$\pi_i = \left(p_i - k_i\right) q_i\left(p_1, \ldots, p_n\right) - F_i.$$

An interior Nash equilibrium is now a vector $(p_1^*, \ldots, p_n^*)$ that solves the following system of first-order conditions,

$$q_i\left(p_1, \ldots, p_n\right) + \frac{\partial q_i\left(p_1, \ldots, p_n\right)}{\partial p_i}\left(p_i - k_i\right) = 0, \qquad i = 1, 2, , \ldots, n, \tag{14}$$

where $\frac{\partial q_i(p_1,\ldots,p_n)}{\partial p_i} = -\mu$ for all i. Using $q_i^* = q_i\left(p_1^*, \ldots, p_n^*\right)$ to denote the equilibrium output of each firm, (14) implies that the equilibrium producer surplus is

$$PS^* = \sum_{i=1}^{n}\left(p_i^* - k_i\right) q_i^* = \frac{1}{\mu}\sum_{i=1}^{n}\left(q_i^*\right)^2.$$

Substituting for $\sum_{i=1}^{n}\left(q_i^*\right)^2 = \mu PS^*$ in (3), noting that consumer surplus is still given by (11), recalling that $\widehat{\gamma} \equiv \frac{\gamma}{\beta}$, and rearranging, yields the following result:

Proposition 7: *In an n-firm differentiated products oligopoly with price competition, where firms have (possibly different) constant marginal costs and face a linear demand system (13),*

$$\frac{PS^*}{CS^*} = \frac{2H}{\widehat{\gamma} + (1 - \widehat{\gamma})H} \times \frac{(1 - \widehat{\gamma})(1 + (n-1)\widehat{\gamma})}{1 + (n-2)\widehat{\gamma}}. \tag{15}$$

[31] Notice that when firms are symmetric, $q_i^* = q^*$ for all i, so $CS^* = \frac{n(\beta - \gamma(n-1))\left(q^*\right)^2}{2}$ and $PS^* = \beta n\left(q_i^*\right)^2$. Hence, $\frac{PS^*}{CS^*} = \frac{2}{1 - \widehat{\gamma}(n-1)}$, which is indeed the right-hand side of (12), when $H = \frac{1}{n}$.

In the Shubik-Levitan case, the equation is similar, except that $\widehat{\gamma} = \frac{\tau}{n+\tau}$.

Proposition 7 shows that under price competition, $\frac{PS^*}{CS^*}$ is also positively related to the HHI, similarly to the case of quantity competition. The difference between the two cases is that under price competition, an increase in the HHI has a smaller effect on $\frac{PS^*}{CS^*}$ because $\frac{(1-\widehat{\gamma})(1+(n-1)\widehat{\gamma})}{1+(n-2)\widehat{\gamma}} \leq 1$ with equality holding only when $\widehat{\gamma} = 0$. In other words, under price competition, the same HHI is associated with a lower $\frac{PS^*}{CS^*}$ ratio, meaning that for a given HHI, the share of consumer surplus in the total surplus in equilibrium is larger under price competition than under quantity competition.

4.3 Changes in the level of the HHI and the distribution of surplus

To examine the implications of changes in the HHI for the distribution of surplus, recall that the HHI is an outcome of firms' strategies. Therefore, it is not immediately obvious from Propositions 6 and 7 that the HHI and $\frac{PS^*}{CS^*}$ necessarily move in the same direction in response to exogenous shocks, because these shocks do not only affect the HHI, but may also affect the relationship between the HHI and $\frac{PS^*}{CS^*}$. In the next proposition, I explore this issue.

Proposition 8: *In an n-firm differentiated products oligopoly with either quantity or price competition, the following holds:*

(i) *An increase in the HHI due to a change in the intercepts of the inverse demand functions or the marginal costs of firms is accompanied by an increase in $\frac{PS^*}{CS^*}$ both when demand is given by the Spence and the Shubik-Levitan specifications.*

(ii) *An increase in the HHI due to a decrease in the number of firms is accompanied by an increase in $\frac{PS^*}{CS^*}$ when demand is given by the Spence specification.*

(iii) *An increase in the HHI due to a decrease in the number of firms may be accompanied by either an increase or a decrease in $\frac{PS^*}{CS^*}$ when demand is given by the Shubik-Levitan specification, but when firms are symmetric, it is accompanied by an increase in $\frac{PS^*}{CS^*}$.*

Proposition 8 shows that when demand is given by the Spence [1976] specification, then holding $\widehat{\gamma}$ fixed, an increase in the HHI (due to either a change in the intercepts of the inverse demand functions or marginal costs, or a change in the number of firms) is accompanied by an increase in $\frac{PS^*}{CS^*}$, both under quantity and price competition. That is, as in the Cournot model, higher values of the HHI are accompanied by a smaller share of consumer surplus in the total surplus. When demand is given by the Shubik-Levitan specification, the same conclusion holds if the increase in the HHI is caused by a change in the intercepts of the inverse demand functions or the marginal costs, or a decrease in the number of firms, provided that firms are symmetric.

5 Conclusion

I show that in either the Cournot model or a differentiated products model with linear demand and either quantity or price competition, the HHI reflects the distribution of total surplus between firms' owners and consumers. When all firms have constant marginal costs (not necessarily identical across firms) the ratio of producer to consumer surplus is an increasing function of the HHI, implying that consumers obtain a lower share of the total surplus when the HHI is higher. This result generalizes to the case of common ownership with the MHHI replacing the HHI. When the marginal cost of at least one firm is increasing with output, the HHI is a lower bound on the ratio of producer to consumer surplus. These results imply that the HHI has a simple and intuitive normative interpretation.

6 Appendix

Following are the proofs of Lemmas 1-3, Propositions 2-4 and 8, and a number of other derivations and proofs.

Proof of Lemma 1: (i) Note that $\eta(Q) \equiv \frac{Q(CS)'}{CS} \geq 1$ is equivalent to $CS' \geq \frac{CS}{Q}$, which holds if and only if $CS'' = -(p'(Q) + p''(Q)Q) \geq 0$. To complete the proof, note that $\eta(Q) > 0$ as $CS' = -p'(Q)Q > 0$.

(ii) Differentiating $\eta(Q)$, yields

$$\begin{aligned} \eta'(Q) &= \frac{(CS' + Q(CS)'')CS - Q(CS')^2}{(CS)^2} \\ &= \frac{\eta(Q)}{Q}\left[1 + \frac{Q(CS)''}{CS'} - \eta(Q)\right]. \end{aligned} \tag{16}$$

Noting that $CS' = -p'(Q)Q$, $CS'' = -(p'(Q) + p''(Q)Q)$, and $\sigma(Q) \equiv -\frac{p''(Q)Q}{p'(Q)}$, it follows that

$$\eta'(Q) = \frac{\eta(Q)}{Q}\left[2 - \sigma(Q) - \eta(Q)\right].$$

Since $\eta(Q) \geq 0$, $\eta'(Q)$ has the same sign as $2 - \sigma(Q) - \eta(Q)$. ■

Proof of Lemma 2: The "if" part is straightforward. If the inverse demand function is given by (6), then

$$CS = \int_0^Q \left(A - bz^{\delta}\right)dz - \left(A - bQ^{\delta}\right)Q = \frac{\delta b Q^{1+\delta}}{1+\delta}.$$

The elasticity of CS with respect to output is

$$\eta(Q) \equiv \frac{Q(CS)'}{CS} = \left(\frac{\delta b Q^{1+\delta}}{\delta+1}\right)' \frac{Q}{\frac{\delta b Q^{\delta+1}}{\delta+1}} = 1 + \delta,$$

which is indeed a constant.

To prove the "only if" part, I first show that a constant $\eta(Q)$ implies a constant pass-through rate. To this end, suppose that $\eta(Q) \equiv \frac{Q(CS)'}{CS} = \overline{\eta}$ for all Q. Since $\eta(Q)$ is constant, $\eta'(Q) = 0$. Hence, (16) implies that,

$$\overline{\eta} = 1 + \frac{Q(CS)''}{CS'} = \frac{2p'(Q) + p''(Q)Q}{p'(Q)}, \tag{17}$$

where the second equality follows because $CS' = -p'(Q)Q$ and $CS'' = -p'(Q) - p''(Q)Q$. This expression is equal to the inverse of the cost pass-through rate. To see why, note that if the market is served by a monopoly with a constant marginal cost k, the profit-maximizing output is implicitly defined by the first-order condition $p(Q)+p'(Q)Q-k=0$. Fully differentiating the first-order condition with respect to Q and k and rearranging, yields

$$\frac{\partial Q}{\partial k}=\frac{1}{2p'(Q)+p''(Q)Q}.$$

Hence, the cost pass-through rate is

$$p'(k)\equiv p'(Q)\frac{\partial Q}{\partial k}=\frac{p'(Q)}{2p'(Q)+p''(Q)Q}. \tag{18}$$

Together with (17), this implies that $p'(k)=\frac{1}{\eta}$. Bulow and Pfleiderer [1983] prove that an inverse demand function exhibits a constant cost pass-through rate if and only if it is represented by (6). Altogether then, a constant $\eta(Q)$ implies a constant $p'(k)$, which in turn implies that the inverse demand function is represented by (6).

Finally, it is easy to verify that when the inverse demand function is represented by (6), the cost pass-through rate is

$$p'(k)=\frac{-b\delta Q^{\delta-1}}{-2b\delta Q^{\delta-1}-b\delta(\delta-1)Q^{\delta-2}Q}=\frac{1}{1+\delta},$$

which is the inverse of $\eta(Q)$. Since the curvature of the demand function is $\sigma(Q)\equiv-\frac{p''(Q)Q}{p'(Q)}$, it follows that $p'(k)=\frac{p'(Q)}{2p'(Q)+p''(Q)Q}=\frac{1}{2-\sigma(Q)}$. Hence,

$$\sigma(Q)=2-\frac{1}{p'(k)}=1-\delta.$$

■

The relationship between $\eta(Q)$, $p'(k)$, and $\sigma(Q)$, for general demand functions: To study the relationship between $\eta(Q)$, the cost pass-through rate, and the curvature of the demand function, for general demand functions, note that $CS'=-p'(Q)Q$ and $CS''=-(p'(Q)+p''(Q)Q)$. Substituting in (18) and rearranging, the cost pass-through rate can be written as

$$p'(k)=\frac{p'(Q)}{p'(Q)-CS''}=\frac{CS'}{CS'+Q(CS)''}.$$

Substituting for CS' from this expression into the definition of $\eta(Q)$, yields

$$\eta(Q)\equiv\frac{Q(CS)'}{CS}=\frac{p'(k)}{1-p'(k)}\times\frac{Q^2(CS)''}{CS}.$$

As for the curvature of the demand function, note that since $CS'=-p'(Q)Q$ and $\sigma(Q)\equiv-\frac{p''(Q)Q}{p'(Q)}$,

$$\eta(Q)=-\frac{p'(Q)(Q)^2}{CS}=\frac{p''(Q)}{\sigma(Q)}\frac{Q^3}{CS}.$$

When the inverse demand function is given by (6), $p'(k) = \frac{1}{1+\delta}$ and $\sigma(Q) = 1 - \delta$, so $\eta(Q) = \frac{1}{p'(k)}$ and $\eta(Q) = 2 - \sigma(Q)$. ■

Proof of Lemma 3: To examine how an increase in δ affects the HHI, arrange firms in an increasing order of marginal costs, i.e., $k_1 \leq \ldots \leq k_n$, and note that when $p(Q) = -bQ^{\delta}$, the first-order condition for firm i is given by

$$-bQ^{\delta} - b\delta Q^{\delta-1} q_i - k_i = 0.$$

Summing over all firms, the equilibrium aggregate output is,

$$Q^* = \left(-\frac{n\widehat{k}}{b(n+\delta)}\right)^{\frac{1}{\delta}},$$

where $\widehat{k} \equiv \frac{1}{n}\sum_{i=1}^{n} k_i$ is the average marginal cost in the industry. Using the first-order conditions again, the market share of each firm i is,

$$s_i = \frac{q_i}{Q} = -\frac{bQ^{\delta} + k_i}{b\delta Q^{\delta}}.$$

Substituting for Q^* in s_i^*, the equilibrium market share of each firm i is:

$$s_i^* = \frac{(n+\delta) k_i - n\widehat{k}}{\delta n\widehat{k}}.$$

To ensure that $s_i^* > 0$ for all i, I will assume that $k_n < \frac{n\widehat{k}}{n+\delta}$ (note that $\frac{n\widehat{k}}{n+\delta} > \widehat{k}$ as $\delta < 0$). Since $\delta < 0$, $k_1 \leq \ldots \leq k_n$ implies that $s_1^* \geq \ldots \geq s_n^*$: firms with lower marginal costs have bigger market shares.

Next, note that the HHI is given by,

$$\begin{aligned} H &= \sum_{i=1}^{n} (s_i^*)^2 = \sum_{i=1}^{n} \left(\frac{(n+\delta) k_i - n\widehat{k}}{\delta n\widehat{k}}\right)^2 \\ &= \frac{1}{\left(\delta n\widehat{k}\right)^2} \sum_{i=1}^{n} \left((n+\delta)^2 k_i^2 + n^2\widehat{k}^2 - 2n(n+\delta)\widehat{k}k_i\right) \\ &= \frac{1}{\left(\delta n\widehat{k}\right)^2} \left((n+\delta)^2 \sum_{i=1}^{n} k_i^2 - n^2(n+2\delta)\widehat{k}^2\right). \end{aligned}$$

Since HHI is independent of the demand parameter b, changes in b do not affect $(1+\delta)H$, and therefore by Corollary 2, do not affect $\frac{PS^*}{CS^*}$ either. Differentiating with respect to δ,

$$\begin{aligned} \frac{\partial H}{\partial \delta} &= \frac{1}{\delta^3 n^2 \widehat{k}^2} \left[\left(2(n+\delta)\sum_{i=1}^{n} k_i^2 - 2n^2\widehat{k}^2\right)\delta - 2\left((n+\delta)^2 \sum_{i=1}^{n} k_i^2 - n^2(n+2\delta)\widehat{k}^2\right)\right] \\ &= \frac{2(n+\delta)\left(n\widehat{k}^2 - \sum_{i=1}^{n} k_i^2\right)}{\delta^3 n\widehat{k}^2} = -\frac{2(n+\delta)\sum_{i=1}^{n}\left(k_i - \widehat{k}\right)^2}{n\delta^3\widehat{k}^2} > 0. \end{aligned}$$

■

Proof of Proposition 2: When firms are symmetric, $H = \frac{1}{n}$. In this case, H can increase only when n decreases, say due to an infinitesimal merger.[32] Noting that there is an inverse relationship between H and n, I prove the proposition by showing that $\frac{1}{n}\eta(Q^*)$ decreases with n (in which case a decrease in n raises both H as well as $\frac{1}{n}\eta(Q^*)$); that is,

$$\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right) = -\frac{\eta(Q^*)}{n^2} + \frac{\eta'(Q^*)}{n}\frac{\partial Q^*}{\partial n} < 0.$$

(Note that $\eta(Q^*)$ depends on n only through Q^* but not directly).

To this end, recall that the first-order condition for firm i is given by (1). Summing up over all firms, the aggregate output, Q^*, is implicitly defined by

$$np(Q) + p'(Q)Q - \sum_{i=1}^{n} k_i = 0. \tag{19}$$

When all firms have the same marginal cost, k, $\sum_{i=1}^{n} k_i = nk$. Fully differentiating (19) with respect to Q^* and n, and rearranging terms,

$$\frac{\partial Q^*}{\partial n} = \frac{p'(Q^*)Q^*}{n\left[p'(Q^*)(n+1) + p''(Q^*)Q^*\right]}.$$

Using the expression for $\eta'(Q^*)$ from (16) and $\frac{\partial Q^*}{\partial n}$, yields

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right) &= -\frac{\eta(Q^*)}{n^2} + \frac{1}{n} \times \underbrace{\frac{\eta(Q^*)}{Q^*}\left[1 + \frac{Q^*(CS^*)''}{(CS^*)'} - \eta(Q^*)\right]}_{\eta'(Q^*)} \times \underbrace{\frac{p'(Q^*)Q^*}{n\left[p'(Q^*)(n+1) + p''(Q^*)Q^*\right]}}_{\frac{\partial Q^*}{\partial n}} \\
&= -\frac{\eta(Q^*)}{n^2}\left[1 - \frac{(1-\eta(Q^*))(CS^*)' + Q^*(CS^*)''}{(CS^*)'} \times \frac{p'(Q^*)}{\left[p'(Q^*)(n+1) + p''(Q^*)Q^*\right]}\right] \\
&= -\frac{\eta(Q^*)}{n^2}\left[1 - \frac{(2-\eta(Q^*))p'(Q^*) + p''(Q^*)Q^*}{p'(Q^*)(n+1) + p''(Q^*)Q^*}\right] \\
&= -\frac{\eta(Q^*)}{n^2}\left[\frac{p'(Q^*)(n-1+\eta(Q^*))}{p'(Q^*)(n+1) + p''(Q^*)Q^*}\right] < 0,
\end{aligned}$$

where the third equality follows because $(CS^*)' = -p'(Q^*)Q^*$ and $(CS^*)'' = -p'(Q^*) - p''(Q^*)Q^*$, and the inequality follows because by assumption, $p'(Q) + p''(Q)Q \le 0$. ■

Examples for demand functions with increasing or decreasing $\eta(Q)$:

[32] The concept of infinitesimal mergers is due to Farrell and Shapiro [1990a]. It may correspond to an economic event, such as the transfer of a small amount of capital from one firm to another or the purchase by one firm of a small ownership stake in another firm. A merger then can be viewed as the composite of many such infinitesimal mergers.

The Logit demand $Q = \frac{e^{\frac{A-p}{b}}}{1+e^{\frac{A-p}{b}}}$, where $A, b > 0$. The associated inverse demand function is $p = A - b\ln\left(\frac{Q}{1-Q}\right)$. Hence, consumer surplus is given by

$$CS = \int_0^Q \left(A - b\ln\left(\frac{z}{1-z}\right)\right) dz - Q\left(A - b\ln\left(\frac{Q}{1-Q}\right)\right) = -b\ln(1-Q).$$

The elasticity of CS is then

$$\eta(Q) = \frac{Q(CS)'}{CS} = -\frac{Q}{(1-Q)\ln(1-Q)}.$$

Differentiating $\eta(Q)$ yields,

$$\eta'(Q) = -\frac{Q + \ln(1-Q)}{((1-Q)\ln(1-Q))^2} \geq 0,$$

where the inequality follows because $-(Q + \ln(1-Q)) = 0$ when $Q = 0$, and $\frac{d}{dQ}(-Q - \ln(1-Q)) = \frac{Q}{1-Q} > 0$ since $Q < 1$.

The Logarithmic demand $Q = \ln\left(\frac{A-p}{b}\right)$, where $A, b > 0$. The associated inverse demand function $p = A - be^Q$. Here,

$$CS = \int_0^Q (A - be^z)\, dz - Q\left(A - e^Q\right) = b\left(1 + (Q-1)e^Q\right).$$

The elasticity of CS is

$$\eta(Q) \equiv \frac{Q(CS)'}{CS} = \frac{Q^2 e^Q}{1 + (Q-1)e^Q}.$$

Hence,

$$\eta'(Q) = \frac{Qe^Q\left(2 + Q - (2-Q)e^Q\right)}{\left(1 + (Q-1)e^Q\right)^2} \geq 0,$$

where the inequality follows because $2+Q-(2-Q)e^Q = 0$ when $Q = 0$ and $\frac{d}{dQ}\left(2 + Q - (2-Q)e^Q\right) = 1 + e^Q(Q-1) \geq 1 + (Q-1) \geq 0$.

The demand function $Q = \frac{A-p}{b-(A-p)}$, where $A, b > 0$. Here, the inverse demand function $p = A - \frac{bQ}{1+Q}$. Then, consumer surplus is

$$CS = \int_0^Q \left(A - \frac{bz}{1+z}\right) dz - Q\left(A - \frac{bQ}{1+Q}\right) = b\left(\ln(1+Q) - \frac{Q}{1+Q}\right).$$

The elasticity of CS is

$$\eta(Q) \equiv \frac{Q(CS)'}{CS} = \frac{Q^2}{(1+Q)((1+Q)\ln(1+Q) - Q)}.$$

Hence,

$$\eta'(Q) = \frac{-2Q\left(\frac{Q(2+Q)}{2+2Q} - \ln(1+Q)\right)}{(1+Q)^3\left(\frac{Q}{1+Q} - \ln(1+Q)\right)^2} \leq 0,$$

where the inequality follows because $\frac{Q(2+Q)}{2+2Q} - \ln(1+Q) = 0$ when $Q = 0$ and $\frac{d}{dQ}\left(\frac{Q(2+Q)}{2+2Q} - \ln(1+Q)\right) = \frac{Q^2}{2(1+Q)^2} > 0$.

To ensure that $p'(Q) + p''(Q)Q \leq 0$, note that if the inverse demand function is $p = A - \frac{bQ}{1+Q}$, then $p'(Q) + p''(Q)Q = \frac{Q-1}{(1+Q)^3}$, which is non-positive only when $Q \leq 1$. Assuming that $A = 1$, the equilibrium aggregate output Q^*, defined implicitly by the solution of equation (19), is

$$Q^* = \frac{n - 1 - 2n\widehat{k} + \sqrt{(n-1)^2 + 4n\widehat{k}}}{2n\widehat{k}},$$

where $\widehat{k} \equiv \frac{1}{n}\sum_{i=1}^{n} k_i$ is the average marginal cost in the industry. It is easy to verify that $0 \leq Q^* \leq 1$ provided that $\frac{2n-1}{4n} \leq \widehat{k} \leq 1$.

In the next table, I summarize the properties of the three demand functions that were discussed here.

"Place Table I approximately here."

■

Proof of Proposition 3: Consider an n-firm industry and let $k = (k_1, \ldots, k_n)$ be the vector of marginal costs. Now consider a mean-preserving spread of k and let $k' = (k'_1, \ldots, k'_n)$ be the vector of marginal costs after the spread.

Recall that the equilibrium aggregate output, Q^*, is implicitly defined by equation (19). This equation implies that Q^* depends on $\sum_{i=1}^{n} k_i$ and is therefore unaffected by a mean-preserving spread of k. As a result, $\eta(Q^*)$ is also unaffected.

To study the effect on the HHI, note from (1) that the equilibrium market share of each firm i before the spread is given by

$$s_i^* = \frac{q_i^*}{Q^*} = \frac{p(Q^*) - k_i}{-p'(Q^*)Q^*}. \tag{20}$$

Hence, the HHI before the mean-preserving spread is given by

$$H = \sum_{i=1}^{n} \underbrace{\left(\frac{p(Q^*) - k_i}{-p'(Q^*)Q^*}\right)^2}_{s_i^*} = \frac{np(Q^*)\left(p(Q^*) - 2\widehat{k}\right) + \sum_{i=1}^{n}(k_i)^2}{(-p'(Q^*)Q^*)^2}.$$

After the mean-preserving spread, the HHI is given by the same expression, except that k_i' replaces k_i. Since k' is a mean-preserving spread of k, $Var(k') > Var(k)$, where $Var(k') = \sum_{i=1}^{n} \left(k_i' - \widehat{k}\right)^2$ is the variance of k' and $Var(k) = \sum_{i=1}^{n} \left(k_i - \widehat{k}\right)^2$ is the variance of k, where $\widehat{k} \equiv \frac{1}{n}\sum_{i=1}^{n} k_i$ is the average marginal cost in the industry.

Hence the change in the HHI due to the mean-preserving spread is

$$\begin{aligned}
\Delta H &= \frac{np(Q^*)\left(p(Q^*) - 2\widehat{k}\right) + \sum_{i=1}^{n}(k_i')^2}{(-p'(Q^*)Q^*)^2} - \frac{np(Q^*)\left(p(Q^*) - 2\widehat{k}\right) + \sum_{i=1}^{n}(k_i)^2}{(-p'(Q^*)Q^*)^2} \\
&= \frac{\sum_{i=1}^{n}(k_i')^2 - \sum_{i=1}^{n}(k_i)^2}{(-p'(Q^*)Q^*)^2} \\
&= \frac{Var(k') - Var(k)}{(-p'(Q^*)Q^*)^2} > 0.
\end{aligned}$$

That is, the mean-preserving spread of the distribution of marginal costs causes an increase in the HHI. ■

Proposition 4: (i) When the marginal costs of all firms increase by a constant x, the aggregate output Q^* is defined implicitly by (19) with $k_i + x$ replacing k_i. Fully differentiating the equation with respect to Q^* and x, and recalling that $p'(Q) < 0$ and $p'(Q) + p''(Q)Q \leq 0$,

$$\frac{\partial Q^*}{\partial x} = \frac{n}{(n+1)p'(Q^*) + p''(Q^*)Q^*} < 0.$$

As for HHI, the market share of each firm i is given by (20) with $k_i + x$ replacing k_i. Notice that $k_1 \leq \ldots \leq k_n$ implies that $s_1^* \geq \ldots \geq s_n^*$: firms with lower marginal costs have bigger market shares. Now,

$$\frac{\partial H}{\partial x} = 2\sum_{i=1}^{n} s_i^* \frac{\partial s_i^*}{\partial x},$$

where,

$$\begin{aligned}
\frac{\partial s_i^*}{\partial x} &= \frac{1}{p'(Q^*)Q^*} + \left(\frac{-(p'(Q^*))^2 Q^* + (p'(Q^*) + p''(Q^*)Q^*)(p(Q^*) - k_i - x)}{(-p'(Q^*)Q^*)^2}\right)\frac{\partial Q^*}{\partial x} \\
&= \frac{1}{p'(Q^*)Q^*} + \left(\frac{-(p'(Q^*))^2 Q^* + (p'(Q^*) + p''(Q^*)Q^*)s_i^*(-p'(Q^*)Q^*)}{(-p'(Q^*)Q^*)^2}\right)\frac{\partial Q^*}{\partial x} \\
&= \frac{1}{p'(Q^*)Q^*} + \left(\frac{p'(Q^*) + (p'(Q^*) + p''(Q^*)Q^*)s_i^*}{-p'(Q^*)Q^*}\right)\frac{\partial Q^*}{\partial x},
\end{aligned}$$

Since $p'(Q^*) + p''(Q^*)Q^* \leq 0$ and $\frac{\partial Q^*}{\partial x} < 0$, $\frac{\partial s_1^*}{\partial x} \geq \ldots \geq \frac{\partial s_n^*}{\partial x}$; that is, lower cost firms which have bigger market shares are more affected by a change in x than smaller firms.[33] Applying Chebyshev's

[33] Note that $\sum_{i=1}^{n} s_i^* = 1$, $\frac{\partial s_1^*}{\partial x} \geq \ldots \geq \frac{\partial s_n^*}{\partial x}$ implies that $\frac{\partial s_i^*}{\partial x} \geq 0$ for large firms and $\frac{\partial s_i^*}{\partial x} \leq 0$ for small firms (indeed, the first term in $\frac{\partial s_i^*}{\partial x}$ is negative and identical across firms, while the second is positive and larger for larger firms).

sum inequality, and noting that $\sum_{i=1}^{n} s_i^* = 1$, it follows that

$$\begin{aligned}\frac{\partial H}{\partial x} &= 2\sum_{i=1}^{n} s_i^* \frac{\partial s_i^*}{\partial x} = 2n\left(\frac{1}{n}\sum_{i=1}^{n} s_i^* \frac{\partial s_i^*}{\partial x}\right) \\ &\geq 2n\left(\frac{1}{n}\sum_{i=1}^{n} s_i^*\right)\left(\frac{1}{n}\sum_{i=1}^{n} \frac{\partial s_i^*}{\partial x}\right) \\ &= \frac{2}{n}\frac{\partial}{\partial x}\left(\sum_{i=1}^{n} s_i^*\right) = 0.\end{aligned}$$

Hence, the HHI increases. Since a cost change does not affect $\eta(Q^*)$ directly and since Q^* falls, Lemma 4 then implies that $\eta'(Q^*) \leq 0$ is sufficient for the increase in the HHI to be accompanied by an increase in $\frac{PS^*}{CS^*}$.

(ii) Now consider a decrease in k_1. From equation (19) it is immediate that since $p'(Q) < 0$ and $p'(Q) + p''(Q)Q \leq 0$, $\frac{\partial Q^*}{\partial k_1} < 0$, so Q^* increases when k_1 falls.

As for HHI, the market share of each firm i, s_i^*, is given by (20). Differentiating s_i^*, where $i \neq 1$, with respect to k_1 and using (20),

$$\frac{\partial s_i^*}{\partial k_1} = \frac{p'(Q^*) + (p'(Q^*) + p''(Q^*)Q^*)s_i^*}{-p'(Q^*)Q^*} \times \frac{\partial Q^*}{\partial k_1} > 0,$$

where the inequality follows because $\frac{\partial Q^*}{\partial k_j} < 0$ and because by assumption $p'(Q^*) + p''(Q^*)Q^* < 0$. Hence a decrease in k_1 causes a decrease in each s_i^*, $i \neq 1$.

Now, denote that the decrease in s_j^* following the decrease in k_1 by Δ_j; since market shares must sum up to 1, the market share of firm 1 increases by $\sum_{j\neq 1}\Delta_j$. Noting from (20) that $k_1 \leq \ldots \leq k_n$ implies $s_1^* \geq \ldots \geq s_n^*$, the resulting change in the HHI is

$$\begin{aligned}\Delta H &= \underbrace{\left(s_1 + \sum_{j\neq 1}\Delta_j\right)^2 + (s_2 - \Delta_2)^2 \ldots + (s_n - \Delta_n)^2}_{\text{The HHI after}} - \underbrace{\left(s_1^2 + \ldots + s_n^2\right)}_{\text{The HHI before}} \\ &= 2s_1(\Delta_2 + \ldots + \Delta_n) + \left(\sum_{j\neq 1}\Delta_j\right)^2 - 2s_2\Delta_2 + \Delta_2^2 + \ldots - 2s_n\Delta_n + \Delta_n^2 \\ &= 2(s_1 - s_2)\Delta_2 + \ldots + 2(s_1 - s_n)\Delta_n + \left(\sum_{j\neq 1}\Delta_j\right)^2 + \sum_{j\neq 1}\Delta_j^2 \geq 0.\end{aligned}$$

Since the HHI and Q^* increase, and since a cost change does not affect $\eta(Q^*)$ directly, Lemma 4 implies that $\eta'(Q^*) \geq 0$ is sufficient for the increase in the HHI to be accompanied by an increase in $\frac{PS^*}{CS^*}$. ■

Consumer surplus and the demand system in the product differentiation case: Starting with the Spence [1976] specification, the demand system (10) is derived from the preferences of a representative consumer, whose utility function is quadratic:

$$u(q_1,\ldots,q_n) = \sum_{i=1}^{n} A_i q_i - \frac{\beta \sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n} \sum_{j\neq i}^{n} q_i q_j}{2} + m, \tag{21}$$

where m is income spent on all other goods, $A_1,\ldots,A_n$ and β, are positive utility parameters, and $0<\gamma<\beta$. Maximizing $u(q_1,\ldots,q_n)$ subject to a budget constraint, $\sum_{i=1}^{n} p_i q_i + m = I$, where p_i is the prices of good i, and I is income, yields the system of inverse demand functions (10).

To derive the associated demand system, note that (10) can be written as,

$$q_i = \frac{1}{\beta}\left(A_i - p_i - \gamma \sum_{j\neq i}^{n} q_j\right).$$

Subtracting $\frac{\gamma}{\beta} q_i$ from both sides of the equation, summing over all firms and recalling that $Q = \sum_{i=1}^{n} q_i$, yields

$$Q = \frac{\sum_{i=1}^{n}(A_i - p_i)}{\beta + (n-1)\gamma}.$$

Substituting for Q in q_i and rearranging, yields (13).

To express consumer surplus, note first that the utility function of the representative consumer can now be written as:

$$\begin{aligned} u(q_1,\ldots,q_n) &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j}{2} + m \\ &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2} + m, \end{aligned}$$

where the last equality follows since $\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j = \left(\sum_{j=1}^{n} q_i\right)^2 = Q^2$. Substituting for m from the budget constraint into (21) and using (10), consumer surplus is given by

$$\begin{aligned} CS(q_1,\ldots,q_n) &= \sum_{i=1}^{n} A_i q_i - \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^* + \gamma Q^2}{2} - \sum_{i=1}^{n} \underbrace{\left(A_i - \beta q_i - \gamma \sum_{j\neq i}^{n} q_j\right)}_{p_i} q_i \\ &= -\frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2} + (\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \\ &= \frac{(\beta-\gamma)\sum_{i=1}^{n} q_i^2 + \gamma Q^2}{2}. \end{aligned}$$

The Shubik-Levitan demand and inverse demand systems and consumer surplus are derived similarly, except that now $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$.

■

Proof of Proposition 8: (i) Note that equations (12) and (15), which relate the HHI with $\frac{PS^*}{CS^*}$, are independent of the demand parameters $A_1, \ldots, A_n$, and the cost parameters $k_1, \ldots, k_n$. Hence, whenever the HHI increases due to changes in these parameters, $\frac{PS^*}{CS^*}$ increases as well, regardless of whether firms engage in quantity or price competition.

(ii) Equation (12) is also independent of n, so under quantity competition, whenever the HHI increases due to a change in n, $\frac{PS^*}{CS^*}$ increases as well. By contrast, equation (15) depends on n, but since the right-hand side of the equation is increasing with H and decreasing with n, a decrease with n, which causes an increase in H, also causes an increase in $\frac{PS^*}{CS^*}$.

(iii) Recalling that in the Shubik-Levitan case, $\widehat{\gamma} = \frac{\tau}{n+\tau}$, equations (12) and (15) become,

$$\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{n + \frac{\tau}{H}}, \qquad \frac{PS^*}{CS^*} = \frac{2n^2(1+\tau)}{\left(n + \frac{\tau}{H}\right)(n + (n-1)\tau)}.$$

The right-hand sides of the two equations are increasing with both H and n. Hence, a decrease in n, which causes an increase in the HHI, may cause either an increase or a decrease in $\frac{PS^*}{CS^*}$. However, in the symmetric case where $H = \frac{1}{n}$, the two equations become

$$\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{n(1+\tau)}, \qquad \text{and} \qquad \frac{PS^*}{CS^*} = \frac{2n}{n(1+\tau) - \tau}.$$

Since the right-hand sides decrease with n, a decreases in n, which causes an increase in H, also causes an increase in $\frac{PS^*}{CS^*}$. ■

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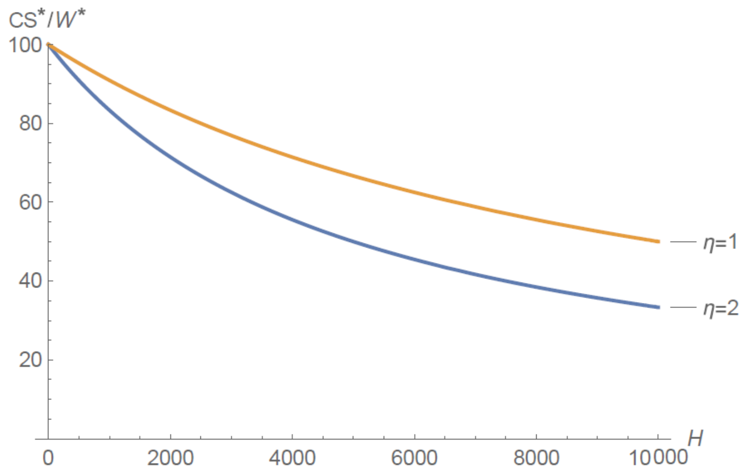

Figure 1: Consumers' share in the total surplus as a function of HHI when $\eta(Q^*) = 1$ and $\eta(Q^*) = 2$

TABLE I

THE THREE EXAMPLES FOR DEMAND FUNCTIONS FOR WHICH $\eta(Q)$ IS NOT CONSTANT

Demand	Inverse demand	CS	$\eta(Q)$	$\eta'(Q)$
$Q = \frac{e^{\frac{A-p}{b}}}{1+e^{\frac{A-p}{b}}}$	$p = A - b\ln\left(\frac{Q}{1-Q}\right)$	$-b\ln(1-Q)$	$-\frac{Q}{(1-Q)\ln(1-Q)}$	$-\frac{Q+\ln(1-Q)}{((1-Q)\ln(1-Q))^2} \geq 0$
$Q = \ln\left(\frac{A-p}{b}\right)$	$p = A - be^{Q}$	$b\left(1+(Q-1)e^{Q}\right)$	$\frac{Q^2 e^Q}{1+(Q-1)e^Q}$	$\frac{Qe^Q\left(2+Q-(2-Q)e^Q\right)}{\left(1+(Q-1)e^Q\right)^2} \geq 0$
$Q = \frac{A-p}{b-(A-p)}$	$p = A - \frac{bQ}{1+Q}$	$b\left(\ln(1+Q) - \frac{Q}{1+Q}\right)$	$\frac{Q^2}{(1+Q)((1+Q)\ln(1+Q)-Q)}$	$\frac{-2Q\left(\frac{Q(2+Q)}{2+2Q}-\ln(1+Q)\right)}{(1+Q)^3\left(\frac{Q}{1+Q}-\ln(1+Q)\right)^2} \leq 0$