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On Symplectic Capacities and Volume Radius

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Abstract: *In this work we present an improvement to a theorem by C. Viterbo, relating the symplectic capacity of a convex body and its volume. This provides one more step towards the proof of the following conjecture: among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the Euclidean ball has maximal symplectic capacity. More precisely, the conjecture states that the best possible constant γ_n such that for any choice of a symplectic capacity c and any convex body $K \subset \mathbb{R}^{2n}$ we have*

$$\frac{c(K)}{c(B^{2n})} \leq \gamma_n \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}$$

is 1*. Until this work, the best estimate known to hold for general convex bodies, coming from Viterbo's work was $\gamma_n = 32n$, and $\gamma_n = 2n$ in the case of centrally symmetric bodies. Our main result in this text is that there exists a universal constant A for which $\gamma_n \leq A \log^2(n)$ for all convex bodies in $\mathbb{R}^{2n}$. Moreover, we show wide classes of convex bodies for which the inequality holds without the logarithmic term.*

1 Introduction and Main results

This paper lies at the meeting point of Asymptotic Geometric Analysis and Symplectic Geometry. In particular we use methods from Asymptotic Convex Geometry (sometimes called the Local Theory of Banach Spaces), to improve a result of Viterbo concerning symplectic capacities of convex bodies. These methods are linear in nature, and the reader should not expect any difficult symplectic analysis. However, we stress that the naive linear approach provides a significant improvement to the known results. The type of improvement we provide is a reduction from order n to

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order $\log(n)$ of a dimension-dependent isoperimetric constant, where n is the dimension of the space involved. Clearly, this reduction becomes especially relevant in large dimensions. Understanding the behavior of convex bodies in dimension tending to infinity is the main subject of Asymptotic Geometric Analysis, which we here join with the symplectic notion of capacity.

Consider the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ with the standard linear coordinates $(x_1, y_1, \ldots, x_n, y_n)$. One equippes this space with the standard symplectic structure $\omega_{st} = \sum_{j=1}^{n} dx_j \wedge dy_j$, and with the standard inner product $g_{st} = \langle \cdot, \cdot \rangle$. Note that under the identification (see notations below) between $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ these two structures are the real and the imaginary part of the standard Hermitian inner product in $\mathbb{C}^n$. In this work we consider the class of convex bodies in $\mathbb{R}^{2n}$. We are interested in comparing the symplectic way of measuring the size of a convex body, using what is called "symplectic capacities", with the standard Riemannian way limited here to volume. In order to make this more precise we need some preliminaries.

Definition 1.1. *A symplectic capacity on $(\mathbb{R}^{2n}, \omega_{st})$ associates to each subset $U \subset \mathbb{R}^{2n}$ a non-negative number $c(U)$ such that the following three properties hold:*

(P1) $c(U) \leq c(V)$ *for* $U \subseteq V$ *(monotonicity)*

(P2) $c\big(\psi(U)\big) = |\alpha|\, c(U)$ *for* $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ *such that* $\psi^* \omega_{st} = \alpha\, \omega_{st}$ *(conformality)*

(P3) $c\big(B^{2n}(r)\big) = c\big(B^2(r) \times \mathbb{C}^{n-1}\big) = \pi r^2$ *(nontriviality and normalization),*

where $B^{2k}(r)$ is the open $2k$-dimensional ball of radius r. Note that the third property disqualifies any volume-related invariant, while the first two properties imply that every two sets $U, V \subset \mathbb{R}^{2n}$ will have the same capacity provided that there exists a symplectomorphism sending U onto V. Recall that a *symplectomorphism* of $\mathbb{R}^{2n}$ is a diffeomorphism which preserves the symplectic structure i.e., $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ such that $\psi^* \omega_{st} = \omega_{st}$. We will denote by $\mathrm{Symp}(\mathbb{R}^{2n}) = \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{st})$ the group of all the symplectomorphisms of $(\mathbb{R}^{2n}, \omega_{st})$.

A priori, it is not clear that symplectic capacities exist. The celebrated non-squeezing theorem of Gromov [6] shows that for $R > r$ the ball $B^{2n}(R)$ does not admit a symplectic embedding into the symplectic cylinder $Z^{2n}(r) := B^2(r) \times \mathbb{C}^{n-1}$. This theorem led to the following definitions:

Definition 1.2. *The symplectic radius of a non-empty set $U \subset \mathbb{R}^{2n}$ is*

$$c_B(U) := \sup \left\{ \pi r^2 \mid \textit{ There exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \textit{ with } \psi\left(B^{2n}(r)\right) \subset U \right\}.$$

The cylindrical capacity of U is

$$c^Z(U) := \inf \left\{ \pi r^2 \mid \textit{ There exists } \psi \in \mathrm{Symp}(\mathbb{R}^{2n}) \textit{ with } \psi(U) \subset Z^{2n}(r) \right\}.$$

Note that both the symplectic radius and the cylindrical capacity satisfy the axioms of Definition 1.1 by the non-squeezing theorem. Moreover, it follows from Definition 1.1 that for every symplectic capacity c and every open set $U \subset \mathbb{R}^{2n}$ we have $c_B(U) \leq c(U) \leq c^Z(U)$.

The above axiomatic definition of symplectic capacities is originally due to Ekeland and Hofer [3]. Nowadays, a variety of symplectic capacities can be constructed in different ways. For several of the detailed discussions on symplectic capacities we refer the reader to [2], [8], [9], [11], [14] and [24].

In this work we are interested in an inequality relating the symplectic capacity of a convex body in $\mathbb{R}^{2n}$ and its volume. By a convex body we mean a convex bounded set in $\mathbb{R}^{2n}$ with non-empty interior. This inequality supports the conjecture that among all convex bodies in $\mathbb{R}^{2n}$ with a given volume, the symplectic capacity is maximal for the Euclidean ball. Note that by monotonicity this is obviously true for the symplectic radius c_B. More precisely, denote by $\mathrm{Vol}(K)$ the volume of K and abbreviate B^{2n} for the open Euclidean unit ball in $\mathbb{R}^{2n}$. Following Viterbo [23] we state

Conjecture 1.3. *For any symplectic capacity c and for any convex body $K \subset \mathbb{R}^{2n}$*

$$\frac{c(K)}{c(B^{2n})} \leq \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n}$$

and equality is achieved only for symplectic images of the Euclidean ball.

The first result in this direction is due to Viterbo [23]. Using John's ellipsoid (which also evolved in Convex Geometric Analysis) he proved:

Theorem 1.4 (Viterbo). *For a convex body $K \subset \mathbb{R}^{2n}$ and a symplectic capacity c*

$$\frac{c(K)}{c(B^{2n})} \leq \gamma_n \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/n}$$

where $\gamma_n = 2n$ if K is centrally symmetric and $\gamma_n = 32n$ for general convex bodies.

In [7], Hermann constructed a starshaped domain in $\mathbb{R}^{2n}$, for $n > 1$, with arbitrarily small volume and some fixed cylindrical capacity. Therefore, in the category of starshaped domains the above theorem with *any* constant γ_n independent of the body K, must fail. In addition, he proved the above conjecture for a special class of convex bodies which admit many symmetries, called convex Reinhardt domains (for definitions see [7]).

Here we provide one more step towards the proof of the above conjecture. Before we state our main results we wish to emphasize that we work exclusively in the category of linear symplectic geometry. That is, we restrict ourselves to the concrete

class of linear symplectic transformations. It turns out that even in this limited category, there are tools which are powerful enough to obtain a significant improvement of Theorem 1.4 above. More precisely, let $\mathrm{Sp}(\mathbb{R}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{st})$ denote the group of linear symplectic transformation of $\mathbb{R}^{2n}$. We consider a more restricted notion of linearized cylindrical capacity, which is similar to c^Z but where the transformation ψ is taken only in $\mathrm{Sp}(\mathbb{R}^{2n})$ namely

$$c^Z_{lin}(U) := \inf \left\{ \pi r^2 \mid \textit{There exists } \psi \in \mathrm{Sp}(\mathbb{R}^{2n}) \textit{ with } \psi(U) \subset Z^{2n}(r) \right\}.$$

Of course, it is always true that for every symplectic capacity c we have $c \leq c^Z \leq c^Z_{lin}$.

Our main result is the following

Theorem 1.5. *There exists a universal constant A_1 such that for every even dimension $2n$ and any convex body $K \subset \mathbb{R}^{2n}$ we have*

$$\frac{c^Z_{lin}(K)}{c(B^{2n})} \leq A_1 \log^2(n) \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}.$$

The theorem clearly implies that Theorem 1.4 holds with $\gamma_n = A_1 \log^2(n)$. We remark that the methods Viterbo used to prove Theorem 1.4 were also linear.

In addition we show that there are certain, quite general, families of convex bodies for which the above inequality is true without the logarithmic factor. Such examples are the unit balls of ℓ_p^n for $1 \leq p \leq \infty$, all *zonoides* (symmetric convex bodies which can be approximated by Minkowski sums of line segments in the Hausdorff sense, in particular all the projections of cubes), and all bodies which have satisfy some more complicated geometric conditions called bounded type-p constant for $p > 1$. The precise definitions and results for special classes of convex bodies will be given in Section 5 below.

Some of the above mentioned examples are based on a strengthened formulation of Theorem 1.5. This strengthening is attained is by bounding from above the symplectic capacity of a convex body K by some parameter of the body which measures, roughly speaking, the "difference" between the Banach space X_K whose unit ball is the body K and a Hilbert space. In order to be more precise we need to introduce the notion of the K-convexity constant of a Banach space X, known also as the Rademacher projection constant. We begin with the following preliminaries. Consider the group $\Omega^m = (\mathbb{Z}/2\mathbb{Z})^m \simeq \{-1, 1\}^m$ and let μ denote the uniform probability measure on Ω^m (i.e., normalized counting measure). For each $A \subseteq \{1, \ldots, m\}$ we define the Walsh function $W_A \in L^2(\Omega^m, \mathbb{R})$ by

$$W_A(t) = \prod_{i \in A} r_i(t) = \prod_{i \in A} t_i,$$

where $r_i : \Omega^m \to \{-1, 1\}$ are the Rademacher functions $r_i(t) = t_i$, i.e. the ith coordinate function (so, $r_i = W_{\{i\}}$). We set $W_\emptyset \equiv 1$. Note that $|W_A| = 1$ for all $A \subseteq \{1, \ldots, n\}$ and for every $A \neq B$ the functions W_A, W_B are orthogonal, namely

$$\langle W_A, W_B \rangle = \int_{\Omega^m} \prod_{i \in A} r_i \prod_{j \in B} r_j \, d\mu = \int_{\Omega^m} \prod_{k \in A \triangle B} r_k \, d\mu = \prod_{k \in A \triangle B} \int_{\Omega^m} r_k \, d\mu = 0$$

The Walsh functions form an orthonormal basis in $L_2(\Omega^m, \mathbb{R})$. Alternatively, $(W_A)_{A \subseteq [1,\ldots,m]}$ is the group of characters of the multiplicative group $(\mathbb{Z}/2\mathbb{Z})^m$. For a Banach space X define

$$L_2(\Omega^m, X) = \Big\{ f : \Omega^m \to X \ ; \ \|f\|_{L_2(\Omega^m, X)} = \Big(\int_{\Omega^m} \|f\|_X^2 \, d\mu \Big)^{1/2} \Big\}.$$

In general $L_2(\Omega^m, X)$ is always a Banach space, and is a Hilbert space if and only if X is. Still, every $f \in L_2(\Omega^m, X)$ can be represented as

$$f = \sum_{A \subseteq \{1,\ldots,m\}} \widehat{f}(A) W_A, \text{ where } \widehat{f}(A) = \int_{\Omega^m} f W_A \, d\mu = \frac{1}{2^m} \sum_{t \in \Omega^m} f(t) W_A(t).$$

We will be interested in a one special subspace of $L_2(\Omega^m, X)$ namely the one spanned by the m Rademacher functions:

$$Rad_m X = \Big\{ \sum_{i=1}^m x_i r_i \ ; \ x_i \in X \ , \ i = 1, \ldots, m \Big\},$$

equipped with the L_2 norm. Consider the Rademacher projection operator $\mathcal{R}_m : L_2(\Omega^m, X) \to Rad_m X$ defined as follows

$$\mathcal{R}_m(f) = \sum_{i=1}^m \widehat{f}(\{i\}) r_i.$$

Definition 1.6. *The Rademacher projection constant of a Banach space X is the supremum of the operator norms of the projections $\mathcal{R}_m$, i.e.*

$$\|\mathrm{Rad}\|_X := \sup_m \|\mathcal{R}_m\|.$$

This is also known in the literature as the K-convexity constant of X. Note that for an infinite dimensional Banach space X, this constant may be infinite. When for some (infinite dimensional) Banach space X the number $\|\mathrm{Rad}\|_X = C$ is finite, X is called K-convex with K-convexity constant C. In what follows we shall deal only with finite dimensional spaces, and it is not difficult to check that they are always K-convex. However, the dependence of the K-convexity constant on the dimension of the space will be of interest to us. For example, if we consider a family of finite

dimensional Banach spaces X_k (with increasing dimension, say) which all arise as subspaces of some fixed infinite dimensional Banach space X which is K-convex, then we know that the K-convexity constants of the spaces X_k are uniformly bounded by the K-convexity constant of X.

In the special case where X is an Hilbert space, each $\mathcal{R}_m$ is an orthogonal projection of norm equal to 1. A fundamental theorem of Pisier, see [18], states that for every Banach space X which is isomorphic to a Hilbert space H

$$\|\mathrm{Rad}\|_X \leq c \log[d(X,H)+1],$$

where c is a universal constant and d denotes the Banach-Mazur distance, defined for two isomorphic normed spaces as $\inf\{\|T\| \, \|T^{-1}\|\}$ where the infimum runs over all isomorphism $T : X \to H$. Notice that for a k-dimensional X, John's Theorem (see [16]) implies that $d(X, \ell_2^k) \leq \sqrt{k}$. Thus, combining John's Theorem and Pisier's result we obtain that Theorem 1.5 actually follows from

Theorem 1.7. *There exists a universal constant A_2 such that for every even dimension $2n$ and any convex body $K \subset \mathbb{R}^{2n}$*

$$\frac{c_{lin}^Z(K)}{c(B^{2n})} \leq A_2 \|\mathrm{Rad}\|_{X_K}^2 \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n}$$

where X_K is the Banach space whose unit ball is the body K.

There are wide classes of Banach spaces which are K-convex, and this implies that so are all of their subspaces, with a uniform bound on their K-convexity constant. Thus they will generate families of convex bodies for which we will have a better bound on γ_n in Theorem 1.4. In fact, X not being K-convex is equivalent to having an ε-isometric copy, with respect to the above mentioned Banach-Mazur distance, of ℓ_1^m inside X for every $\varepsilon > 0$ and every m (see e.g. [17]). In particular the (infinite dimensional) space ℓ_p of infinite p-summable sequences for $1 < p < \infty$ is K-convex (with a K-convexity constant depending on p) and therefore for each $1 < p < \infty$, for all finite dimensional ℓ_p^n we will have a uniform bound on γ_n which will depend only on p. In section 5 we shall discuss these examples together with other families of convex bodies for which Theorem 1.7 holds without the logarithmic factor (and in particular show that for ℓ_p^n we can have γ_n independent also of p).

Symmetric vs. non-symmetric case: Throughout the paper, it would be enough to assume that our body K is centrally symmetric i.e., $K = -K$. Indeed, assume we have a general convex body $K \subset \mathbb{R}^{2n}$, we consider the difference body $K' = K - K$, that is, $K' = \{x - y : x, y \in K\}$. Of course, $K \subset K'$, so that for any symplectic capacity $c(K) \leq c(K')$, and K' is centrally symmetric. Moreover, the Rogers-Shephard inequality [19] implies that

$$\mathrm{Vol}(K') \leq 4^{2n} \mathrm{Vol}(K),$$

Thus, knowing the inequalities in Theorem 1.5 and Theorem 1.7 for K' implies the same inequalities with an extra factor 16 for a general convex body K.

Non-linear methods: It is worthwhile to mention that there are non-linear methods of symplectic embedding constructions, known as "symplectic folding" and "symplectic wrapping", which one might use when approaching Conjecture 1.3. We refer the readers to [12], [20] for more details on this subject. These methods have been successfully used in some symplectic embedding constructions of concrete convex bodies, see e.g. [20], [22].

Notations: In this paper the letters $A_0, A_1, \ldots$, are used to denote universal positive constants which do not depend on the dimension nor the body involved. In what follows we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by associating to $z = x + iy$, where $x, y \in \mathbb{R}^n$, the vector $(x_1, y_1, \ldots, x_n, y_n)$, and consider the standard complex structure given by complex multiplication by i, .e. $i(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n)$. Note that under this identification $\omega(v, iv) = \langle v, v\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^{2n}$. We will use the notion "holomorphic plane" for a real 2-dimensional plane generated by two vectors of the form v, iv. We shall denote $P_E K$ the orthogonal projection of a body K on a subspace E, and by $\alpha B_E \equiv B_E(\alpha)$ the open ball of radius α in the subspace E. We will denote the volume of the $2n$-dimensional Euclidean ball by κ_{2n}, and sometimes use the estimates

$$\left(\frac{\pi e}{n}\right)^n \left(\frac{1}{\sqrt{2\pi n e}}\right) \leq \kappa_{2n} = \frac{\pi^n}{n!} \leq \left(\frac{\pi e}{n}\right)^n \left(\frac{1}{\sqrt{2\pi n}}\right) \leq \left(\frac{\pi e}{n}\right)^n. \tag{1.1}$$

We will use the notation ℓ_p with $1 \leq p < \infty$ for the space of infinite p-summable sequences endowed with the norm $(\sum_i |x_i|^p)^{1/p}$, and ℓ_∞ for the space of bounded infinite sequences with norm given by $\sup_i |x_i|$. We use ℓ_p^n to denote $\mathbb{R}^n$ with the norm $(\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $\sup_{i=1,\ldots,n} |x_i|$ for $p = \infty$.

Structure of the paper: The paper is organized as follows. We first consider some concrete examples, where we can show directly the validity of Conjecture 1.3 up to a universal constant. In Section 3 we introduce some of the main ingredients in the proof of Theorem 1.7. In Section 4 we show how the method that works in the concrete examples of Section 2 enables us to use some simple linear algebra, together with the deep bounds following from Pisier's work mentioned above and the ingredients from Section 3, to prove Theorem 1.7. Finally, in the last section we discuss some families of convex bodies where Theorem 1.5 holds in improved form i.e., without the logarithmic factor.

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2 Some Concrete Examples

In this section we estimate the cylindrical capacity of some concrete convex bodies. We show that for these examples Conjecture 1.3 holds up to a constant factor. These examples, in particular the techniques used for the examples of the ellipsoid and of the cross polytope, will guide us in the proof of the general case. Other, more general examples, will be discussed in Section 5.

2.1 Ellipsoids

The fact that for an ellipsoid E one has $c_B(E) = c^Z(E)$ (and therefore Conjecture 1.3 holds) is well known (see, e.g., [9], [15]). However, we wish to remark on its proof since it provides some preliminary intuition for the general case.

First of all, consider the case where E is a *symplectic ellipsoid*, given by $E = DB^{2n}$ where $D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n)$ with $0 < r_1 < r_2 < \ldots < r_n$. In this case we clearly have that E lies "between" a ball and a cylinder of the same radii, i.e., $B^{2n}(r_1) \subset E \subset Z^{2n}(r_1)$, and hence, $c_B(E) = c^Z(E)$.

For E an ellipsoid in general position, one can find suitable (linear) symplectic coordinates in which E becomes a symplectic ellipsoid. We discuss this well known fact in Section 1.7, and as in the proof of Corollary 4.3 below, it means that one has $SE = DB^{2n}$ for some symplectic matrix S and a positive diagonal matrix $D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n)$. Using the above we see that

$$c_B(E) = c_B(SE) = c_B(DB^{2n}) = c^Z(DB^{2n}) = c^Z(SE) = c^Z(E).$$

2.2 The Cube

Let $Q = [-1, 1]^{2n}$ be the unit cube in $\mathbb{R}^{2n}$. In this case it is not hard to check that

$$c^Z(Q) \leq 2\pi, \quad \text{while} \quad \left(\frac{\mathrm{Vol}(Q)}{\mathrm{Vol}(B^{2n})}\right)^{1/n} = \frac{4}{\sqrt[n]{\kappa_{2n}}} \geq \frac{4n}{\pi e}.$$

Indeed, the inequality on the left follows since $Q \subset Z^{2n}(\sqrt{2})$ and the inequality on the right follows from the estimate (1.1) above. Thus, in the case of the cube, we see that Theorem 1.4 holds with a constant $\gamma_n \simeq 1/n$, in particular (at least asymptotically, but it is not hard to check the constants in general) with constant 1.

Remark 2.1. In fact, using that $Q \subset [-1, 1]^2 \times \mathbb{R}^{2n-2}$, we see that $c^Z(Q) = 4$; however, to embed the cube into $Z^{2n}(r)$ with $\pi r^2 = 4$ one steps out of the linear category.

For the following linear image of a cube: $\widetilde{Q} = DQ$, where $D = diag(a_1, b_1 \dots a_n, b_n)$ for some positive real numbers a_i, b_i we have

$$c^Z(\widetilde{Q}) \leq 2\pi \min_i a_i b_i \quad \text{while} \quad \left(\frac{\mathrm{Vol}(\widetilde{Q})}{\mathrm{Vol}(B^{2n})}\right)^{1/n} = \frac{4}{\sqrt[n]{\kappa_{2n}}} \sqrt[n]{\prod a_i b_i} \geq \frac{4n}{\pi e} \sqrt[n]{\prod a_i b_i}$$

Indeed, the linear symplectic transformation $S = diag\Big(\sqrt{\frac{b_1}{a_1}}, \sqrt{\frac{a_1}{b_1}}, \dots, \sqrt{\frac{b_n}{a_n}}, \sqrt{\frac{a_n}{b_n}}\Big)$ satisfies that $S\widetilde{Q} \subset Z^{2n}(\sqrt{2}\min a_i b_i)$ (Again, the bound can be improved by using a non-linear symplectomorphism to $c^Z(\widetilde{Q}) \leq 4\min_i a_i b_i$). We see thus that applying a diagonal transformation to the cube only improves the inequality, and Theorem 1.4 holds for this body with a constant $\gamma_n \simeq 1/n$.

2.3 The Cross-Polytope

The cross-polytope in dimension $2n$ is the polytope corresponding to the convex hull of the $4n$ points $\pm e_i, i = 1, \dots, 2n$ formed by permuting the coordinates $(\pm 1, 0, 0, ..., 0)$. Since this is the unit ball of the ℓ_1 norm on $\mathbb{R}^{2n}$, we shall denote it by B_1^{2n}, so, $B_1^{2n} = \mathrm{Conv}\{\pm e_i\}$. We claim that

$$c^Z(B_1^{2n}) \leq \frac{\pi}{n} \quad \text{and} \quad \left(\frac{\mathrm{Vol}(B_1^{2n})}{\mathrm{Vol}(B^{2n})}\right)^{1/n} = \left(\frac{\frac{2^{2n}}{2n!}}{\kappa_{2n}}\right)^{1/n} \geq \frac{2}{\pi n}.$$

The inequality on the right hand side follows from a direct computation (using the above estimate (1.1)). In order to estimate from above the cylindrical capacity of B_1^{2n} we will find an holomorphic 2-dimensional plane E such that the projection $P_E B_1^{2n}$ of B_1^{2n} into E is "small". More precisely, consider the following unit vectors

$$v = \frac{1}{\sqrt{2n}}(1, \dots, 1), \quad iv = \frac{1}{\sqrt{2n}}(-1, 1, \dots, -1, 1).$$

Note that $v \perp iv$ and that $w_{std}(v, iv) = 1$. Denote $E_0 = \mathrm{span}\{e_1, e_2\}$ and $E = \mathrm{span}\{v, iv\}$. Note that $E = UE_0$ for some $U \in U(n)$. Next we consider the projection of B_1^{2n} to the subspace E. It follows from the definition that

$$P_E B_1^{2n} = P_E \mathrm{Conv}\{\pm e_i\} = \mathrm{Conv} P_E\{\pm e_i\}.$$

A direct computation shows that

$$P_E e_j = \langle e_j, v\rangle v + \langle e_j, iv\rangle iv = \begin{cases} \frac{1}{n}(1, 0, \dots, 1, 0) & j \text{ is odd}, \\ \frac{1}{n}(0, 1, \dots, 0, 1) & j \text{ is even}. \end{cases}$$

For every j we have that the Euclidean norm $\|P_E e_j\| = \frac{1}{\sqrt{n}}$, and thus the diameter of $P_E B_1^{2n}$ is equal $\frac{1}{\sqrt{n}}$ which in turn implies that $c^Z(B_1^{2n}) = c^Z(U^{-1}B_1^{2n}) \leq \frac{\pi}{n}$.

Moreover, a direct calculation shows that the cross-polytope B_1^{2n} includes the Euclidean ball $\frac{1}{\sqrt{2n}}B^{2n}$, so that $c_B(B_1^{2n}) \geq \frac{\pi}{2n}$. In particular, we get that up to constant 2 the two capacities c_B and c_Q are equivalent, and hence Theorem 1.4 holds with $\gamma = 2$. By using the bound (1.1) for the volume of the Euclidean ball we get the slightly better constant $\gamma = e/\pi$.

Next we consider the linear image $\widetilde{B}_1^{2n} = DB_1^{2n}$ where $D = diag(a_1, b_1 \dots a_n, b_n)$ for some positive real numbers a_i, b_i such that $\det D = 1$. By applying the linear symplectomorphism S from the above example of the cube we can assume without loss of generality that $a_i = b_i$ for $i = 1 \dots n$. As before we are looking for a 2-dimensional holomorphic plane E such that the projection of $\widetilde{B}_1^{2n}$ into E has small diameter. We choose the direction

$$\widehat{v} = \gamma(a_1^{-1}, a_1^{-1}, \dots, a_n^{-1}, a_n^{-1}), \quad \text{where} \quad \gamma = \Big(\sum_{i=1}^{n} 2a_i^{-2}\Big)^{-\frac{1}{2}},$$

which is easily checked to be a direction on which the projection of $\widetilde{B}_1^{2n}$ has minimal length. Together with it we take $i\widehat{v} = \gamma(-a_1^{-1}, a_1^{-1}, \dots, -a_n^{-1}, a_n^{-1})$ (both are unit vectors). Note that $\widehat{v} \perp i\widehat{v}$ and $\omega_{std}(\widehat{v}, i\widehat{v}) = 1$. Let $E = \text{span}\{\widehat{v}, i\widehat{v}\}$. In order to bound the diameter of the projection of $\widetilde{B}_1^{2n}$ into the subspace E we compute the projections:

$$\begin{cases} P_E a_k e_{2k-1} = a_k \langle e_{2k-1}, \widehat{v}\rangle \widehat{v} + a_k \langle e_{2k-1}, i\widehat{v}\rangle i\widehat{v} = \gamma^2(2a_1^{-1}, 0, \dots, 2a_n^{-1}, 0) \\ P_E a_k e_{2k} = a_k \langle e_{2k}, \widehat{v}\rangle \widehat{v} + a_k \langle e_{2k}, i\widehat{v}\rangle i\widehat{v} = \gamma^2(0, 2a_1^{-1}, \dots, 0, 2a_n^{-1}) \end{cases}$$

It follows that the Euclidean lengths $\|P_E a_k e_{2k-1}\| = \|P_E a_k e_{2k}\| = \sqrt{2}\gamma$ for $k = 1, \dots n$. We notice that $n(\sqrt{2}\gamma)^2$ is the harmonic mean of the numbers $a_i, i = 1, \dots, n$, and therefore is smaller than their geometric mean which equals 1. Thus, $\gamma\sqrt{2} \leq 1/\sqrt{n}$, and hence, as in the not-distorted B_1^{2n}, we have that $c^Z(\widetilde{B}_1^{2n}) \leq \frac{\pi}{n}$. So, in this case, again, Theorem 1.5 holds with a constant instead of a logarithmic factor.

3 First estimates

One of the main ingredients in the proof of Theorem 1.7 is a connection between the symplectic measure of a convex body K, given by its symplectic capacity, and classical notions of the "width" of a convex body K. To be more precise we need several more definitions. For a non-empty centrally symmetric convex body K in $\mathbb{R}^{2n}$ we denote by $\|\cdot\|_K$ the norm on $\mathbb{R}^{2n}$ induced by K, that is, $\|x\|_K = \inf\{r : x \in rK\}$. We set

$$M(K) = \int_{S^{2n-1}} \|x\|_K \sigma(dx),$$

for the average of the norm $\|\cdot\|_K$ on the sphere S^{2n-1}, and define $M^*(K) := M(K^\circ)$ where K° is the polar body of K defined by $K^\circ := \{x \in \mathbb{R}^{2n} : \langle x, y\rangle \leq 1, \forall y \in K\}$. The number $M^*(K)$ is called half the mean width of K because

$$M^*(K) = \int_{S^{2n-1}} \sup_{y\in K} \langle x, y\rangle \sigma(dx),$$

where, we integrate over all unit directions x, half the distance between two parallel hyperplanes touching K and perpendicular to the vector x (which is called the width of K in direction x). We may assume without loss of generality that K is indeed centrally symmetric, as was explained in the introduction.

We will sometimes prefer to use instead of the mean width $M^*(K)$ a discrete analogue called the (normalized) Rademacher average, which is defined as follows.

$$s^*(K) = \mathrm{Ave}_{\varepsilon_i = \pm\frac{1}{\sqrt{2n}}} \sup\Big\{\sum_i \varepsilon_i x_i : x \in K\Big\} = \mathrm{Ave}_{\varepsilon\in\{\frac{-1}{\sqrt{2n}}\frac{1}{\sqrt{2n}}\}^{2n}} \|\varepsilon\|_{K^\circ}$$

That is, we average the dual norm of the body K (or half the width of K), not on the whole sphere of directions as in the definition of the mean width $M^*(K)$, but only on the vertices of the normalized cube (in Banach Space theory one usually use the non-normalized version, $r^* = \sqrt{2n}s^*$).

This parameter is much more similar to M^* than it seems at first. Indeed one can show (see [16]) that there exists a universal constant A_3 such that for every dimension $2n$ and every symmetric convex body $K \subset \mathbb{R}^{2n}$ one has

$$A_3^{-1} s^*(K) \leq M^*(K) \leq A_3 \|\mathrm{Rad}\|_{X_K} s^*(K), \tag{3.1}$$

where X_K is the Banach space whose unit ball is K. Below we use only the left hand side inequality, which is not difficult to prove, and is true with constant $A_3 = \sqrt{\pi/2}$. In fact, we could instead use below the trivial fact that $\int_{U(n)} s^*(UK) = M^*(K)$. The following theorems give an upper bound for the cylindrical capacity of a convex body K in terms of $M^*(K)$ and $s^*(K)$ respectively. (Notice that by the above remarks Theorem 3.2 is formally stronger than Theorem 3.1. However, they admit an almost identical proof).

Theorem 3.1. *There exists a universal constant A_4 such that for every even dimension $2n$ and any centrally symmetric convex body $K \subset \mathbb{R}^{2n}$, there exist a holomorphic plane $E \subset \mathbb{R}^{2n}$ such that*

$$P_E(K) \subset A_4 B_E^2(M^*(K)).$$

In particular, it follows from the monotonicity property of symplectic capacities that

$$c(K) \leq \pi A_4^2 M^*(K)^2.$$

Theorem 3.2. *There exists a universal constant A_5 such that for every even dimension $2n$ and any centrally symmetric convex body $K \subset \mathbb{R}^{2n}$, there exist a holomorphic plane $E \subset \mathbb{R}^{2n}$ such that*

$$P_E(K) \subset A_5 B_E^2(s^*(K)).$$

In particular, it follows from the monotonicity property of symplectic capacities that

$$c(K) \leq \pi A_5^2 s^*(K)^2.$$

Before we prove these two theorems let us discuss the relation between Theorem 3.1 and Theorem 1.7. First notice that the estimate in Theorem 3.1 is weaker than Conjecture 1.3: Indeed, Urysohn's inequality (see e.g. Corollary 1.4 in [17]) gives

$$\left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/2n} \leq M^*(K).$$

However, one can ask whether there are bodies for which there is equivalence of the two. In some sense this is indeed the case. In particular, there exists a universal constant A_6 such that every symmetric convex body $K \subset \mathbb{R}^{2n}$ has a position K' (that is, a volume preserving linear transformation T of $\mathbb{R}^{2n}$ and $K' = TK$) in which

$$M^*(K') \leq A_6 \|\mathrm{Rad}\|_{X_K} \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/2n},$$

where X_K is the Banach space whose unit ball is K. We will discuss this well known fact in more detail in Section 4. Recall that for a body $K \subset \mathbb{R}^{2n}$ we have $\|\mathrm{Rad}\|_{X_K} \leq c\log(2n+1)$ for a universal c. Unfortunately, the above mentioned transformation T need not be symplectic, and we will address this problem too in Section 4. Further, for some classes of bodies, the term $\|\mathrm{Rad}\|_{X_K}$ above can be eliminated. We discuss this in Section 5.

In the remainder of this section we prove Theorems 3.1 and 3.2. Recall that Markov's inequality states that if (X, S, μ) is a measure space, f is a measurable real-valued function, and $t > 0$, then

$$\mu(\{x \in X \mid |f(x)| \geq t\}) \leq \frac{1}{t}\int_X f d\mu.$$

Proof of Theorem 3.1. Consider the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ equipped with the canonically defined normalized Haar measure μ. Define $f : S^{2n-1} \to \mathbb{R}$ by $f(x) = \|x\|_{K^\circ}$. It follows from Markov's inequality above that for $t = \alpha M^*(K)$

$$\mu(\{x \in S^{2n-1} \;\Big|\; \|x\|_{K^\circ} \geq \alpha M^*(K)\}) \leq \frac{1}{\alpha M^*(K)} \int_{S^{2n-1}} \|x\|_{K^\circ} d\mu = 1/\alpha. \qquad (3.2)$$

Next, for every unit vector $x = (x_1, x_2, x_3, x_4, \ldots, x_{2n-1}, x_{2n}) \in S^{2n-1}$ consider $ix = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1}) \in S^{2n-1}$. Note that $x \perp ix$ and $\omega(x, ix) = 1$.

By substituting α large enough, say 3, in (3.2) above, we get that for measure $1 - 1/\alpha = 2/3$ of the vectors x on the sphere we have that $\|x\|_{K^\circ} \leq \alpha M^*(K)$. So, at least $1 - 2/\alpha = 1/3$ of the couples (x, ix) satisfy that this is true for *both* of them. Hence, for at least $1/3$ of the vectors $x \in S^{2n-1}$ we have

$$P_E K \subset \alpha\sqrt{2} M^*(K) B_E^2, \quad \text{where } E = \text{span}\{x, ix\}.$$

Note that the subspace E is a unitary image of the subspace $E_0 = \text{span}\{e_1, e_2\}$, where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$. Since a unitary transformation preserves the symplectic structure and since the symplectic capacity is monotone, we have that the capacity $c^Z(K)$ is at most the capacity of a ball of radius $\alpha\sqrt{2}M^*(K)$. This completes the proof of the theorem, with $A_4 = 3\sqrt{2}$. □

Remark 3.3. Actually, this theorem can be deduced directly from Dvoretzky's theorem (see e.g. [16]) about random sections or projections of convex bodies which implies, roughly speaking, that most projections of a convex body K of the appropriate dimension or lower are Euclidean balls of diameter approximately $M^*(K)$. This can be made precise: for two dimensional sections given by UE_0 for $U \in U(n)$ and $E_0 = span\{x_1, y_1\}$, it can be shown that under mild assumptions the above is true for large (Haar) measure of $U \in U(n)$ (see, e.g., [1]). However, since all we need is a two dimensional subspace, and since we are willing to sacrifice universal constants (such as $\sqrt{2}$ above), a much more low-tech approach which is based only on Markov's inequality worked equally well.

Proof of Theorem 3.2. This proof is almost identical to the above. Instead of considering the unit sphere we consider $\{-1/\sqrt{2n}, 1/\sqrt{2n}\}^{2n} \subset \mathbb{R}^{2n}$ equipped with the uniform (normalized) counting measure μ. It follows from Markov's inequality that for $t = \alpha s^*(K)$

$$\mu(\{\varepsilon \in \{\pm 1/\sqrt{2n}\}^{2n} \;\Big|\; \|\varepsilon\|_{K^\circ} \geq \alpha s^*(K)\}) \leq 1/\alpha. \tag{3.3}$$

As before, for every vector $x = (x_1, x_2, x_3, x_4, \ldots, x_{2n-1}, x_{2n}) \in \{\pm 1/\sqrt{2n}\}^{2n}$ we consider $ix = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1})$, which is also in $\{\pm 1/\sqrt{2n}\}^{2n}$. Again $x \perp ix$ and $\omega(x, ix) = 1$, and with $\alpha = 3$ in (3.3) above we get that at least $1 - 2/\alpha = 1/3$ of the 2^{2n} couples (x, ix) satisfy that this is true for *both* of them. In particular we have one such couple with

$$P_E K \subset \alpha\sqrt{2} s^*(K) B_E^2, \quad \text{where } E = \text{span}\{x, ix\}.$$

As before the subspace E is a unitary image of the subspace $E_0 = \text{span}\{e_1, e_2\}$, so the capacity $c^Z(K)$ is at most the capacity of a ball of radius $\alpha\sqrt{2}s^*(K)$. This completes the proof of the theorem, with $A_5 = 3\sqrt{2}$. □

4 Proof of Theorem 1.7

As mentioned before, a key ingredient in the proof of Theorem 1.7 is the upper estimate of the symplectic capacity of a convex body K in terms of its mean width i.e.,

$$c(K) \leq \pi(A_4 M^*(K))^2$$

given by Theorem 3.1. In order to find an upper bound for the mean width of a convex body in terms of its volume radius we use a result by Figiel and Tomczak-Jaegermann [4], which uses a previous result of Lewis [13], stating that every centrally symmetric convex body $K \subset \mathbb{R}^n$ has a position TK, where T is a volume preserving linear transformation, such that

$$M(TK)M^*(TK) \leq A_6 \|\mathrm{Rad}\|_{X_K},$$

where X_K is the Banach space whose unit ball is the body K and A_6 is universal. Combine this with the fact that

$$\frac{1}{M(TK)} \leq \left(\frac{\mathrm{Vol}(TK)}{\mathrm{Vol}(B^n)}\right)^{1/n},$$

where B^n is the n-dimensional Euclidean unit ball, which follows from polar integration and Hölder's inequality, we conclude that

Theorem 4.1. *There exists a universal constant A_6 such that for every symmetric convex body $K \subset \mathbb{R}^{2n}$ there exists a position TK, where T is a volume-preserving linear transformation, for which*

$$M^*(TK) \leq A_6 \|\mathrm{Rad}\|_{X_K} \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\right)^{1/2n}$$

This is already close to our goal. However, it is important to note that the above mentioned transformation T need not be symplectic, and hence Theorem 1.7 does not follow directly from the combination of Theorem 3.1 and Theorem 4.1. However, these theorems serve as a tool and as motivation for the line of proof of Theorem 1.7. Our next step is therefore to deal with the "non-symplectivity" of the transformation T. To this end, we shall need the following well known fact about the simultaneously normalization of a symplectic form and an inner product, which we already used in the example of an ellipsoid in Section 2 (see e.g. [15]).

Lemma 4.2. *Let (V, ω) be a symplectic vector space and let $g : V \times V \to \mathbb{R}$ be an inner product. Then there exists a basis $\{u_1, v_1, \ldots, u_n, v_n\}$ of V which is both g-orthogonal and ω-standard, that is, $g(v_i, u_j) = 0$, $g(u_i, u_j) = c_j \delta_{i,j}$, $g(v_i, v_j) = d_j \delta_{i,j}$, and $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$, $\omega(u_i, v_j) = \delta_{i,j}$. Moreover, this basis can be chosen such that $c_j = d_j$ for all j.*

Straightforward linear algebra gives

Corollary 4.3. *Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic structure and the standard inner product. Let T be a volume preserving $2n$-dimensional real matrix. Then there exists a linear symplectic matrix $S \in \mathrm{Sp}(\mathbb{R}^{2n})$ and an orthogonal transformation $W \in O(2n)$ such that*

$$T = WDS, \quad \text{where} \quad 0 < D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n), \text{ and } \prod_i r_i = 1.$$

We postpone the proof of Corollary 4.3 to the end of this section. Combining Theorem 4.1 with Corollary 4.3 and Equation (3.1), and using the fact that M^* is invariant to orthogonal transformations we get

Theorem 4.4. *There exist universal constants A_3 and A_6 such that for any dimension $2n$, for every symmetric convex body $K \subset \mathbb{R}^{2n}$ there exists a positive diagonal transformation $D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n)$ with $\prod r_i = 1$ and a symplectic position $K' = SK$ where S is a linear symplectic transformation such that*

$$s^*(DK') \leq A_3 M^*(DK') \leq A_3 A_6 \|\mathrm{Rad}\|_{X_K} \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/2n}.$$

In order to complete the proof of Theorem 1.7 we shall need the following proposition.

Proposition 4.5. *There exists a universal constant A_7 such that for any convex body $K \subset \mathbb{R}^{2n}$ and for every diagonal matrix $D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n)$ with $\prod r_i = 1$ there exists a holomorphic plane $E = \mathrm{span}\{v, iv\}$ such that the orthogonal projection of the body K to the subspace E satisfies*

$$P_E(K) \subset A_7 B_E^2(s^*(DK))$$

Postponing the proof of Proposition 4.5 we first use it to prove Theorem 1.7

Proof of Theorem 1.7. Let $K \subset \mathbb{R}^{2n}$ be a symmetric convex body. It follows from Theorem 4.4 that there exists a symplectic linear image $K' = SK$ of K where $S \in \mathrm{Sp}(\mathbb{R}^{2n})$ and a positive diagonal matrix $D = \mathrm{diag}(r_1, r_1, r_2, r_2 \ldots, r_n, r_n)$ with $\prod r_i = 1$ such that

$$s^*(DK') \leq A_8 \|\mathrm{Rad}\|_{X_K} \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/2n}$$

where A_8 is universal $A_8 = A_3 A_6$. Next we apply Proposition 4.5 for the body K' and to the above mentioned diagonal matrix D we conclude that there exists a holomorphic plane E such that

$$P_E(K') \subset A_7 B_E^2(s^*(DK'))$$

Since E is holomorphic and since the symplectic capacity is monotone, we have that for every symplectic capacity c we have

$$c(K) = c(K') \leq c^Z_{lin}(K') \leq \pi A_7^2 A_8^2 \|\mathrm{Rad}\|^2_{X_K} \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/n},$$

and the proof of Theorem 1.7 is complete. □

The rest of this section is devoted to the proofs of Proposition 4.5 and of Corollary 4.3.

Proof of Proposition 4.5. Let K be a centrally symmetric convex body and let $D = \mathrm{diag}(r_1, r_1, \ldots, r_n, r_n)$ be a positive diagonal matrix with $\det D = 1$. It follows from the proof of Theorem 3.2 applied to the body DK that there exists an holomorphic plane $\widehat{E} = \mathrm{span}\{v, iv\}$, where v and iv are vertex of the cube $\{\frac{1}{\sqrt{2n}}, \frac{-1}{\sqrt{2n}}\}^{2n}$ such that

$$\|v\|_{(DK)^\circ} = \sup_{z \in DK} |\langle v, z\rangle| \leq A_5 s^*(DK), \quad \|iv\|_{(DK)^\circ} = \sup_{z \in DK} |\langle iv, z\rangle| \leq A_5 s^*(DK) \tag{4.1}$$

Next, denote by $\|\cdot\|_2$ the Euclidean norm and consider the vectors

$$v' = \frac{Dv}{\|Dv\|_2} = \Big(\frac{n}{\sum_{i=1}^n r_i^2} \Big)^{1/2} (v_1 r_1, v_2 r_1, v_3 r_2, v_4 r_2, \ldots, v_{2n-1} r_n, v_{2n} r_n)$$

and

$$iv' = \frac{iDv}{\|Dv\|_2} = \frac{Div}{\|Dv\|_2} = \Big(\frac{n}{\sum_{i=1}^n r_i^2} \Big)^{1/2} (-v_2 r_1, v_1 r_1, -v_4 r_2, v_3 r_2, \ldots, -v_{2n} r_n, v_{2n-1} r_n)$$

Note that these two unit vectors satisfy of course that $v' \perp iv'$ and $w_{std}(v', iv') = 1$. Moreover by the geometric arithmetic mean inequality we see that

$$\|Dv\|_2 = \Big(\frac{1}{n} \sum_{i=1}^n r_i^2 \Big)^{1/2} \geq \Big(\prod_{i=1}^n r_i \Big)^{1/2n} = 1. \tag{4.2}$$

Next, denote $E = \mathrm{span}\{v', iv'\}$. A straightforward computation shows that

$$\begin{aligned}
P_E K &= \{\alpha v' + \beta iv' \mid \alpha = \langle v', y\rangle,\ \beta = \langle iv', y\rangle, \quad \text{where } y \in K\} \\
&= \{\alpha v' + \beta iv' \mid \alpha = \frac{1}{\|Dv\|}\langle v, Dy\rangle,\ \beta = \frac{1}{\|Dv\|}\langle iv, Dy\rangle, \quad \text{where } y \in K\} \\
&= \{\alpha v' + \beta iv' \mid \alpha = \frac{1}{\|Dv\|}\langle v, z\rangle,\ \beta = \frac{1}{\|Dv\|}\langle iv, z\rangle, \quad \text{where } z \in DK\} \\
&\subseteq \{\alpha v' + \beta iv' \mid \alpha = \langle v, z\rangle,\ \beta = \langle iv, z\rangle, \quad \text{where } z \in DK\},
\end{aligned}$$

where the last inclusion follows from inequality (4.2) above. Combine this with Equation (4.1) above we get that

$$P_E(K) \subseteq \sqrt{2} A_5 s^*(DK) B_E^2.$$

The proof of the proposition is now complete with $A_7 = \sqrt{2} A_5$. □

Proof of Corollary 4.3. It follows from the Lemma 4.2 that there exists a linear symplectic matrix S and a positive diagonal matrix $D^2 = \mathrm{diag}(r_1^2, r_1^2, r_2^2, r_2^2 \ldots, r_n^2, r_n^2)$ such that $T^tT = S^tD^2S$ (since T^tT is symmetric and positive definite). Note that for the ellipsoid $E = T^{-1}B^{2n}$ we have

$$\begin{aligned} E &:= \{x \mid \langle x, T^tTx\rangle \le 1\} = \{x \mid \langle x, S^tD^2Sx\rangle \le 1\} = S^{-1}\{y \mid \langle S^{-1}y, S^tD^2y\rangle \le 1\} \\ &= S^{-1}D^{-1}\{z \mid \langle S^{-1}D^{-1}z, S^tDz\rangle \le 1\} = (DS)^{-1}\{z \mid \langle S^{-1}D^{-1}z, (DS)^tz\rangle \le 1\} \\ &= (DS)^{-1}\{z \mid \langle DS(DS)^{-1}z, z\rangle \le 1\} = (DS)^{-1}B^{2n} \end{aligned}$$

Thus we get that $B^{2n} = TE$ and $DS(E) = B^{2n}$, so we conclude that there exists an orthogonal transformation $W \in O(2n)$ such that $T(\sqrt{D}S)^{-1} = W$. We thus conclude that $T = WDS$, as stated. □

5 Improvements for special families of bodies

In this section we describe wide classes of convex bodies for which the logarithmic term in Theorem 1.5 can be disposed of.

We begin with describing the example of zonoids. The class of zonoids consists of symmetric convex bodies which can be approximated, in the Hausdorff sense, by Minkowski sums of line segments. Bodies which are Minkowski sums of segments are called zonotopes, and any zonotope in $\mathbb{R}^n$ can be realized as a linear image of an m-dimensional cube for some m. An example for a zonoid is the Euclidean ball in $\mathbb{R}^n$, which can be approximated in the Hausdorff distance up to ε by (random) orthogonal projections of m dimensional cubes for $m = C(\varepsilon)n$ (this follows from Dvoretzky's Theorem, see [16]). In the paper [5] it was shown that every zonoid $Z \subset \mathbb{R}^{2n}$ with volume 1 has some linear image TZ for $T \in SL(2n)$ such that $M^*(TZ) \le A_9\sqrt{n}$ for some universal constant A_9. After normalization we see that this means $M^*(TZ) \le A_9'\big(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})}\big)^{1/2n}$. Thus, applying the same methods as in the proof of Theorem 1.5 we conclude that for zonoids the estimate in Theorem 1.5 is true without the logarithmic factor.

Next, we consider the unit ball of ℓ_p^{2n} for $1 \le p \le \infty$, denoted B_p^{2n}. The case of $p = 1, \infty$ was discussed in Section 2 above, where it was shown that Theorem 1.5 holds with a universal constant instead of the logarithmic factor. For $1 < p < \infty$, one option, which we *do not* use, is to simply use the well known computations for $M^*(B_p^{2n})$ (see e.g. [16]) and for $\mathrm{Vol}(B_p^{2n})$,

$$\mathrm{Vol}(B_p^{2n}) = \frac{(2\Gamma(\frac{1}{p}+1))^{2n}}{\Gamma(\frac{2n}{p}+1)}.$$

In what follows we take a different, more geometric approach, and avoid using the above mentioned estimates.

We begin with the case of $1 < p < 2$, where we can invoke the following inclusion which is easy to check

$$(2n)^{1/2-1/p} B^{2n} \subset B_p^{2n} \subset (2n)^{1-1/p} B_1^{2n}.$$

The left hand side inclusion implies that

$$\left(\frac{\mathrm{Vol}(B_p^{2n})}{\mathrm{Vol}(B^{2n})} \right)^{1/n} \geq (2n)^{1-2/p},$$

and the right have side inclusion, together with the results for B_1^{2n} from Subsection 2.3 implies that

$$c^Z(B_p^{2n}) \leq (2n)^{2-2/p} c^Z(B_1^{2n}) \leq 2\pi (2n)^{1-2/p}.$$

This completes the case $1 < p < 2$. Moreover, we see that the constant we get is universal and does not depend on p.

We turn to the case $p > 2$. Here we are even better off, because we will use the estimates for the cube from Subsection 2.2 where there was a big difference between the capacity and the ratio of volumes. Indeed, we may use the inclusions for $2 < p < \infty$ (where the cube is denoted now by B_∞^{2n})

$$B^{2n} \subset B_p^{2n} \subset B_\infty^{2n}.$$

Therefore

$$\left(\frac{\mathrm{Vol}(B_p^{2n})}{\mathrm{Vol}(B^{2n})} \right)^{1/n} \geq 1,$$

and using the results of Subsection 2.2 we also have

$$c^Z(B_p^{2n}) \leq c^Z(B_\infty^{2n}) \leq 4,$$

which completes the case $2 < p < \infty$. Had we used the exact estimates for M^* and for the volume we would have gotten that the theorem holds with a much better constant (equal to 1, and getting smaller as p grows). We remark again that we arrived at a universal constant independent of both n and p.

Other examples for which the logarithmic factor can be omitted arise from Theorem 1.7. Indeed, whenever we have good bounds on $\|\mathrm{Rad}\|_X$ we can replace the logarithmic factor by these bounds. We could have applied this scheme in the ℓ_p^n case, however that would *not* guarantee that the constants do not depend on p, and in fact, since for $X = \ell_1, \ell_\infty$ we know that $\|\mathrm{Rad}\|_X$ is not bounded, we would have gotten constants that depend on p and explode as $p \to 1, \infty$.

We emphasize that K-convexity is an infinite dimensional notion, and indeed since we are concerned with the dependence on dimension in Theorem 1.5, if we wish to prove a stronger bound in special cases then these particular cases have to

be families of convex bodies in dimension tending to infinity. Some such examples are concrete bodies such as ℓ_p^n, but another way to construct such families is to look at some infinite dimensional Banach space and consider for example all of its finite dimensional subspaces, and the convex bodies which are their unit balls. In this case K-convexity of the original space X will promise a uniform bound, independent of dimension, in Theorem 1.5, instead of the logarithmic factor.

For example, the Rademacher constant $\|\mathrm{Rad}\|_X$ is bounded for $X = L_p(Y)$ for any (fixed) Banach space Y and $1 < p < \infty$, which is defined for a probability measure μ on a set Ω as

$$L_2(\Omega, Y) = \Big\{ f : \Omega \to Y \ ; \ \|f\|_{L_p(\Omega,Y)} = \Big(\int_\Omega \|f\|_Y^p \, d\mu \Big)^{1/p} \Big\}.$$

Thus, all convex bodies which are unit balls of (finite dimensional) subspaces of the space $L_p(Y)$ will satisfy the main theorem without the logarithmic factor (but the bound will depend on the K-convexity constant of $L_p(Y)$).

More generally, X is K-convex if and only if X is of type-p for some $p > 1$. The definition of type-p is as follows (see, e.g. [17]): A Banach space X is called of type-p for $1 \le p \le 2$ if there is a constant C such that for all m and all $x_1, \ldots, x_m \in X$ we have

$$\| \sum_{i=1}^m r_i x_i \|_{L_2(X)} \le C \left(\sum_{i=1}^m \|x_i\|^p \right)^{1/p}. \tag{5.1}$$

The smallest constant C for which this holds is called the type-p constant of X, and is denoted $T_p(X)$. For a Hilbert space obviously $T_2(X) = 1$, and we remark that Kahane's inequality (see e.g. [16]) states that for $1 \le p < \infty$ there are constants K_p depending only on p so that for any X and any $x_1, \ldots, x_m \in X$

$$\| \sum r_i x_i \|_{L_1(X)} \le \| \sum r_i x_i \|_{L_p(X)} \le K_p \| \sum r_i x_i \|_{L_1(X)},$$

so that the notion of type-p does not depend on the choice of L_2-average in the left hand side of (5.1). It is easily seen from the triangle inequality that every Banach space has type 1 with $T_1(X) = 1$, and it follows easily, say from Khinchine inequality, that no non-zero normed space has type $p > 2$.

The theorem we stated above, that X is K-convex if and only if it has some non-trivial type (i.e., type-p for $p > 1$) is due to Milman and Pisier, see e.g. Theorem 11.3 in [17]. Many more equivalent formulations are possible, for example the same theorem also states that X is K-convex if and only if it is locally π-euclidean (for the definitions see [17]).

One more example for convex bodies where this method shows that the logarithmic factor is not necessary are unit balls of the Schatten classes C_p^n for $1 < p < \infty$. To define the Schatten class spaces let u be an $n \times n$ matrix, so $u^* u$ is positive definite

and symmetric, thus it is orthogonally diagonalizable with nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$. The Schatten class C_p^n is the n^2-dimensional space of all $n \times n$ real matrices equipped with the norm

$$\|u\|_{C_p^n} = \left(\sum_{i=1}^n \lambda_i^{p/2} \right)^{1/p}.$$

(Of course we will consider C_p^{2n} when we want to discuss the symplectic capacity of the unit ball of this space.) Tomczak-Jaegermann showed in [21] that Schatten classes C_p^n have the same type/cotype properties as the L_p spaces, so that we have a uniform estimate in Theorem 1.5 also for the unit balls of C_p^n when $1 < p < \infty$ (but the constant which replaces the logarithm can depend on p).

Remark 5.1. In a paper by Giannopoulos, Milman and Rudelson [5] they prove a theorem which gives a bound on the minimal $M^*(TK)$ over $T \in SL(n)$, which is slightly better than the simple bound we used above. Recall that we argued that

$$\min_T M^*(TK) \leq \min_T (M(TK)M^*(TK)) \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B^{2n})} \right)^{1/2n},$$

and then we used the estimate $\min_T (M(TK)M^*(TK)) \leq A_6 \|\mathrm{Rad}\|_{X_K}$ for a universal A_6. They show that (stating their Theorem B in dual form): For every body K there is a position $K' = TK$ for $T \in SL(n)$ such that (for a universal constant c), denoting by d half the diameter of K' (so that $K' \subset dB^{2n}$) one has

$$\frac{cM^*(K')}{\log(d/M^*(K'))} \leq \left(\frac{\mathrm{Vol}(K')}{\mathrm{Vol}(B^{2n})} \right)^{1/2n}.$$

It is easily seen that one *always* has $d(K')/M^*(K') \leq C'\sqrt{n}$ (since K' includes a segment of length $2d$, and so $M^*(K') \geq M^*([-d,d]) = c'/\sqrt{n}$). Therefore their result clearly implies that there is a position for which

$$M^*(K') \leq C_1(\log C_2 n) \left(\frac{\mathrm{Vol}(K')}{\mathrm{Vol}(B^{2n})} \right)^{1/2n},$$

which is exactly the estimate we derived and used above. However, it is plausible that in many cases the position TK they use has a better ratio of diameter and mean width, and then the above estimates gives an improved result.

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