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Optimal search auctions with correlated bidder types

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Abstract

We study optimal auctions when contacting prospective bidders is costly and the bidders' values are correlated. Although full surplus extraction is, in general, impossible, we can construct a search mechanism that fully extracts the surplus with an arbitrarily high probability.
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1. Introduction

It is well known that, in auction environments with risk neutral bidders and correlated values, the seller can generically extract the entire social surplus (see Crémer and McLean, 1985, 1988). But if the seller must incur (search) costs in order to contact prospective bidders, then the seller's optimal mechanism is in the form of a *search mechanism* that, contingent on history, specifies the order in which prospective bidders are contacted, the time at which the process ends, and the participating bidders' payments. While the sequential nature of the mechanism economizes on the seller's search costs, it may prevent the seller from using Crémer–McLean

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lotteries that condition a bidder's payment on all his rivals' reports. We show that although the seller cannot always fully extract the surplus, he can nonetheless achieve full extraction with an arbitrarily high probability.

Our paper contributes to the small, but growing, literature on optimal search auctions. So far, this literature has only considered independently distributed bidders' types.[1] Our paper by contrast, deals with correlated bidders' types.

2. The model

A seller wishes to sell an indivisible good to one out of a finite set I of n prospective bidders. The seller's value is normalized to zero. Bidder i's value from winning the good (the bidder's type) is $x_i \in X_i$, where X_i is a finite set. A vector of types $x \equiv (x_i)_{i \in I}$ is called a *realized state*. Nature draws states, x, from the set $X \equiv \times_{i \in I} X_i$, according to a strictly positive probability measure f. Everyone's discount factor is $\delta \in (0,1]$.

2.1. Search costs

In order to inform bidder i about the auction, the seller incurs a *search cost*, $c_i > 0$. After being contacted by the seller, each bidder i privately learns his type x_i.

The cost c_i has several possible interpretations. First, the good might be very complex (e.g., the controlling block of a state-owned enterprise). The seller then needs to meet potential bidders in person (e.g., hold a road show). Second, the seller may have goals other than profit maximization and would like to ensure that bidders meet certain criteria (e.g., ensure that the privatized state-owned enterprise will be controlled by a qualified buyer). Third, our framework can be easily modified to a procurement environment with a set I of potential sellers; if the procurer's needs are hard to describe, he would need to understand exactly what each supplier can offer before asking for bids.

2.2. Search mechanisms[2]

To economize on search costs, the seller needs to design a contingent plan, called *search mechanism*. This mechanism works as follows: In period 1, the seller contacts a set of entrants, who privately learn their types and decide whether to participate. Each participating entrant signs a binding contract and sends a message. Given these messages, the mechanism either stops or continues to period 2. If it continues, new entrants are invited, privately learn their types, decide whether to participate, and send messages. The mechanism continues similarly until it stops and the good is allocated.

A *search procedure* is the operation-research part of a search mechanism. Given the bidder's messages, it determines whether to continue the mechanism, the identity of new entrants when the

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[1] See McAfee and McMillan (1988), Burguet (1996), Crémer et al. (in press), Ye (2004), Bergemann and Pesendorfer (2001) and Bergemann and Välimäki (2002). On the other hand, Crémer et al. (2003) allow for very general correlation.

[2] This section is based on Sections 2.3–2.5 in Crémer et al. (in press).

mechanism continues, and the winner's identity when the mechanism stops (but not the bidders' payments). If every invited bidder participates and is truthful, then given any realized state x, a search procedure induces the following objects:

$E^t(x)\equiv$ the set of bidders who enter in period t;

$q_i(x)\equiv$ the probability with which player i consumes the good;

$\tau(x)\equiv$ the period at which the search terminates.

Hence a search procedure can be denoted by $((E^t)_{t=1}^{\infty}, (q_i)_{i\in I})$.[3] The sequential nature of a search procedure imposes the following constraints:

1. The set of period 1 entrants, E^1, is constant on X.
2. Realized states that generate the same history up to period t induce the same decisions in period $t+1$.
3. $q_i(x)=0$ for every nonparticipating bidder i.

Given any search procedure $((E^t)_{t=1}^{\infty}, (q_i)_{i\in I})$, if all invited bidders participate and are truthful, then the discounted expected social surplus is

$$\Pi((E^t)_{t=1}^{\infty},(q_i)_{i\in I})\equiv\mathbb{E}_x\left[\delta^{\tau(x)-1}\sum_{i\in I}q_i(x)x_i-\sum_{t=1}^{\infty}\delta^{t-1}\sum_{i\in E^t(x)}c_i\right]. \tag{1}$$

Note that $q_i(x)x_i$ can be summed over all potential bidders because $q_i(x)=0$ for all nonparticipating bidders. We will say that a search procedure $((E^t)_{t=1}^{\infty}, (q_i)_{i\in I})$ is *efficient* if $\Pi((E^t)_{t=1}^{\infty}, (q_i)_{i\in I})$ is maximized over all search procedures.

The revelation principle extends to search mechanisms – see Crémer et al. (in press). We therefore consider *revelation search mechanisms* in which each bidder i's message space is i's type space. The mechanism consists of a randomization over search procedures and a payment scheme, $(p_i)_{i\in I}$, where, due to the sequential nature of the mechanism, $p_i(x)=0$ for every nonparticipating bidder i. We assume that the bidders are told the identity of the entrants in previous periods, but not the messages which these entrants sent.

3. The impossibility of full extraction of the surplus

Since search procedures continue or stop depending on the incumbents' reports, an incumbent bidder can potentially prevent the entry of rivals. Consequently the seller may be unable to fully extract the surplus using Crémer–McLean lotteries that condition each bidder's payment on the reports of other bidders.

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[3] Note that $\tau(x)$ is determined by $(E^t)_{t=1}^{\infty}$ since $\tau(x)\equiv\max\{s=1, 2, \ldots, E^s(x)\neq\emptyset\}$.

To illustrate, consider two ex ante identical bidders whose types, x_1 and x_2, are drawn from the set $\{L, M, H\}$, where $L<M<H$. The search cost of contacting each bidder is $c>0$ and $\delta=1$. The joint probability distribution of (x_1, x_2) is:

x_1	x_2		
	L	M	H
L	f_{LL}	f_{LM}	f_{LH}
M	f_{LM}	f_{MM}	f_{MH}
H	f_{LH}	f_{MH}	f_{HH}

where, for instance, $Pr(x_1=M, x_2=H)=f_{MH}$. To ensure that it is efficient to invite at least one bidder, assume that

$$f_L L + f_M M + f_H H > c, \tag{2}$$

where $f_x := f_{xL} + f_{xM} + f_{xH}$ for $x=L, M, H$. Since $\delta=1$, it is efficient to invite only bidder 1, say, in period 1. If $x_1=H$, the optimal search stops and bidder 1 gets the good. Otherwise, if

$$\frac{f_{LM}M + f_{LH}H - (f_{LM} + f_{LH})L}{f_L} < c < \frac{f_{MH}(H - M)}{f_M}, \tag{3}$$

it is optimal to stop if $x_1=L$ but continue if $x_1=M$.[4] Therefore, whenever (3) holds, the mechanism stops in period 1 if $x_1 \in \{L, H\}$. Hence, bidder 1's payment does not depend on x_2 when $x_1 \in \{L, H\}$ and by incentive compatibility must be equal. Because of the participation constraint, bidder 1's payment cannot exceed L when $x_1=L$. Hence, the seller cannot fully extract the surplus when $x_1=H$.

4. Almost full extraction of the surplus

Although full surplus extraction may be impossible, one can modify the efficient search procedure and achieve almost full extraction. This modification requires that the procedure always continues with a positive probability, thereby eliminating the ability of early entrants to exclude rivals.

We begin with the previous example. Assume that (i) both (2) and (3) hold, and (ii) the above joint probability matrix satisfies the cone condition for full extraction (Crémer and McLean, 1988; stated below as Assumption 1). Pick any small $\epsilon>0$. If bidder 1 reports M, continue. Otherwise, stop with probability $1 - \epsilon$ but continue with probability ϵ. Since bidder 2 is reached with a positive probability and the Crémer–McLean cone condition holds, it is possible to design a Crémer–McLean lottery for bidder 1 that induces him to be truthful in period 1. Given our assumption that bidders are not told the

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[4] If $x_1=L$, the stopping yields a surplus L while continuation yields $\frac{f_{LL}L + f_{LM}M + f_{LH}H}{f_L} - c$. If $x_1=M$, stopping yields M, while continuation yields $\frac{(f_{LM} + f_{MM})M + f_{MH}H}{f_M} - c$. When (3) holds, then $L > \frac{f_{LL}L + f_{LM}M + f_{LH}H}{f_L} - c$ and $M < \frac{(f_{LM} + f_{MM})M + f_{MH}H}{f_M} - c$.

messages of previous entrants, the posterior joint probability measure from bidder 2's viewpoint if the mechanism reaches him is:

		x_2		
		L	M	H
x_1	L	$\dfrac{\epsilon f_{LL}}{\epsilon f_{LL}+f_{LM}+\epsilon f_{LH}}$	$\dfrac{\epsilon f_{LM}}{\epsilon f_{LM}+f_{MM}+\epsilon f_{MH}}$	$\dfrac{\epsilon f_{LH}}{\epsilon f_{LH}+f_{MH}+\epsilon f_{HH}}$
	M	$\dfrac{\epsilon f_{LM}}{\epsilon f_{LL}+f_{LM}+\epsilon f_{LH}}$	$\dfrac{\epsilon f_{MM}}{\epsilon f_{LM}+f_{MM}+\epsilon f_{MH}}$	$\dfrac{\epsilon f_{MH}}{\epsilon f_{LH}+f_{MH}+\epsilon f_{HH}}$
	H	$\dfrac{\epsilon f_{LH}}{\epsilon f_{LL}+f_{LM}+\epsilon f_{LH}}$	$\dfrac{\epsilon f_{MH}}{\epsilon f_{LM}+f_{MM}+\epsilon f_{MH}}$	$\dfrac{\epsilon f_{HH}}{\epsilon f_{LH}+f_{MH}+\epsilon f_{HH}}$

Since the original joint probability matrix satisfies the Crémer–McLean cone condition, the conditional probability matrix also does. One can induce bidder 2 to be truthful by offering him a Crémer–McLean lottery contingent on bidder 1's report. Therefore, the seller can fully extract the surplus with probability $1 - \epsilon$, where ϵ can be arbitrarily small.

As each possible state has a positive prior probability, $f_{-i}(x_{-i}|x_i)$, the conditional probability of $x_{-i} \in X_{-i} \equiv x_{j \neq i} X_j$ given x_i is well defined. The next assumption is exactly the cone condition in Crémer and McLean (1988).

Assumption 1. For any bidder i and $x_i \in X_i$, the vector $f_{-i}(\,\cdot\,|x_i)$ does not belong to the cone generated by the vectors in the family $\{f_{-i}(\,\cdot\,|x_i') : x_i' \in X_i \setminus \{x_i\}\}$, i.e., there does not exist $\{\rho_i(x_i';x_i)\}_{x_i' \in X_i \setminus \{x_i\}}$ such that

$$f_i(x_{-i}|x_i) = \sum_{x_i' \in X_i \setminus \{x_i\}} \rho_i(x_i';x_i) f(x_{-i}|x_i') \qquad \text{for all } x_{-i} \in X_{-i}.$$

As in the above example, the main idea in the next theorem is to ensure a positive probability for the event of full participation, in which case Crémer–McLean lotteries can be carried out. Although this probability may be tiny, the lotteries can be scaled up to deter lying. The only complication is due to the fact that entrants can learn from the history of entry. To achieve full extraction, the seller needs to ensure that every entrant's posterior belief will satisfy the condition for full extraction, which requires a bidder's posterior conditional probabilities to be well defined. To guarantee that, we generalize the above ϵ-deviation technique into totally mixed strategies at the end of every period so that every entrant always assigns a positive posterior probability to any possible realized state.

Theorem 1. *Given* Assumption 1*, for any $\eta > 0$ there exists a search mechanism with which the seller obtains the maximum social surplus of the symmetric-information search problem with a probability at least $1 - \eta$.*

Proof. Pick a sufficiently small $\epsilon > 0$ such that $1 - \eta < (1 - \epsilon)^{n-1}$, where n is the size of I. Consider the following mechanism: In period 1, invite the entrants prescribed by the efficient procedure. In every period t, offer a menu of Crémer–McLean lotteries, specified below, to every period-t entrant, then

solicit secret reports from them. If all bidders have participated, stop. Otherwise, with probability $1-\epsilon$ follow the efficient search procedure in period $t+1$, and with probability ϵ randomly pick, with equal probability, a nonempty set of non incumbents and invite them in period $t+1$. If search stops, sell the good to a highest-value participant at a price equal to his reported value.

If the lotteries ensure zero expected payoff for truth-tellers and sufficiently large negative payoffs for liars, then they induce truth-telling. Therefore, the efficient procedure is implemented, and the seller obtains the entire social surplus, with probability at least $(1-\epsilon)^{n-1}$. The proof is completed by constructing such lotteries. To this end, consider any bidder i who enters at period $t=1, 2, \ldots$ Given i's report, $\hat{x}_i$, suppose that i is offered the following lottery: if search ends before all potential bidders participate, bidder i gets zero payoff; otherwise (full participation) bidder i gets a payoff equal to $\gamma_i(\hat{x}_i)g_i(\hat{x}_i, x_{-i})$ for some functions γ_i and g_i, where x_{-i} is the profile of reports from all potential bidders but i.

By the totally mixed strategy described above, given any profile x_{-i} there is a unique positive probability $a(x_{-i})$ with which the mechanism, coupled with x_{-i}, leads to the observed sequence of entry up to the current period.[5] Derived from the design of the mechanism, $a(x_{-i})$ is commonly known. Likewise, given any x_{-i} and $s=0, 1, 2, \ldots$, there is a unique positive probability $\beta(\hat{x}_i, x_{-i}, s)$ with which the mechanism, coupled with $(\hat{x}_i, x_{-i})$, leads to the observed sequence of entry up to the current period and will end with full participation in period $t+s$. Given $(\hat{x}_i, x_{-i}, s)$, this probability is commonly known. If bidder i's actual type is x_i, then his expected payoff from the lottery (viewed from the current period) is

$$\gamma_i(\hat{x}_i) \sum_{x_{-i} \in X_{-i}} \frac{f_{-i}(x_{-i}|x_i)}{b(x_i)} G_i(\hat{x}_i, x_{-i}), \tag{4}$$

where

$$b(x_i) \equiv \sum_{x'_{-i} \in X_{-i}} a(x'_{-i}) f_{-i}(x'_{-i}|x_i),$$

$$G_i(\hat{x}_i, x_{-i}) \equiv \sum_{s=0}^{\infty} \delta^s \beta(\hat{x}_i, x_{-i}, s) g_i(\hat{x}_i, x_{-i}).$$

We claim that the family $\left\{ \frac{f_{-i}(\cdot|x_i)}{b(x_i)} : x_i \in X_i \right\}$ of vectors satisfies the cone condition for full extraction (Assumption 1 with $\frac{f_{-i}(\cdot|x_i)}{b(x_i)}$ taking the role of $f_{-i}(\cdot|x_i)$); otherwise, there exists $x_i \in X_i$ and a nonnegative vector $(\lambda(x'_i))_{x'_i \in X_i \setminus \{x_i\}}$ such that

$$\frac{f_{-i}(x_{-i}|x_i)}{b(x_i)} = \sum_{x'_i \in X_i \setminus \{x_i\}} \lambda(x'_i) \frac{f_{-i}(x_{-i}|x'_i)}{b(x'_i)}, \qquad \forall x_{-i} \in X_{-i},$$

which implies that

$$f_{-i}(\cdot|x_i) = \sum_{x'_i \in X_i \setminus \{x_i\}} \frac{\lambda(x'_i)b(x_i)}{b(x'_i)} f_{-i}(\cdot|x'_i).$$

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[5] Actually $a(x_{-i})$ depends only on the reports of the incumbents before i enters.

This contradicts Assumption 1. Since the cone condition is satisfied, Farkas' lemma implies that there exists a function $G_i(\hat{x}_i, \cdot)$ that makes (4) zero if $\hat{x}_i = x_i$ and negative if $\hat{x}_i \neq x_i$. Then the lottery $\gamma_i(\hat{x}_i) g_i(\hat{x}_i, \cdot)$ is obtained by setting

$$g_i(\hat{x}_i, x_i) \equiv \frac{G_i(\hat{x}_i, x_{-i})}{\displaystyle\sum_{s=0}^{\infty} \delta^s \beta(\hat{x}_i, x_{-i}, s)},$$

where $\gamma_i(\hat{x}_i)$ is a sufficiently large scalar to ensure that bidder i's negative payoff when $\hat{x}_i \neq x_i$ outweighs the bidder's gain from buying the good. Thus, a Crémer–McLean lottery exists for i, as desired.   □

5. Discussion

We showed that although the dynamic nature of optimal search auctions precludes full extraction of the surplus as in the static auction case, it is nonetheless possible to fully extract the surplus with an arbitrarily high probability. It should be noted however that in the Crémer–McLean model, the size of necessary transfers is given and finite for any environment (albeit it need not be uniformly bounded when the environment varies). In Theorem 1, by contrast, the transfers need to be large in the low-probability event that full participation occurs and a bidder's report matches the others' poorly. When the seller reduces the probability of this inefficient event to arbitrarily close to zero, he needs to increase the transfers in this event without a bound.

Acknowledgement

References

Bergemann, Dirk, Pesendorfer, Martin, 2001. Information Structure in Optimal Auctions, Cowles Foundation Discussion Paper, vol. 1323.

Bergemann, Dirk, Välimäki, Juuso, 2002. Information acquisition and efficient mechanism design. Econometrica 70, 100–1033.

Burguet, Roberto, 1996. Optimal repeated purchases when sellers are learning about costs. Journal of Economic Theory 68, 440–455.

Crémer, Jacques, McLean, Richard, 1985. Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. Econometrica 53, 345–361.

Crémer, Jacques, McLean, Richard, 1988. Full extraction of the surplus in Bayesian and dominant strategy auctions. Econometrica 56, 1247–1257.

Crémer, Jacques, Spiegel, Yossi, Zheng, Charles, 2003. Optimal selling mechanisms with costly information acquisition, Mimeo.

Crémer, Jacques, Spiegel, Yossi, Zheng, Charles, in press. Optimal search auctions. Journal of Economic Theory.

McAfee, Preston, McMillan, John, 1988. Search mechanisms. Journal of Economic Theory 44, 99–123.

Ye, Lixin, 2004. Optimal auctions with endogenous entry. Contributions to Theoretical Economics 4, 8 (Article).